PALACKÝ UNIVERSITY OLOMOUC

FACULTY OF SCIENCE

MASTER'S THESIS

Non-traditional Methods of Teaching Upper Secondary Mathematics

Bc. Klára Spilková

Department of Algebra and Geometry

Supervisor:	**doc. RNDr. Petr Emanovský, Ph.D.**
Study programme:	**Teaching Training in Mathematics for Secondary Schools**
Field of study:	**Teaching Training in Mathematics for Secondary Schools – English Philology**
Form of study:	**Full-time study**
Year of submission:	**2023**

BIBLIOGRAPHICAL IDENTIFICATION

Author:	Bc. Klára Spilková
Title:	Non-traditional Methods of Teaching Upper Secondary Mathematics
Type of thesis:	Master’s
Department:	Department of Algebra and Geometry
Supervisor:	doc. RNDr. Petr Emanovský, Ph.D.
The year of presentation:	2023
Abstract:	This thesis is focused on the problem-based and project-based learning methods in teaching upper secondary mathematics. Its aim is to summarize literature and research dealing with the characteristics and effectiveness of these methods. The practical part includes problem-based tasks and research on the effectiveness of problem-based learning.
Key words:	teaching methods, problem-based learning, project-based learning, upper secondary mathematics
Number of pages:	77
Number of appendices:	4
Language:	English

BIBLIOGRAFICKÁ IDENTIFIKACE

Autor:	Bc. Klára Spilková
Název práce:	Netradiční metody ve výuce středoškolské matematiky
Typ práce:	Diplomová práce
Pracoviště:	Katedra algebry a geometrie
Vedoucí práce:	doc. RNDr. Petr Emanovský, Ph.D.
Rok obhajoby práce:	2023
Abstrakt:	Tato práce je zaměřena na problémovou a projektovou metodu ve výuce středoškolské matematiky. Jejím cílem je shrnout literaturu a výzkumy zabývající se charakteristikou a efektivitou těchto metod. Praktická část obsahuje úlohy zpracované problémovou metodou a výzkum zkoumající efektivitu této metody.
Klíčová slova:	výukové metody, problémové vyučování, projektové vyučování, středoškolská matematika
Počet stran:	77
Počet příloh:	4
Jazyk:	anglický

Declaration

I declare that I have worked on my master's thesis independently under the supervision of doc. RNDr. Petr Emanovský, Ph.D. and that all primary and secondary sources are listed in the bibliography.

In Olomouc on   Author’s signature:

Table of Contents

Acknowledgements

I would like to thank my supervisor doc. RNDr. Petr Emanovský, Ph.D. for his time and helpful recommendations. I am also very grateful to my family and friends for their support and encouragement during my studies.

Introduction

“Give the pupils something to do, not something to learn; and the doing is of such a nature as to demand thinking, or the intentional noting of connections; learning naturally results.”

John Dewey (1916, p. 181)

Many current educators and education institutions emphasize that education should not only consist of gaining new knowledge but should also develop students’ skills and competencies which are transferable into various context (European Commission, 2018). Making learning closer to real life and fostering students’ competencies are also the goals of many innovative teaching methods such as problem-based and project-based learning. These methods value and try to promote authentic real-world problems, collaboration, problem solving, critical thinking, and many other skills and concepts connected to everyday life. In line with Dewey’s quote above, they attempt to do so by giving the students something to do or explore, and hence actively involve them in the learning process. Some teachers, however, may have doubts about the effectiveness of these methods and their appropriatness in teaching mathematics. Thus, this thesis seeks to investigate the use of problem-based and project-based learning in teaching mathematics by reviewing literature and research discussing these methods.

The theoretical part firstly aims at defining terms connected to teaching methods and secondly at summarizing the different perspectives on problem-based and project-based learning and the research of their effectiveness. The practical part consists of problem-based tasks which I created and of my research on the effectiveness of problem-based learning method conducted at one upper secondary school.

1 Teaching Methods and Forms of Instruction

1. 1 Defining a Teaching Method

Teaching methods govern the activity of students and their teachers in the classroom and should lead to the fulfillment of educational objectives and realization of the content of learning. According to Skalková (2007), the term "teaching method" refers to "ways of intentional arrangement of teacher's as well as student's activity aiming at set objectives" (Skalková, 2007, p. 181). This is also the definition I will be using.

Starčár (Turek, 2010) understands the term "teaching method" similarly as "intentional arrangement of curriculum, teacher's and students' activity, which are directed towards achieving the objectives of pedagogical process while respecting methodological principles" (Starčár, 1979, as cited in Turek, 2010, p. 240). This definition further acknowledges the importance of following general principles of good teaching. Čapek in his Modern methodology (2015) does not provide any definition, rather he helps us to understand the purpose of teaching methods by comparing them to "teacher's instruments, tools for carrying out his main job" (Čapek, 2015, p. 34). He also emphasizes the importance for teachers to be familiar with a wide range of teaching methods including their advantages and drawbacks and be able to employ them in their classes and choose the right ones for specific content and a group of students (Čapek, 2015).

1. 2 Selecting a Teaching Method

When selecting a suitable teaching method, several characteristics of the pedagogical process and the environment should be taken into account. These include educational objectives, the subject and the discussed topic, the characteristics, and abilities of the students, learning environment, teacher's experience, and his theoretical training as well as the amount of new learning content (Nezvalová, 2008). Moreover, the effectiveness of teaching methods including their popularity among students should be considered. Students' preferences should not be downplayed, it may be more advisable to employ a method of average effectivity yet one attractive to the majority of students than a method of exceptionally high effectivity, but which is not welcomed by most students (Petty, 2009). Lastly, variability in teaching methods, as noted above, also appears to be beneficial for developing a wider range of skills, increasing students' attention and interest, and respecting diverse types of students (Čapek, 2015; Petty, 2009).

Skalková (2007) suggests that the system of teaching methods which teachers wield should consist of methods leading to acquiring ready-made knowledge but also of those methods developing independent productive thinking. When it comes to learning to be successful there is a need for both, a good command of basic knowledge as well as the capability of self-directed thought processes. Employing the former types of methods enables efficient and systematic deliverance of core knowledge while the latter types develop students' cognitive abilities, equipping them for efficient self-education. Furthermore, systematic use of these methods allows teachers to gradually lead students towards more demanding individual tasks which can also help to increase their self-efficacy (Skalková, 2007).

1. 3 Learning Styles and Teaching Methods

Geoffrey Petty in his book *Teaching today: A Practical Guide* (Petty, 2009) discusses the relationship between learning styles and teaching methods. He points out that there used to be a belief that students learn best through their dominant learning style; and thus, they should mainly be using their preferred style of learning. Each learning style was thought to be equally successful in fulfilling any learning objective. Nevertheless, Petty (2009) suggests that it has been shown that these styles are not interchangeable. Each person may have a preferred way of learning, yet it seems to be beneficial to learn also through other styles. He adds that the more ways the learning material is presented, the better understanding students seem to have. Different styles complement one another. Teacher's job is then to prepare lessons which will be varied and intriguing, will enable students to develop a rich variety of skills, and will include multiple representations for the benefit of all students regardless of their preferred learning style (Petty, 2009).

1. 4 Educational Objectives

Podlahová at al. (2012) defines educational objective as the intended and expected result of the teaching process, towards which teachers and students are directed. The whole teaching should be focused on fulfilling these objectives. These objectives do not solely consist of cognitive domain but also include affective and psychomotor aspects of learning. Educational objectives should be complex, consistent, verifiable, and adequate, i.e., demanding but achievable. When teachers are defining educational objectives, they ought to formulate them in terms of students' activity using active verbs (Podlahová et al., 2012).

In the Czech Republic general educational outcomes are defined in the Framework Education Programme (*RVP – Rámcové vzdělávací programy*), their more concrete definition is provided in school education frameworks (*ŠVP*). Teachers, however, ought to formulate objectives for every teaching unit and their overarching topic. Based on these objectives, teachers choose the educational content, its extent, the teaching methods adequate for the objective, content of learning as well as specific characteristics of their students. Teachers should further be able to assess whether the expected objectives were met, that is whether their teaching led to effective learning. Learning should be manifested in positive changes in students' knowledge, skills, attitudes, and personality traits (Nezvalová, 2008). The extent to which the result of pedagogical process complies with educational objectives provides teachers with the information on the suitability of the expected objectives, and the methods and ways of evaluation employed**.**

1. 5 Direct versus Indirect Instruction

Closely connected to teaching methods are forms of instruction. There are two distinct approaches which will be discussed: direct and indirect instruction (Borich, 2017). Direct instruction seems to be still the most common type. According to Borich (2017), direct instruction is suitable for teaching pieces of information, and rules. Contrastingly, indirect instruction is best for teaching concepts, inquiry or problem solving. In direct instruction the teacher is the one who presents new information and explains the subject matter. Students' task is to understand the given explanation, ask for clarification and remember what was taught. On the other hand, in indirect instruction students are active participants in constructing knowledge, they try to find similarities, generalize, and draw conclusions based on the information they are given and their prior knowledge (Borich, 2017).

There has been an ongoing discussion among educators about the kind of instruction which is the most effective and beneficial one since at least the 1950s (Kirschner et al., 2006). Supporters of direct instruction often refer to cognitive science as one of their main arguments (Barton, 2018; Kirschner et al., 2006). Kirschner et al. (2006) proposed arguments for the superiority of direct instruction referring to "human cognitive architecture, expert–novice differences, and cognitive load" (Kirschner et al., 2006, p. 75). First, in terms of human cognitive architecture the relationship between the working memory and long-term memory is discussed. When processing new information, working memory is very limited; however, these limitations are no longer in action when the information is stored in long-term memory. They further argue that minimal guidance instruction such as problem-based learning or discovery learning fails to take into account these characteristics of working memories, and thus is likely to be ineffective. Second, they point out that there are differences between novices and experts in the strategies that are most effective for their learning. According to Kirschner et al. (2006) novice learners benefit more from guided, direct instruction whereas for experts less guided approaches seem to be more effective. The third argument refers to cognitive load and is somewhat similar to the first one. According to cognitive load theory (Barton, 2018; Kirschner et al., 2006) student's working memory is limited and can easily be filled with voices, movements, noise, or pictures in the classroom, symbols on the board or worksheet, or anxiety. In line with this theory, instruction should be clear not to overload students' working memories, since cognitive overload may lead to inability to process given information, and thus, hinder learning. Furthermore, they argue that indirect instruction may lead to this cognitive overload. Direct instruction is much clearer and allows all students to learn effectively (Barton, 2018).

Alternatively, proponents of indirect learning, such as problem-based, inquiry-based, or project-based learning, often emphasize the importance of developing life skills such as collaboration, self-directed learning, problem-solving, and critical and creative thinking. These skills seem to be successfully fostered through these approaches in tasks which are more complex, ill-structured, closer to real-life problems (Mustaffa et al., 2014; Chen & Yang, 2019; Hafeez, 2022; more sources can be found in paragraphs addressing research on problem-based and project-based learning). However, the importance of providing deep content knowledge is not overlooked (Hmelo-Silver et al., 2007). In the previous paragraph Kirschner et al.'s article (2006) criticizing constructivist approaches such as problem-based, experiential, or inquiry-based teaching was summarized. In response to this text Hmelo-Silver et al. (2007) published an article where they advocate for the effectiveness of such approaches. The first problem of Kirschner et al.'s (2006) arguments, they point out, is that problem-based (PBL) and inquiry-based learning (IL) are regarded as approaches with minimal guidance from the teacher. However, as they claim both PBL and IL rely heavily on scaffolding to facilitate students' learning (Hmelo-Silver et al., 2007). There are different scaffolding strategies that can be employed, and their goal is to direct students' attention towards productive thinking, to structure the task, or to provide explanation or information needed for the task, when some form of direct instruction may be used. They argue that scaffolding can reduce cognitive load and enables students to learn productively, and thus, undermine another argument proposed by Kirschner et al. (2006). Lastly, they present a body of research which shows the effectiveness of PBL and IL and that these approaches can be in fact beneficial even to lower-performing students (Hmelo-Silver et al., 2007).

This discussion about the superiority of direct or indirect learning, however, is not yet resolved as can be demonstrated on the recent incident between Jo Boaler, a mathematics educator at Stanford University advocating innovative approaches, and few academics supporting traditional teaching of mathematics through direct instruction (Boaler, 2023).

1. 6 Real-World Problem Solving in School Mathematics

An issue which is connected to the discussion above is whether to use real-world problems in school mathematics which are often complex and ill-structured, or whether to design tasks which are clearer, more structured, and thus, preventing cognitive overload (Barton, 2018).

Jurdak (2016) advocates for the former emphasizing that mathematical literacy is not solely the ability to recall and use mathematical concepts when asked to do so, but instead refers to UNESCO definition of mathematical literacy which "may be framed as the ability to identify, understand, interpret, create, communicate, and compute mathematical information, using printed and written materials associated with varying contexts" (Jurdak 2016, p. 44). This definition stresses the importance of the ability to use mathematics in various contexts outside school. Furthermore, referring to activity theory he argues that school math and real-world problem solving are two distinct skills and that students struggle with transferring problem-solving skills to different contexts. He proposes incorporating various contexts, such as modeling, critical mathematics education, or workplace context, into mathematics curriculum. In his perspective, this should help students to develop the ability to transfer problem solving taught at school into real life (Jurdak, 2016).

In contrast, Barton (2018) suggests that using real-life contexts in math may be problematic and even hinder learning. He mentions three common types of problematic real-life contexts in school math. Some contexts may be pointless, for example when used to practice calculations we want to be automatic, others may be confusing and contain many ambiguities which teachers may overlook. Moreover, he remarks that real-life tasks frequently demand simplification of the discussed phenomena or situation and may give a false impression that reality is quite simple. He also notes that motivation can be increased by feelings of success and purposefulness of learning which can be encouraged in other ways beside real-life examples (Barton, 2018).

As can be observed, there are many perspectives on what the best educational practices are. Personally, though I gained valuable insight from both perspectives, I believe that focus on developing the ability to use problem-solving skills learned at school in everyday life, which also includes introducing real-world problems, is more in line with the kind of education many educational institutions have promoted for at least the past two decades (Anon, 2007; Balada et al., 2007; European Commission, 2018) and perhaps is more likely to prepare students for their future lives and careers since the focus is more on skills and competencies which are transferable into various contexts. Nevertheless, I would say that teachers should still be aware of concepts such as cognitive load and reflect it in their teaching. For example, automaticity of some mathematical operations may help to prevent cognitive overload in more demanding mathematical problems (Barton, 2018). In the following section I will present and describe methods for teaching mathematics which use mostly indirect instruction; however, before that I included my summary of Jo Boaler's study on the effectiveness of traditional and reform teaching approaches which I found very engaging, and which motivated me to explore more recent research on this topic.

1. 7 Jo Boaler's (2002) study on the Effectiveness of Traditional and Reform Teaching Approaches

Experiencing School Mathematics: Traditional and Reform Approaches to Teaching and Their Impact on Student Learning (Boaler, 2002) was the first book which presented direct evidence for the effectiveness of traditional and progressive approaches to teaching mathematics (Experiencing school mathematics, n.d.). Jo Boaler, an author on education and a Professor of Mathematics Education at Stanford University (Jo Boaler, n.d.), carried out a 3-year long longitudinal study in which she thoroughly analyzed traditional and reform approach to teaching mathematics as adapted in two English schools, Amber Hill and Phoenix Park (Boaler, 2002). Though this research is rather dated, I believe that it can still provide some useful insight into the outcomes of these two contrasting teaching approaches.

Both schools had a significant percentage of students from working-class families and a similar proportion of students having the same socioeconomic status (SES), both were located on the outskirts of large cities within mostly white, working-class communities. Amber Hill mathematics teaching practice was characterized by mainly “exposition and practice” approach where the teacher explained the particular learning matter and subsequently the students practiced similar math problems. They followed one textbook which was focused on preparing them for the final exams. The mathematics taught in this manner could be described as rule bound and closed, since students were supposed to remember many formulas and the teachers posed mainly closed questions. The students were disciplined though the level of students’ engagement varied.

Conversely, Phoenix Park, adopted a reform approach, their mathematics lessons were project-based with very little direct instruction. Typically, the students were given an open problem and the freedom to follow any direction they chose. If they did not know how to continue, teachers would never tell the students explicitly what to do next. Rather they supported them in finding the solution or a new direction by posing good questions. Only when the need arose the teacher would explain new formulas or principles to groups or individual students. The students had the freedom to participate as well as not to do any work at all; and thus, the lessons were frequently noisy. However, the teaching approach changed in the middle of the final year when they adopted a more traditional approach in order to prepare students for the style of the final exam and to cover some topics that students may not have encountered through their projects. Students from both schools took the same standardized exam at the end of the final year (GCSE). Boaler as a part of her study analyzed the results of both schools.

She observed that even though a similar proportion of students from both schools attained the grades A*- C, more Phoenix Park students passed the exam. These results were surprising since Amber Hill mathematics was focused on the examination, the students were more motivated to achieve well and had calculators. In Phoenix Park, though the style of the exam was very distinct from the manner mathematics was taught, the exam itself was not viewed as seriously as at Amber Hill, some students lacked calculators and the achievement on the entry to Phoenix Park was lower than a national average, still the percentage of students who passed their final exam at the end of their studies was higher than in Amber Hill and even than a national average.

There were also other differences, such as in the proportion of correctly answered procedural versus conceptual questions, or the distribution of the grades based on the SES. First, Phoenix Park students answered a greater number of conceptual questions than Amber Hill students who answered many more procedural questions. This led Boaler to conclude that at each school students developed a different type of mathematical knowledge. Amber Hill's approach led to better command of mathematical rules and procedures. This knowledge, however, tended to be forgotten over a longer period of time, and was not very flexible. The mathematical knowledge of Phoenix Park students did not predominantly consist of vast amounts of mathematical facts, rules, and procedures. Their knowledge was flexible and could be adapted into many different situations, and thus, enabled students to solve problems they did not see before. Second, a greater proportion of Amber Hill underachieving students were of working class, in Phoenix Park middle-class students had a similar probability to be underachievers as working-class students. Boaler hypothesized that this could be due to the lack of grouping, since in Phoenix Park students worked in mixed-ability groups, and the opportunity for all students to learn more demanding content (Boaler, 2002).

Since writing this book Jo Boaler has published many books and research on mathematics education (Jo Boaler, n.d.). I included the summary of this book since I believe that it provides valuable insight into the two distinct approaches to teaching mathematics and into their outcomes. This book also made me wonder whether contemporary research confirms or contradicts her findings in some ways. Thus, in the following section I will discuss two innovative approaches: problem-based learning and project-based learning. My attempt is to provide a theoretical framework of these two methods, describe their main characteristics and discuss the findings of more recent research.

2 Non-traditional Methods of Teaching Mathematics

Schunk (2012) describes traditional teaching as a kind of teaching where the "teacher prepares a lesson, presents it to a class, gives students assignments and feedback, and evaluates their learning" (Schunk, 2012, p. 272). Traditional methods, such as the lecture method, are teacher centered, and thus, students may become passive learners (Ullah & Iqbal, 2020). They are frequently associated with direct teaching strategies, many of which are regarded as a fundamental part of effective teaching (Direct instruction definition, 2013), these include "explanation, example, practice, and feedback in the context of a presentation and recitation format" (Borich, 2017, p. 251). There are also other instructional strategies such as "indirect instruction, constructivist and self-directed instruction, and collaborative learning" (Borich, 2017, p. 251). These forms of instruction are more commonly associated with non-traditional teaching methods though direct instruction many also be used to some extent (Direct instruction definition, 2013). This chapter will provide an overview of two non-traditional methods: problem-based and project-based learning.

2. 1 Problem-based Learning

2. 1. 1 Definition

Problem-based learning (PBL) is a method which has been promoted by many educators as an effective alternative to traditional teaching methods. According to Savery (2006), PBL "is an instructional (and curricular) learner-centered approach that empowers learners to conduct research, integrate theory and practice, and apply knowledge and skills to develop a viable solution to a defined problem" (Savery 2006, p. 9). It is a constructivist approach; thus, students are expected to find a solution of a given problem based on their own research, discussion with their peers and adequate help from the teacher.

2. 1. 2 History of PBL

The PBL was first adopted in the medical field in the late 1960s. Previously, medical education consisted mainly of long lectures and students were expected to absorb excessive amounts of information not all being relevant to their future practice. According to Barrows (1996, as cited in Pagander & Read, 2014), this often led to students' demotivation and boredom and moreover did little to prepare medical students for clinical practice. Adoption of PBL meant that students more frequently encountered situations similar to those in their future medical career. The teaching became closer to practice, students' task was to diagnose a patient; thus, they practiced hypothetical-deductive reasoning as well as expert knowledge. This approach spread in the following two decades to more medical schools in North America and Europe (Savery, 2006) despite the fact that even in the beginning of the 21st century some systematic reviews such as those by Newman (2003, as cited in Savery 2006) or Lynagh (2005, as cited in Savery 2006) concluded that available evidence for effectiveness of this approach is not of high quality, and thus, called for further studies. Since then, PBL was further adapted to various disciplines including mathematics and to different levels of education, from elementary schools to universities, some even developing PBL curriculum for the whole school (Savery, 2006).

2. 1. 3 Theoretical Framework of PBL

According to Pagander and Read (2014), all learning methods are based on theories which try to explain how we learn. Regarding PBL they suggest that "similarities can be found in, as well as parts borrowed from, Donald Schön's 'reflective practitioner', John Dewey's 'learning by doing', David Kolb's learning circle, Piaget's cognitive development theory, Krashen's input hypothesis, and Vygotsky's concept of the Zone of Proximal Development and scaffolding" (Pagander & Read, 2014, pp. 7-8). Zhang (2012, as cited in Arbo & Ching, 2022) likewise states that PBL is based on constructivist ideas about learning which stem from theories by Piaget and Vygotsky.

2. 1. 4 Characteristics and Key Elements of PBL

There are various approaches to PBL which differ in certain aspects; however, there are several features they share. Arbo and Ching (2022) list the following four elements of PBL approach in the classroom: *authentic problems, collaboration, developing expertise, authentic assessment*. Problems in the real world are complex, not clearly defined nor structured. Hence students should also learn how to solve *authentic problems*, identify problems, and search for information and the best strategy to solve the problem. PBL focuses on *collaboration* among students as well as with their teacher. This develops students' communication and teamwork skills. *Expertise* is developed through practice which is an inherent part of PBL. The tasks are *assessed authentically*. This type of assessment helps students in their future work life since the given tasks and set expectations are close to those in the workplace (Arbo & Ching, 2022).

Moreover, PBL helps to develop problem-solving skills which are crucial for improving in mathematics and mathematical reasoning. Polya (1945, as cited in Arbo & Ching, 2022) mentions four principles of problem solving, these include *understanding the problem*, *finding a strategy*, *applying the strategy,* and *reflecting the solution*. Students may have trouble understanding the terminology; thus, in order to solve the given problem, it is important that all presumably problematic terms are explained. Only when students *understand the problem* thoroughly, can they try to *find the* right *strategy* to solve it. This step is crucial for successfully fulfilling the task. Then the *strategy* is *applied*, this phase is done collaboratively, and hence, helps learners develop communication, collaboration as well as critical and creative thinking. The last step is a *reflection on the problem*, this should help students reinforce what they learned while solving the problem. They reflect on whether their strategy and its application were effective and may identify how they could improve (Polya, 1945, as cited in Arbo & Ching, 2022).

Sammamish Collaborative assembled 7 key elements of PBL classroom (Sammamish, 2015). This was a five-year collaborative work of Sammamish High School teachers, school leaders, and researchers from the University of Washington and Knuth Research with the aim of giving guidance to other teachers embracing PBL approach. They name 7 elements whose implementation should lead to quality PBL experience. They believe that "education focused on these elements is really about fostering creativity, communication and leadership in students" (Sammamish, 2015, p. 3). These elements include four already listed above: *authentic problems*, *collaboration*, *developing expertise*, *authentic assessment,* furthermore they mention *culturally responsive instruction*, *student voice and leadership,* and *academic discourse* (Sammamish, 2015).

According to this document, *problems*, just as real-world problems are, should be multi-layered, ill-structured, *authentic* as well as relevant to the students, which should increase their motivation. Problems can be created either by the teacher, or even by the students. Quality *collaboration* should include both good interpersonal and project management behaviors. Students should not simply work side by side but rather collectively solve the task. Building trust and respect in the classroom is essential for successful collaboration. Teachers should also ensure that all students are included in the group interaction and provide feedback on their group process. *Expertise* requires continual practice and refinement of skills and discipline-specific knowledge, and an access to appropriate social and cultural capital. Moreover, well-timed and clear feedback is also essential for developing expertise. *Authentic assessment* should reflect standards of quality work and performances in professional environments. The assessment can consist of presentations, portfolios, pieces of writing as well as traditional formats such as multiple-choice exams. Students collaborate on the assessment. *Culturally responsive instruction* takes into consideration that all students were raised in different environments and may belong to various cultural, ethnic, religious, economic, and other groups whose values influence their education. Incorporating students' background into curriculum and classroom practices makes learning more authentic and relevant for all students and contributes to building a democratic society. Encouraging s*tudent voice* and *leadership* further helps students to adopt democratic values and principles. All student voices regardless of their cultural, racial, gender, or linguistic identity should be included and valued. This also improves their communication and leadership skills, which is helpful for their future careers. PBL should furthermore help students learn and practice *academic discourse*. Students should participate in academic discussion, give a whole class presentation, write different materials, professional emails and letters. They should also learn what the appropriate style of communication in various circumstances is (Sammamish, 2015).

Concerning the role of teachers, Barrows and Tamblyn (1980, as cited in Pagander & Read 2014) suggest that it is to facilitate learning rather than convey knowledge. This may, however, prove to be problematic since many teachers are used to having the latter role. Moreover, as Cazzola (2008 b) points out, working with teachers' beliefs is essential since many teachers encounter only traditional methods in their studies and may still regard them as the most effective ones. Thus, allowing pre-service teachers to experience innovative methods and properly preparing them for their new role as facilitators seem crucial (Cazzola, 2008 b).

Hmelo et al. (2007) also comment on the role of teachers as facilitators. They present several scaffolding strategies that teachers in PBL and IL (inquiry-learning) environments use to help students think productively and to prevent cognitive overload. First, teachers pose leading questions, provide diagrams, investigation models, or worksheets to model expert reasoning and provide support. Second, teachers may explain some concepts directly or give a lecture when the need arises. Third, teachers help students to structure the task by working with a model of problem-solving process (Hmelo et al. 2007). An example of such a model is introduced in the following paragraph.

2. 1. 5 PBL Lesson

According to Bokonjic (2009) and Egidius (1999) (as cited in Pagander & Read, 2014), PBL lesson should include the following seven steps:

1. **Clarifying terms** – There is a table on the board with four columns: Facts in the text, Problem, Hypotheses about cause and effect, Learning objectives. The teacher hand outs a task to the students and clarifies any unknown or problematic terms. Students identify the facts in the text and write it in the first column of the table.
2. **Defining the problem** – Each group has a discussion in order to identify the problem and propose methods for finding the solution. The problem is written in the second column.
3. **Brainstorming –** Students brainstorm ideas in groups, they try to come up with various ideas which would solve the problem. Ideas are not evaluated; all are written down.
4. **Structuring and hypothesis –** Based on the step 2 and 3 hypotheses about the nature of the problem and its solution are made. Students have to agree on the best hypothesis, then structure it and write it in the third column of the table.
5. **Learning objectives – S**tudents have to set their learning objectives, i.e., the pieces of information they need in order to work on their hypothesis. They write it into the last column of the table.
6. **Searching for information –** Students search for information individually for at least two days. It may be better to give more time so that students are able to find reliable sources.
7. **Synthesis –** Finally, students share their findings in their group. Having new information, the group again analyzes the problem and ideally finds its best solution.
8. (**Feedback**) **–** Both students and the teacher give feedback in order to improve the next lessons. Feedback is given on the group and individual work, their organization and the guidance of the teacher.

Table 1: Example of a table used in PBL lessons.

Facts in the text	Problem	Hypothesis and cause and effect	Learning objectives

Cazzola (2008) offers a shorter list of four steps which PBL lessons usually follow. First, students are given a problem. Second, the small group discussion follows, and then each student tries to find information needed to be able to solve the problem. Next, students meet and put together all their findings and discuss it. They try to draw conclusions based on their discussion. Finally, if a new problem emerges, the group goes back to the first step (Cazzola, 2008).

2. 1. 6 Research on PBL

The research on effectiveness of PBL approach seem to be both scarce and ambiguous. One can find research supporting PBL; however, there are others being neutral or even against it. Pagander and Read (2014) in their review of literature suggest that research shows that PBL increases the "feeling" of learning more. Their other findings are summarized in the following text. Most Pro-PBL research suggests that collaboration and solving problems in mixed ability groups is beneficial even for weaker students. PBL also seems to improve students' problem-solving skills at school and practical skills such as planning, independent learning, thinking, and reasoning which are valuable for their future careers. Presumably, PBL also positively affects the mood in the classroom. Conversely, some research is against uncritical adoption of PBL. It shows that PBL is beneficial mostly for stronger students and may demotivate weaker ones. Moreover, they question the reliability of PBL research and suggest that regarding the development of problem-solving and life skills the evidence is weak. They further suggest that research shows that adoption of PBL is subject specific and that for those courses where the subject material needs to be learned in sequence, such as engineering, it may not be suitable at all (Pagander & Read, 2014).

They further list some factors influencing the effectiveness of PBL. First, it is the size of the class, they should ideally be smaller than the norm is. Large classes make PBL difficult since the teacher is not able to help all groups effectively. Second, there are more definitions and realizations of PBL which makes the implementation challenging. The effectiveness of this method is also hard to assess since the most prevalent form of assessment is standardized testing; however, self-assessment is crucial for PBL. Next, the research agrees on the significance of the teachers who should be appropriately trained for PBL and that the school curriculum should be in line with the style and objectives of PBL. Overall, as Pagander and Read (2014) suggest, PBL is a method which is suitable for certain students and situations since the research on PBL carried out so far cannot be generalized. However, they point out that most research agrees on the fact that PBL develops problem-solving skills, independent thinking, their active participation as well as life skills. PBL, thus, seems to enhance students' competencies rather than only knowledge, which corresponds to what many educators currently emphasize. Nevertheless, summative assessment, which is still the main assessing format in many classrooms, and PBL are not easily reconcilable; thus, in many classes it may be challenging to implement this method and much guidance and training would be needed in order to implement PBL successfully and reach all students regardless of their abilities (Pagander & Read, 2014).

2. 1. 6. 1 Mathematics Education Research on PBL

Mustaffa et al. (2014) carried out a meta-analysis on the existing research of the impact of PBL in mathematics. They analyzed the effectiveness of PBL implementation according to different school levels. Their conclusion was that implementing PBL in primary school rarely impacted mathematics positively. There were only 2 out of 21 papers whose report was a positive one. The challenges that teachers have to face at this level are that group work is heavily dependent on the guidance from the teacher, and it may be difficult to sustain a positive attitude in students as well as teachers' emotions. Contrastingly, implementing PBL at secondary levels had a positive impact both on cognitive and affective domains. According to most of the research they analyzed, PBL enhances problem-solving and reasoning skills, students' knowledge and in-depth understanding, and furthermore, increases students' motivation. Some difficulties, however, may occur during the implementation of PBL. For example, many students are used to traditional methods, and thus, may prefer these familiar ones over PBL. Moreover, it is crucial that teachers choose a correct problem for a particular learning objective. Finally, this meta-analysis found that PBL also had a positive impact on tertiary education both on cognitive as well as affective domain. They further suggest that PBL also enhances problem-solving and effective communication skills, collaboration, performance, conceptual knowledge, perceived value, useful learning, cooperation, retention, and confidence. Thus, it develops generic skills, which are needed at the workplace (Mustaffa et al., 2014).

Their later paper (Mustaffa et al. 2016), which is a review of literature on the impact of implementing PBL in mathematics, supports their earlier conclusion that PBL positively impacts high school students both in cognitive and affective domain. Positive impact was found both for long as well as short-term implementation of PBL; however, it also seems to be determined by the learning objectives and particular mathematics domain (Mustaffa et al. 2016).

One more recent systematic literature review conducted by Laine and Mahmud (2022) is also optimistic about the effects of PBL on secondary mathematics. This review indicates that implementation of PBL enhances achievement in mathematics at secondary as well as tertiary level. Furthermore, many positive effects on the affective domain and social interaction are mentioned. Overall, in line with their findings Laine and Mahmud (2022) recommend using PBL in classrooms.

These conclusions seem to be in contrast with Pagander and Read's (2014) analysis which indicated that research on PBL is ambiguous. Pagander and Read's review (2014), however, is of an earlier date and was not focused solely on mathematics education. Thus, these factors may explain this apparent discrepancy.

2. 1. 7 PBL Example Tasks

First, I selected a few example tasks from *Awesome Math: Teaching Mathematics with Problem-Based Learning* (Andreescu et al., 2020). More problem-based tasks as well as whole units can be found in this book, moreover, the third chapter of this thesis includes PBL tasks which I created.

Example 1:

A typical adult gorilla is 5 ft tall and weighs 400 pounds. If King Kong is 20 ft tall, how much does he weigh approximately? This problem is about basic measurements; however, many students get it wrong by rushing to provide an answer without *thinking* about what really is being asked. Many will quickly respond, "1,600 pounds," which is completely illogical if they have a sense of weight.

Only 1,600 pounds for such an enormous gorilla? A typical black and white cow weighs that much! A hippopotamus can weigh up to 4,000 pounds. While this is a simple exercise rather than a complicated problem, it illustrates a larger problem in mathematical thinking.

When mathematics pedagogy is reduced to checking a box, guessing an answer, or completing repetitive exercises, then students are rewarded for quickly reaching a solution over thoughtfully working through a problem.

Solution

You know the height difference (one dimension), but you need to translate that to the difference in volume (three dimensions) of the gorillas. If King Kong is four times bigger in each of the three dimensions (4 times taller, 4 times wider, and 4 times longer) than the average gorilla, that equals 400 multiplied by 4 multiplied by 4 multiplied by 4, i.e., $400 \times (20/5)^3 = 25\,600$ pounds.

Picture 1: Think Tank – Gorilla (Andreescu et al., 2020, p. xix)

Example 2:

Area and Volume of a Sphere

OVERVIEW

The area of a sphere of radius r is $A = 4\pi r^2$.

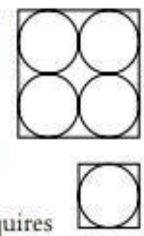
The volume of a sphere of radius r is $V = (4/3)\pi r^3$.

LEARNING OBJECTIVES

- Review the formulas for the area and volume of a sphere.
- Offer an application of these formulas into to a real-life problem.
- Provide an exercise of visualizing a sectional view of a sphere.

PROBLEMS

A box holds eight chocolates (assumed spherical). A front/top/side view of the box is shown. An identical box holds a single chocolate having twice the diameter of the chocolates in the original box. A view of the box is shown:



1. Determine which box contains more chocolate by comparing the volume of chocolate in each box.
2. Each chocolate is wrapped up in paper. Determine which box requires more wrapping paper to wrap the chocolate. Justify your answer.

Picture 2: Mini-Unit – Area and Volume of a Sphere (Andreescu et al., 2020, p. 172)

Second, I compiled a list of resources with PBL tasks or with sources which can inspire teachers to create their own PBL problems, I discovered some these resources thanks to Andreescu et al.'s (2020) book on problem-based learning and two books on project-based learning which recommend sources useful both for problem-based as well as project-based learning (Krauss & Boss, 2013; Wolpert-Gawron, 2016):

- *Awesome Math: Teaching Mathematics with Problem-Based Learning* (Andreescu et al. 2020)
- Three-Act Math – Dy/Dan
  https://blog.mrmeyer.com/
- Desmos | Let's learn together.
  https://www.desmos.com/
- The Learning Network – The New York Times
  https://www.nytimes.com/spotlight/learning-article-of-the-day
- Smile Program
  https://smileprogram.info/
- NRICH Maths – Cambridge University
  https://nrich.maths.org/teacher-secondary
- Real World Math Lessons | 3 Act Math Tasks | Math Worksheets & Practice (makemathmoments.com)
  https://learn.makemathmoments.com/tasks/
- Free Resources – COMAP
  https://www.comap.com/resources/free-materials
- North American Computational Linguistics Open Competition (nacloweb.org)
  https://www.nacloweb.org/practice.php
- Sample Problems | Society for Industrial and Applied Mathematics (siam.org)
  https://m3challenge.siam.org/resources/sample-problems
- Old Contests (purplecomet.org)
  https://purplecomet.org/?action=resource/oldcontests
- Regain your spark for teaching maths – Maths Teacher Circles
  https://www.mathsteachercircles.org/maths/
- Teaching with Gapminder | Gapminder
  https://www.gapminder.org/teaching/

2. 2 Project-based Learning

2. 2. 1 Definition

Project-based learning (PjBL[1]) is a student-centered instructional approach (Kokotsaki et al. 2016), in which students explore and engage in solving authentic problems with the goal of producing a final product. Projects require students' initiative, the teacher's role is to be an advisor rather than authority, projects often continue over a longer period of time (Helle et al. 2006). However, there is no single definition of project-based learning and different authors may value certain aspects of PjBL over others (Condliffe et al., 2017). For example, the Buck Institute for Education (BIE), standing behind a website PBLWorks (2022) which offers project-based learning services and tools, defines project-based learning as "a systematic teaching method that engages students in learning knowledge and skills through extended inquiry process structured around complex, authentic questions and carefully designed products and tasks" (Pecore 2015, p. 159). As this definition shows, PjBL is often defined rather broadly; and thus, implementation of PjBL may also be quite distinct. For instance, John L. Pecore (2015) even includes problem-based learning, case-based learning, and game-based learning among the types of project-based learning (Pecore, 2015). Conversely, Krauss and Boss (2013) although they acknowledge that there are more similarities than differences between problem-based and project-based learning, they discuss the differences in terms of focus, duration, and outcomes. PBL often focuses on a problem within a specific subject and takes several lessons. PjBL, in contrast, is frequently interdisciplinary and students may work on one project from a few days up to several weeks. Further, there is one or more desired answers in PBL, yet in PjBL the outcome depends to a great extent on the students. There is a greater variety in the final products as well as their quality, teachers set minimal standards, but it is up to the students whether they simply fulfill the minimal requirement or exceed them (Krauss & Boss, 2013).

[1] Some authors use the abbreviation PBL both for problem-based and project-based learning, however, I decided to use the abbreviation PjBL for project-based learning to make a clear distinction. This abbreviation can be also found for example in Ralph (2016). (Ralph, Rachel. (2016). Post secondary project-based learning in science, technology, engineering, and mathematics. Journal of Technology and Science Education.)

2. 2. 2 History of PjBL

The concept of learning by projects was encouraged already at the end of the 19th century in many schools for manual training, industrial arts, or agriculture. However, it was William H. Kilpatrick who helped to define and popularize this method in his essay "The Project Method" in 1918. He was inspired by John Dewey's thoughts on education, and Edward L. Thorndike's psychological perspective on learning (Pecore, 2015).

John Dewey was an American philosopher and educator in the beginning of the 20th century who viewed education as an "active and constructive process" (Dewey 1916, p. 46) needed to be experienced. He also emphasized the importance of making connections among school subjects since our world is also not divided into a world of mathematics, physics, history etc. (Krauss & Boss, 2013). The other influence, Edward L. Thorndike, was an educational psychologist and Dewey's colleague for 25 years. In contrast to Dewey's views, he was more interested in quality rather than innovation of schools. He is known for his laws of learning and his findings improved the instruction in the classroom (Hilgard, 1996).

In line with Dewey's and Thorndike's writings Kilpatrick promoted projects which were student-initiated, motivated students intrinsically, and thus, improved learning and increased students' enjoyment and self-confidence at school. He also believed that project learning can further build students' moral character and lead to their independence. Kilpatrick defined projects very broadly as a "whole-hearted purposeful activity" (Pecore, 2015, p. 158) and was convinced, unlike Dewey, that projects should be determined predominantly by the students, not the teacher (Pecore, 2015).

The 21st century brought the revival of the project method, now advanced as project-based learning which is based on constructivism (Pecore, 2015).

2. 2. 3 Theoretical Framework of PjBL

Project-based learning stems from constructivist learning theory which draws on works by Jean Piaget, Lev Vygotsky, and Jerome Bruner. Constructivism states that each individual constructs knowledge by interacting with his physical and social environment. Learners come to class with unique prior understanding and experiences, some of which may be in conflict with the views of the new social environment. Learners then construct new understanding by engaging with the learning environment provided by the school. Teachers should, therefore, assist as guides and support when learners struggle rather than transmit ready-made knowledge (Pecore, 2015). This constructivist nature of project-based learning is clearly pronounced in definition chosen by Kokotsaki et al. (2016) where PjBL is defined as “a student-centered form of instruction which is based on three constructivist principles: learning is context-specific, learners are involved actively in the learning process, and they achieve their goals through social interactions and the sharing of knowledge and understanding” (Kokotsaki et al. 2016).

2. 2. 4 Characteristics and Key Elements of PjBL

The list of characteristics of project-based learning may also vary depending on the author. However, that is not to say that they are at odds with broad definitions of PjBL, such as the one proposed by the Buck Institute for Education (Pecore 2015). According to Condliffe et al. (2017) one aspect that seems to connect different sets of features across various definitions is that all seem to emphasize that students should actively construct their knowledge. However, the lack of consensus on other characteristics poses a challenge to implementation of PjBL by teachers as well as to research on its effectiveness (Condliffe et al., 2017). In the following text a few selected frameworks will be introduced. For instance, according to Mallibhat and Joshi (2020) PjBL has the following four features: thanks to project learning learners may improve in organization of their learning, become more responsible for their learning, customize the projects to be more in line with their interests, and are given space for authentic expression (Mallibhat & Joshi, 2020).

Helle at al. (2006) discuss those features of PjBL which are relevant to some concepts from cognitive psychology, and thus, may be used as arguments for its implementation. These features are *problem orientation*, *constructing a concrete artifact*, *learner control of their learning process*, *contextualisation*, and *multiple forms of representation* (Helle at al., 2006). First, *problem orientation* is in line with the research on expertise which suggests that experts often use pattern recognition in their problem solving (Schunk, 2012). However, in order to form these patterns and identify typical cases, students need to encounter and solve many concrete cases where they use specific domain knowledge. Problem orientation permits this through exposing students to ill-defined real-life problems rather than well defined ones (Helle at al., 2006). Second, the advantage of *constructing a concrete artifact* over traditional learning is that students need to use the information not only remember it. Therefore, if they lack some important knowledge, they may be more motivated to fill these gaps. Moreover, working with the concepts students are less likely to form misconceptions (Helle et al., 2006). Third, throughout the project *learners* have *control of the learning process,* which enables them to pace their own learning and make the activation of prior knowledge and experience more likely to take place since they have some choice over the content and pace of their learning. Thus, new information gets connected to students' already acquired knowledge structures, which is important for its later retrieval (Schunk, 2012). Fourth, learning in projects is *contextualized*. According to the encoding specificity hypothesis we recall things better when the cues in the learning environment and the environment for the retrieval of information are similar (Schunk, 2012). Nevertheless, one cannot know what aspects of the learning environment will work as students' cues, and therefore, also to what extent PjBL learning environment will indeed facilitate retrieval (Helle et al., 2006). Fifth, *multiple forms of representation* are often used at the workplace; however, connecting these representations may prove to be difficult. The advantage of PjBL may be in connecting these models through integrating different discipline content as well as practice and theory (Helle at al., 2006).

Some frameworks, however, not only attempt to describe what the main characteristics of PjBL are, but they further aspire to identify the features which high quality PjBL ought to possess. Two websites promoting project-based learning, PBLWorks (2022) and High Quality PBL (HQPBL, 2022), defined essential elements of quality projects. These models for PjBL are research-informed and should reflect the experience of many educators (Gold Standard PBL, 2022). PBL Works in their article Gold Standard PBL (2022) identified seven essential project design elements: *a challenging problem or question*, *sustained inquiry*, *authenticity*, *student voice & choice*, *reflection*, *critique & revision*, and *public product* (Gold Standard PBL, 2022). First, the *problem* or the *question* driving the project should be engaging and *challenging*. However, not overly challenging, otherwise students may be quickly discouraged. *Sustained inquiry* is the process where students continuously pose questions, search for resources and use required information. *Authenticity* means that projects should be embedded in real-world practices and be relevant to students' lives. *Student voice* and *choice* are essential since they help to increase students' motivation, give space for individual expression, and teach students to take responsibility for their own learning. *Reflection* should be a part of the learning process, students should get accustomed to reflecting on the project and their learning regularly, this can be done either informally throughout the lesson or as an explicit part of the working process, for example as a part of formative assessment. Reflection may help students improve the process of deciding on the right strategy and may help them discover other areas where they can apply their new knowledge. Another important element is *critique* and *revision*, students participate in both activities. Peer feedback should be an integral part of classroom practices; thus, students should be guided on how to give and receive constructive feedback. Feedback, however, comes also from teachers and even experts and other adults outside the classroom may provide authentic feedback. Every project should culminate in a *public product*, there are three reasons for doing so. Firstly, making the product public is likely to increase students' motivation and help them to take the project more seriously. Secondly, it invites other people to be a part of the learning community and discuss students' products. Thirdly, parents and wider community can better understand what PjBL means and what its benefits are (Gold Standard PBL, 2022).

The other initiative, High Quality PBL, also created a framework for high quality project-based learning consisting of six criteria: *intellectual challenge and accomplishment*, *authenticity*, *public product*, *collaboration*, *project management*, and *reflection* (HQPBL, 2022). As can be observed, some of these criteria are similar to the ones defined by PBLWorks. However, even there some differences or different emphases may be found. The first criteria, *intellectual challenge and accomplishment*, is similar to the first element of Gold Standard PBL, however, it highlights even more that the project should not be intellectually effortless, but rather require critical thinking, rigorous study of the subject matter and should motivate students to produce a high-quality product. *Authenticity* means that projects should be connected to the world outside of the classroom, students should be involved in the decision-making process including the choice of the topic, activities, or products. The tools, techniques, and digital technologies employed in the project should correspond to those used in real-life practice. Projects should lead to *public products*. This criterion is also very similar to the one in the previous framework: it increases students' motivation and enables a further discussion with the wider community. The following two criteria, *collaboration* and *project management*, are not listed among the seven elements of Gold Standard PBL; however, they do include them among the learning goals for projects (Gold Standard PBL, 2022). *Collaboration* both within the team as well as with adult mentors, experts and other community members is considered essential, students should learn how to complete the task as a team, not as individuals. *Project management* is another skill which is highly valuable in the workplace, and school projects can help develop it. Students should learn how to efficiently work in a team and use project management processes, tools, and strategies. Finally, students should learn through *reflection* on their work as well as on that of other students'. Students should reflect continuously throughout the project in order to retain acquired knowledge and skills for a longer period of time, increase awareness of their own learning and improve their self-confidence (HQPBL, 2022).

2. 2. 5 Research on PjBL

Examining the effectiveness of PjBL has been the aim of many studies. The majority of these studies used a quasi-experimental pretest-posttest design (Kokotsaki et al., 2016; or Hafeez, 2022, these literature reviews can also be consulted to obtain the list of reviewed studies). Some authors of literature reviews or meta-analyses on PjBL conclude that the evidence for PjBL's effectiveness is not proven and more quality research should be carried out (Condliffe et al., 2017; Kokotsaki et al., 2016). Alternatively, others suggest that enough evidence shows that PjBL is indeed more effective than traditional instruction (Chen & Yang, 2019).

Chen and Yang (2019) in their meta-analysis analyzed existing research for the past 20 years focused on comparing PjBL and traditional instruction. They found that PjBL had a positive effect on academic achievement which was significantly greater than that of traditional instruction. The amount of the effect, however, was affected by subject area, school location, hours of instruction, and the support of information technology. PjBL had a greater effect in social sciences than in science and mathematics, in Western contexts than in East Asian ones. Moreover, the implementation of PjBL was more effective when it was used for at least 2 hours per week and learning was supported by technology. However, based on their analysis they do not advise to eliminate traditional instruction altogether, rather they suggest using traditional instruction when appropriate during a project to teach some basic skills (Chen & Yang, 2019).

Other researchers are not as confident in their claims. For instance, Condliffe et al. (2017) point out that the evidence for effectiveness of PjBL in improving students' academic achievements is still not sufficient; however, they recognize that some studies indicate that PjBL positively affects students' involvement, their motivation and self-efficacy (Condliffe, 2017). According to Chen and Yang (2019) other benefits of PjBL are better attitudes towards learning in general as well as to the subject content, improvements in self-regulation, self-monitoring, self-directed and self-regulated learning, and self-assessment (Chen & Yang, 2019). Furthermore, Hafeez (2022) in his literature review concludes that PjBL effectively develops learners' creative and critical thinking skills, and Indrawan et al. (2019) also mention improvement in problem solving skills.

2. 2. 5. 1 Mathematics Education Research on PjBL

Jacques (2017) conducted a literature analysis on the effectiveness of PjBL in the mathematics classroom. He selected twenty-seven articles for the review, the grade levels ranging from kindergarten to college level. The most frequent topic within the domain of mathematics was geometry and measuring, next were functions and formulas, followed by data and statistics. Regarding non-mathematical topics engineering was the most popular area for PjBL, followed by science and finance. Only twelve articles out of the selected twenty-seven measured what the gains in mathematical skills are, the rest analyzed students' attitudes or did not focus on students themselves. The majority of these twelve studies showed improvement in the assessment of the particular topic with the exception of three studies on fractions or word problems. However, state and standardized exams showed gains in the achievement only provided that PjBL was integrated in the entire school curriculum. Thus, Jacques (2017) concluded that studies show mixed results and that there is a lack of research on this topic to be able to make any generalizations.

According to Holmes and Hwang (2016) PjBL's positive effects have already been established for science and elementary level of mathematics education, however, not yet for secondary mathematics. Thus, in their longitudinal study they explored the effects of PjBL in secondary mathematics education. They collected both quantitative and qualitative data in PjBL school as well as in traditional high school which served as a control group. They have not found any statistical difference between the schools in the overall development of mathematical skills. However, there was a difference in the size of achievement gap among students according to their socioeconomic status (SES) and race. In comparison with the traditional school, the gap based on the SES was noticeably diminished and there was no statistically significant difference in the achievement according to race in the PjBL school. They ascribed this finding to the collaborative aspect of PjBL, since students work in teams with classmates of different backgrounds, can find new friends and help one another to understand the learning matter. This study also found that PjBL has a positive impact on students' intrinsic motivation, promotes peer learning and helps to develop critical learning better than traditional teaching. Nevertheless, they suggest that more research should be conducted to confirm and clarify these findings (Holmes & Hwang, 2016).

2. 2. 5. 2 Limitations of Existing Studies on PjBL

Some authors highlight the limitations of studies on PjBL and argue that there is still not enough quality evidence for establishing effectiveness of PjBL (Kokotsaki et al., 2016; Condliffe et al., 2017). For instance, Condliffe et al. (2017) suggest that "the evidence for PBL's [PjBL's] effectiveness in improving students' outcomes is 'promising but not proven'" (Condliffe et al. 2017, p. iii). Kokotsaki et al. (2016) note in their literature review on PjBL, that in most studies they reviewed there was no random allocation of participants to control and experimental groups, and as a result, these studies cannot provide as strong and reliable evidence for the effectiveness of PjBL. Moreover, some lower quality studies did not involve any control group (Kokotsaki et al., 2016). Condliffe et al. (2017) also point out that making generalizations about the effectiveness of PjBL based on existing research proves to be problematic since there is no uniform view on what PjBL entails and what competencies it tries to promote. Thus, models of PjBL frequently differ depending on the study. Furthermore, they claim that the evaluation design of many studies made it possible for other factors to have influenced the outcomes of their research (Condliffe et al., 2017).

2. 2. 6 Approaches to PjBL Implementation

Condliffe et al. (2017) described three basic approaches to implementing PjBL in the classroom: "Teachers or schools can access externally developed PBL [PjBL] curricula, teachers can develop their own PBL [PjBL] approach, or PBL [PjBL] can be a part of a whole-school reform effort or a critical piece of a school's structure" (Condliffe et al. 2017, p. 13). First, teachers may use externally developed PjBL curricula such as *Investigating and Questioning our World through Science and Technology* (IQWST), *Knowledge in Action* (KIA), or *Project-Based Inquiry Science* (PBIS) (Condliffe et al., 2017). Such curricula make PjBL implementation more accessible for many teachers; however, they can also be viewed as too restrictive. Nevertheless, for instance the developers of the KIA curriculum acknowledge this potential drawback and emphasize the importance of teacher's adaptation of these curricula. Second, teachers may develop their own PjBL curriculum. This approach is common since for many teachers an externally developed curriculum is not accessible. These teachers, however, may access online teaching resources or professional development services. Condliffe et al. (2017) mention for example the following online platforms which can support teachers' effort to use PjBL in their classroom: Edutopia, Mathalicious, or PBLU; and these two professional development resources: *Project-Based Learning Institute* and *Engage! Learning Inc.* (Condliffe et al., 2017). However, these resources are in English, and therefore, may not be accessible for all teachers. Moreover, the lack of research in effective PjBL strategies and measurable principles make the implementation of PjBL more difficult. Third, some schools adopt PjBL as an instructional approach for all classes. For example, the following four school networks implemented PjBL: *Expeditionary Learning Schools* (EL), *New Tech*, *High Tech High*, and *Envision Schools*. One advantage of these models is that some schools, such as New Tech Network and High Tech High, also describe what PjBL means in their schools. This makes the implementation of PjBL easier, and teachers are more likely to be indeed using PjBL, not only claiming, or believing to do so (Condliffe et al., 2017).

2. 2. 6. 1 Challenges in the Implementation of PjBL

According to Condliffe et al. (2017) the review of the existing research "have suggested that PBL [PjBL] implementation is hindered when teachers' beliefs about the process of learning, students' capacity to engage in student-directed inquiry, and educational goals do not align with the deeper learning aspirations of a PBL [PjBL] approach" (Condliffe et al. 2017, p. 32). The school environment strongly affects the teacher's beliefs. For example, some studies found that PjBL implementation is easier when most teachers in the same building also try to use it. On the other hand, if most teachers use teacher-directed inquiry and traditional forms of assessment, implementation of PjBL may prove to be challenging since PjBL demands student-directed inquiry and non-traditional assessment forms. Moreover, quality training and support during PjBL implementation, teacher's first-hand experience of PjBL and having the tools PjBL requires, seem to be essential for successful implementation. Especially given that according to this review implementation of any PjBL model without professional development proves to be difficult (Condliffe et al., 2017).

Hafeez (2019) lists other possible challenges such as students' attitude to working collaboratively, lack of technical and labor skills, information technology problems, time management, social challenges when the project affects the population, and funding problems. Some of these challenges, however, such as the lack of technical and labor skills and information technology problems, pose a greater problem to underdeveloped countries (Hafeez, 2019).

2. 2. 7 Project Design

Designing a high-quality project may also prove to be a very challenging task. This paragraph offers few insights into how to design engaging, learning-rich projects. According to Krauss and Boss (2013, pp. 54-56) the process of designing a project can be divided into six steps:

1. **Identify project-worthy concepts** – find four or five fundamental concepts for each subject,
2. **Explore their significance and relevance** – consider what the students should remember about the concepts for a lifetime, how relevant these concepts are in everyday life, produce a list of topics which are meaningful and worthwhile,
3. **Find real-life contexts** – for each concept identify five to seven professions which use these concepts, consider what the overlaps with other subjects are,
4. **Engage critical thinking** – reflect on how students can be engaged in critical thinking during the project, ask students to: "compare and contrast, predict, make a well-founded judgment or informed decision, understand causal relationships (cause and effect), determine how parts relate to the whole (systems), identify patterns or trends, examine perspectives and alternate points of view, extrapolate to create something new, evaluate reliability of sources" (Krauss & Boss 2013, p. 55),
5. **Write a project sketch** – write two or three project sketches including the overview, describe the activities in which the students will most likely be involved and clarify what the students should learn,
6. **Plan the setup** – write a project title, an entry event which should get everyone's attention (such as a mysterious letter or an intriguing video), and a driving question awakening students to investigate,

Lastly, discuss and workshop your project idea to clarify and improve your project sketch (Krauss & Boss, 2013).

2. 2. 8 PjBL Example Tasks

The first two example project tasks are taken from Boaler's study (2002) discussed at the end of the first chapter. The first project described in this study is called Volume 216 (p. 21), the second 36 pieces of fencing (pp. 51-52). The following two examples of PjBL tasks are taken from Krauss and Boss's book *Thinking Through Project-based Learning* (2013) where many other project sketches can be found for subjects such as social studies, science, math and language arts.

Example 1: Volume 216

Driving Question: The volume of a shape is 216. What kind of shape could it be?

Students look for shapes with a volume 216. They should extend their work for example by further discussing what shapes cannot have this volume and why or other related questions.

Example 2: 36 Pieces of Fencing

Driving Question: “What sort of shapes can you make with 36 pieces of fencing?” (Boaler, 2002, pp. 51-52)

There are 36 pieces of fencing. Each piece is 1 meter wide and can be connected to another piece at any angle. After posing the driving question, rules are written on the board. The first rule is that all pieces of fencing must be used. Other rules are added as students’ questions concerning other possible restrictions arise. These rules may concern for example whether the shapes must be regular or do not have to be. With each question, it is up to the teacher to decide what restrictions will be at play. Students first discuss either as the whole class or in groups what are the possible shapes that can be obtained. Next, students are asked to find areas for all shapes, order them in a way that makes sense to them and write down in their own words their understanding of the problem.

Example 3: 20 Years Old and in Debt! (Grades 9-12)

Driving Question: How can we prosper and not go into debt?

College freshmen carry an average of $1,585 in credit card debt, the cost of three iPads! Students create a scenario that shows they understand what it takes to manage their own credit cards and also consider alternatives to credit cards.

Picture 3: 20 Years Old and in Debt! (Krauss & Boss, 2013, p. 167)

Example 4: Angry Birds Physics (Grades 9-12)

Driving Questions: What laws of physics hold in Angry Birds World? How do software engineers make decisions?

A best-selling video game becomes the basis for an investigation of projectile motion as physics students investigate the question: "What laws of physics hold in Angry Birds World?' Students seek answers questions like these: *Does the white bird conserve momentum when it drops its bomb? Why would the game designer want the white bird to drop its bomb the way that it does? The yellow bird changes velocity with the tap of a finger. Analyze more than one flight path to answer this: What are the details of its change in velocity?* To investigate these questions, students make screencasts of game play using Jing, Screencast-O-Matic, or Camtasia Studio, then do analysis. To support their scientific thinking, they use tools for data analysis and modeling, such as Logger Pro and Tracker Video. Credit: Frank Noschese, Cross River, New York, and John Burk, Delaware.

Picture 4: Angry Birds Physics (Krauss & Boss, 2013, p. 166)

The following list of online resources can be discussed for more examples of PjBL tasks. The list includes rich project tasks or can be used as an inspiration for creating one's own projects, some of these resources were recommended in Krauss and Boss's book (2013) on project-based learning:

- STEM – Secondary mathematics resource packages
  https://www.stem.org.uk/secondary/resources/collections/maths/secondary-maths
- youcubed – Graduate School of Education, Stanford University
  https://www.youcubed.org/tasks/
- Projects | MyPBLWorks
  https://my.pblworks.org/projects
- Examples of Project-Based Learning at High Tech High
  https://www.hightechhigh.org/student-work/projects/
- PBL Essentials – Explore the Core PBL Teaching Practices
  https://pblessentials.org/

3 *Tasks on Problem-based Learning*

3. 1 Problem-based Learning Tasks

This section consists of example problem tasks which I created. It covers three topics: exponential functions, the Sine and Cosine Rule, and volume. The first unit, focused on exponential function, is inspired by PBL learning units in *Awesome Math* (Andreescu et al., 2020), a book discussing teaching mathematics through PBL. I searched for real-world examples on exponential increase and decrease on the internet where I sought to find examples of exponential growth or decay in real life. I used these findings to make problems which would be engaging to students.

The following units focused on the Sine and Cosine Rule, and volume are inspired by Meyer's Three-Act Math tasks (Meyer, 2016). In his blog *Dy/Dan* Dan Meyer shares his approach to teaching mathematics. His goal is to "help students develop a question before answering it and to create a headache before offering them aspirin" (Meyer, 2016). Thus, his math problems aim at capturing students' interest and utilizing their intuition before making actual calculations (Meyer, 2009). The structure of the problems I created is similar to Meyer's tasks, especially in the attempt to activate students' intuition by asking them to make predictions first and in providing the information they need gradually.

3. 1. 1 Exponential Functions

Definition

Let $a > 0$, $a \neq 1$. *Exponential functions* are defined by the following equation: $y = a^x$ for all real numbers.

Exercise A: **Graph**

1) $y = 2^x$

x	-3	-2	-1	0	1	2	3
y							

2) $y = 2^{x-1}$

x	-3	-2	-1	0	1	2	3
y							

3) $y = 2^{x+2}$

x	-3	-2	-1	0	1	2	3
y							

4) $y = 2^x + 3$

x	-3	-2	-1	0	1	2	3
y							

5) $y = 2^x - 3$

x	-3	-2	-1	0	1	2	3
y							

6) $y = 2^{-x}$

x	-3	-2	-1	0	1	2	3
y							

Compare with 1):

7) $y = -2^{x}$

x	-3	-2	-1	0	1	2	3
y							

Compare with 1):

8) $y = \frac{1}{2}^{x}$

x	-3	-2	-1	0	1	2	3
y							

Compare with 7):

9) $y = 3^{x}$

x	-3	-2	-1	0	1	2	3
y							

10) $y = 2.3^{x}$

x	-3	-2	-1	0	1	2	3
y							

11) $y = \frac{1}{2}3^{x+2} - 1$

x	-3	-2	-1	0	1	2	3
y							

Reflection:

How does the graph change according to the value of different parameters?

Exercise B:

At 9 am there was one bacterium which doubled every hour. How many bacteria will be there at 6 pm on the same day?

Find the formula for the number of bacteria as a function of time.

Exercise C:

At 8 am bacteria culture consists of 300 bacteria. Each bacteria doubles every two hours. How many bacteria will be there at 2 pm on the same day?

Find the formula for the number of bacteria as a function of time.

Exercise D:

Without any prevention bacteria replicate exponentially when the temperature is within the range of $4 - 60$ °C, called the "danger zone". They can double even every 20 minutes depending on the bacteria and other conditions. Some food which is prone to contamination (such as fresh salads, rice, pasta, potato dishes, meat, unwashed fruits, and vegetables etc.) may be unsafe to eat after leaving it for two hours within this temperature range. With temperatures lower than 4 °C the reproduction of bacteria is slower if it is below -18 °C the bacteria do not replicate altogether.

a) You leave a piece of chicken on the table, in the beginning there are 70 000 bacteria of one kind, the temperature is 25 °C. Up to how many bacteria can there be in two hours?

b) You leave a piece of fresh sushi on a table in a room of 20 °C. In this temperature bacteria grow by 37 % each hour. Your friend put her piece of sushi into the fridge where the bacteria grow by 6 % each hour. Suppose there are 100 harmful bacteria in both pieces of sushi, how many bacteria will be in your piece and how many in your friend's after leaving it for 12 hours?

Exercise E:

Until 2020 the number of Netflix subscribers grew exponentially with the average year growth rate of 29.4 %. In 2008 there were 9 160 000 Netflix subscribers. Based on this data make an estimate on how many subscribers Netflix had in 2020.

Exercise F (data for this task can be found in the Appendix A):

The population of Delhi in India has increased significantly over the past 70 years. Based on the data in *Delhi population growth.xlsx* determine in which years the population grew exponentially and in which the increase was linear. Try to produce a similar graph to the one in the document using exponential and linear functions.

Exercise G:

Peter transfers 4000 Kč into a savings account with a yearly interest rate of 2 %. How much money will Peter have there in 5, 15, 25 and 35 years?

Exercise H:

Monica saved 5000 Kč from her summer job, she wants to save it for a new laptop which she will need when she goes to the university in 3 years. What will be the purchase value of 5000 Kč in 3 years if the average inflation over the next 3 years is 1.5 %?

Exercise I:

a) Depreciation of iPhone value

You buy a new iPhone for 20 000 Kč. Electronics age quickly, their worth decreases immediately after the purchase and then each month. iPhones lose 25 % of their value after the purchase and then 3 % each month. What is the worth of your iPhone after 6 months? What is the percentage decrease?

b) Depreciation of value of Samsung mobile phone

You buy a new Samsung mobile phone for 6 000 Kč. Electronics age quickly, their worth decreases immediately after the purchase and then each month. Samsung mobile phones lose 30 % of their value after the purchase and then 4 % each month. What is the worth of your Samsung after 6 months? What is the percentage decrease?

3. 1. 2 Sine and Cosine Rule

3. 1. 2. 1 Orientation Run

You are on an orientation run. You are given only a map, a compass, and a calculator. You found the first two control points; you know that the third one is 220 meters N30°W from the second one. There is a very dense thicket which would slow you down, you decide to go north 300 meters and then turn. What direction do you have to take to find the control point? How far will it be?

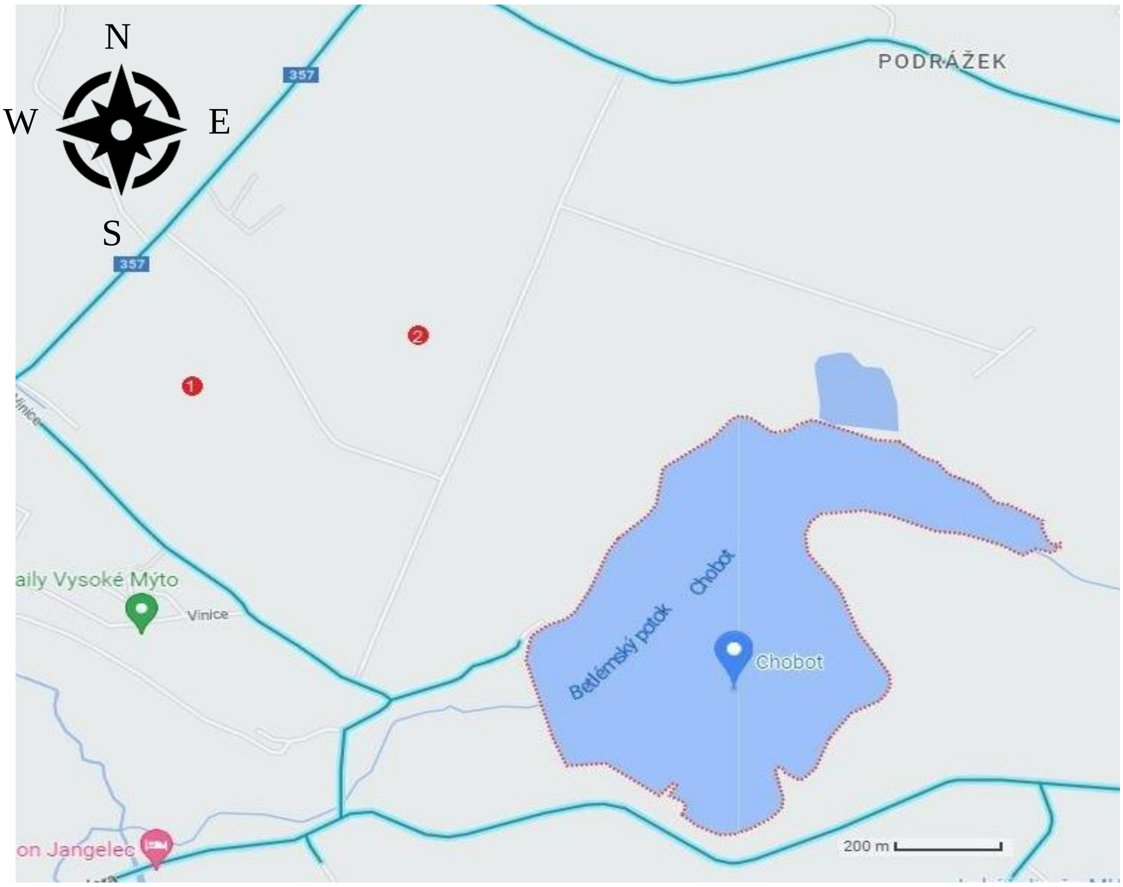
Before doing more precise calculations, make a guess. What is your best guess? What guess would definitely be too small or too big?

Direction (angle):

Your best guess: Too small: Too big:

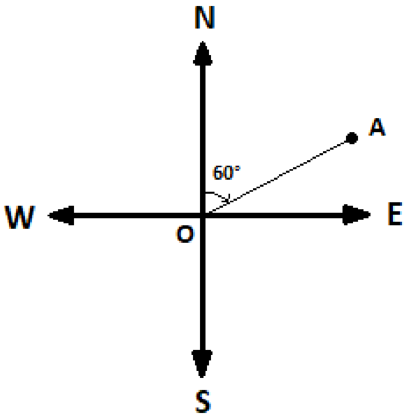
Distance:

Your best guess: Too small: Too big:



Picture 5: Chobot (Google, n.d.-a, and my adjustment of the picture)

What additional information do you need?



Picture 6: Map directions (my picture, inspired by (Directions and Bearings, n.d.))

3. 1. 2. 2 Praděd

You are in a race where you are competing to be the fastest person to reach the top of Praděd. You are very close to the top but there is another runner ten meters ahead of you. You decide to take a shortcut and run straight uphill outside the designated path. However, it was raining yesterday and the hill is quite muddy. Who will be the winner?

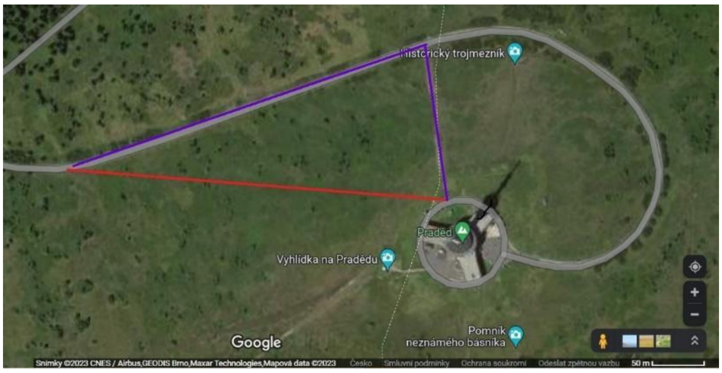
How far is the top of the hill? Make a guess before you do more precise calculations.

The shortcut (red path)

Your best guess: Too small: Too big:

The designated path (blue)

Your best guess: Too small: Too big:



Picture 7: Google n.d.-b, and my adjustments (highlighting the path)

What information do you need to be more precise in your calculations?

The average speed of

Running uphill on a road: 12.5 km/h
Running uphill on a clay path: 10.5 km/h
Running uphill on a muddy grass: 8 km/h

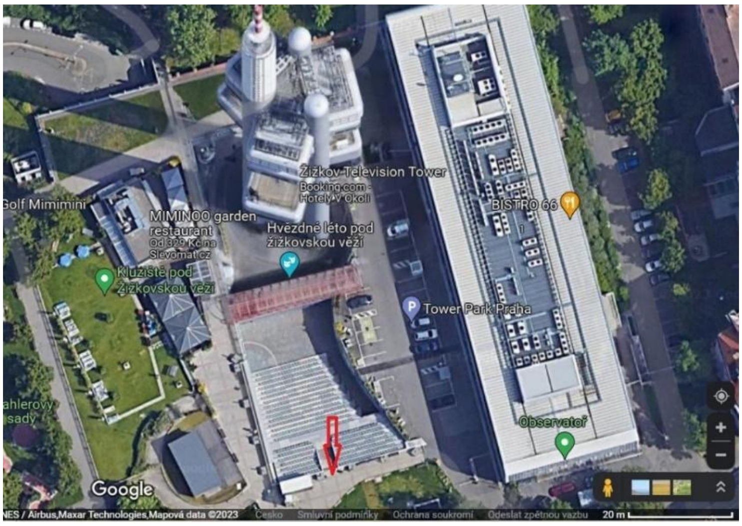
Picture 8: Praděd (Google n.d.-b, and my adjustments – highlighting the path, adding dimensions)

3. 1. 2. 3 Žižkov Television Tower (Žižkova televizní věž)

How tall is the Žizkov television tower? What is your best guess? What guess would definitely be too small or too big?

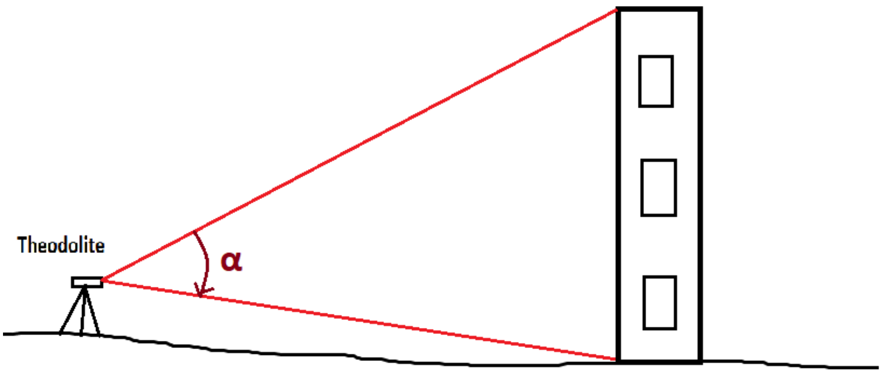
Your best guess: Too small: Too big:

You are standing at the point designated by the arrow below, you measured the angle of elevation of the building (from the bottom to the top) using a theodolite (1.7 m high), the angle is 82.91°.



Picture 9: Žižkov television tower (Google, n.d.-c, my adjustments – adding red arrow)

Picture 10: Žižkov television tower (my photo)

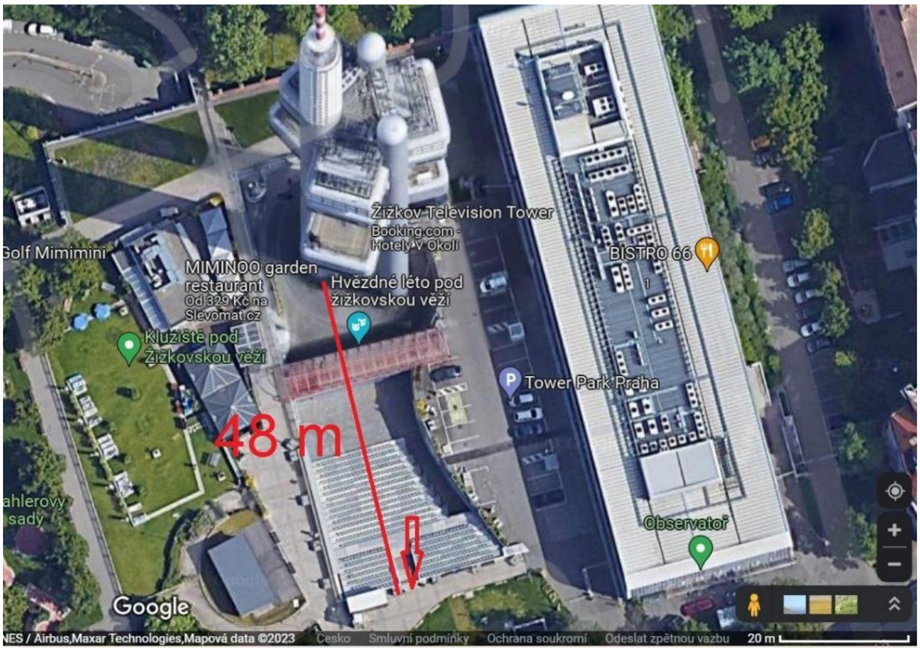


Picture 11: Theodolite (my picture, inspired by Param Visions (2021))

The ground level of the building is 4,8 m lower than your altitude. You are standing at the same altitude as the red bridge.

Picture 12: Towek Park Praha entrance (my photo)

Do you need any additional information?



Picture 13: Žižkov television tower (Google, n.d., my adjustments – adding the red arrow, line and dimensions)

3. 1. 3 Volume

Picture 14: Observation Tower Bára (my photo)

3. 1. 3. 1 Observation Tower Bára

The first observation tower Bára was built in 2008. Sadly, it was demolished by a massive storm only four days after its opening to the public. A year later the new tower Bára II was opened. Now consisting of more security components, such as steel cables. There are also three weights filled with crushed stone.

What is the weight of each weight (picture 3.1.2)? *What is a guess that is too small? What is a guess that is too large?*

Your best guess: Too small: Too big:

Now Calculate the approximate weight. *What pieces of information do you need?*

Picture 15: Weight at the Observation Tower Bára (my photo)

The crushed stones are inside a 10x10 steel grid.

Picture 16: Weight at the Observation Tower Bára (my photo and adjustments)

The density of crushed stones of this diameter is approximately 1500 kg/m^3.

4 Research on the Effectiveness of PBL

The aim of this research is to examine the effect of problem-based learning (PBL) on students' achievement in mathematics. The secondary goal was to gain insight into what may be the possible hindrances for newly trained mathematics teachers from implementing PBL in their classrooms.

Research question:

1) Is there any effect of PBL on students' mathematics achievement?
2) What may hinder newly trained mathematics teachers from implementing PBL in their classrooms?

Hypotheses:

1) The PBL method enhances students' achievement in mathematics more than traditional teaching methods.
2) The possible hindrances of PBL implementation are the size of the class, the lack of teachers' training for PBL, the school curriculum which is not in line with PBL's objectives, and summative assessment.

The first hypothesis was based on two literature reviews conducted by Mustaffa et al. (2016) and Laine and Mahmud (2022). Both reviews concluded that PBL has a positive effect on students' achievement in mathematics. Moreover, the latter review suggested that PBL method leads to better learning outcome than traditional methods (Laine & Mahmud, 2022). The research takes only 2 weeks; however, according to Mustaffa et al. (2016) even a short-term implementation of PBL lasting as little as 2 weeks seems to positively affect both affective as well as cognitive domain.

The second hypothesis is mostly based on Pagander and Read's (2014) summary of factors influencing the effectiveness of PBL. First, they mention that the size of the class should be ideally smaller than the norm is. They further point out that research agrees on the importance of teachers' training in PBL, the appropriately designed school curriculum as well as the assessing format (Pagander & Read, 2014).

4. 1 Methodology

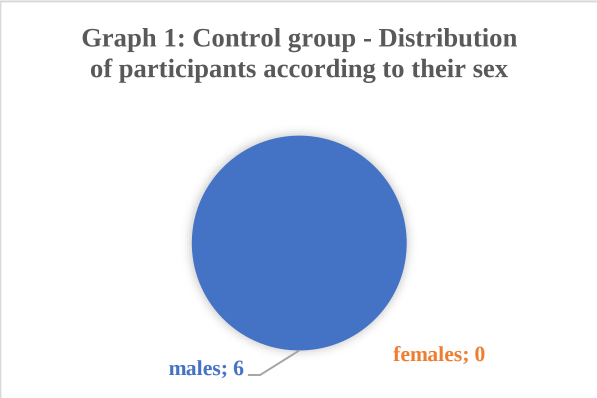
The headlines which are used to describe the research are partly inspired by Nurbavliyev et al.'s study (2020) on the impact of PjBL on students' achievement in mathematics. All tasks and the pre-test and post-test used in this research can be found in the Appendix or in the section on PBL tasks. I included these tasks since while I was researching the existing literature on this topic, some studies which I read did not contain the tasks they used, nor they explained the way they implemented PBL. Similarly, Pagander and Read (2014) mentioned that the problem they encountered during researching for their literature review was the incomplete and vague information about the research as well as the methods used. This made the results of these studies rather unreliable and hard to interpret since one cannot be sure how exactly PBL was implemented.

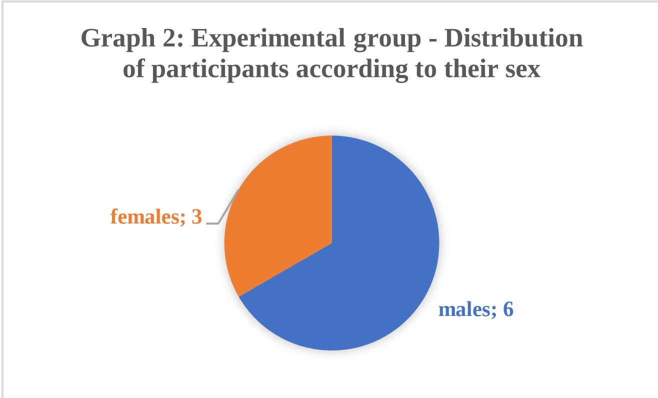
4. 1. 1 Methods

This study used a pre-test and post-test research design and included an experimental as well as a control group. The research was conducted at a technically oriented upper secondary school in April 2023 and it lasted 2 weeks. Subsequently, the pre-test and post-test results were statistically analyzed in Excel performing a Mann-Whitney U test or Wilcoxon signed-rank test. The Mann-Whitney U test examines two independent samples determining whether they have the same distribution. This test can be conducted when the assumption of normality is not met (Hron et al., 2018). For paired samples, the Wilcoxon signed-rank test can be used (Hendl, 2004). These tests were selected instead of t-test since this study analysed only 15 test results (8 from the experimental group, 6 from the control group), and thus, conducting a normality test, a condition for t-test, was not appropriate for such a small sample size.

4. 1. 2 Participants

The experimental and the control group both came from the same class, they were 2nd year high school students of a technical field. The two groups were created randomly by the school in the beginning of their first year. The experimental group consisted of 10 students, of whom 7 were males and 3 females. The control group consisted of 8 students, all males. Their age ranged from 16 to 17 years. However, the consent form was not returned by one male student in the experimental group and two students in the control group. Thus, their test results were not included in the analysis.





4. 1. 3 Lesson Design

Both groups learned the same subject matter, had the same number of classes during the same time period and was taught by the same teacher (myself). However, the material was presented using two distinct methods. The experimental group learned through PBL whereas the control group was taught traditionally by direct instruction.

The control group was taught using direct instruction which followed Angelillo (2008, as cited in Borich, 2017) and Marzano (2009, as cited in Borich, 2017) description of this type of instruction. First, the goals of the unit were stated, second, the new content was presented in rather small steps, next, the teacher modelled how to calculate specific problems, students had chance to ask for clarification and subsequently practised similar problems. The solutions for most problems were shown on the blackboard either by the teacher or students. Finally, students were checked for understanding using questions and by examining students' work. Instructions were given to all students at the same time; however, if needed, support was provided during practice to individual students. Students were also able to dicsuss the problems in pairs.

The experimental group was taught using PBL. Students worked individually or in mixed-ability groups of three to four students, though in some instances the subject matter was discussed with the whole class. First, students were given problem tasks with instructions. Some tasks were given to individual students; however, most tasks were solved in groups. They were given time to think about the problem and its solution. If they felt stuck, support was provided by the teacher using leading questions. Sometimes the content was summarized by the teacher on the blackboard; however, this happened only after students' attempt to solve the particular problem. During group work stronger students were asked to explain the solution and their reasoning behind it to weaker students if they still struggled understanding the problem. The solution was checked by the teacher and other task was provided. Some groups were faster than others, hence, solved more tasks.

4. 1. 4 Lesson Progression

The part of mathematics which was taught during this research was the Sine and Cosine Rule. It was the first time that students encountered these two rules in mathematics. The following table shows the lesson progression.

Table 2: Lesson progression

Lesson	Experimental group	Control group
1	**Pre-test** *Interaction pattern:* individual work **Sine Rule Discovery** *Interaction pattern:* individual work	**Pre-test** *Interaction pattern:* individual work **Sine Rule presentation and deduction** *Interaction pattern:* whole class work
2	**Problems on the Sine Rule in a triangle** *Interaction pattern:* individual work, pair work **Cosine rule presentation** *Interaction pattern:* whole class work	**Problems on Sine Rule in a triangle** *Interaction pattern:* whole class work, individual work **Cosine Rule presentation** *Interaction pattern:* whole class work **Problems on the Cosine Rule** *Interaction pattern:* whole class work, individual work
3	**Problems on the Cosine Rule in a triangle** *Interaction pattern:* individual work **Problem task on the Sine and Cosine Rule – Chobot** *Interaction pattern:* group work	**Summary of the Sine and Cosine Rule** *Interaction pattern:* whole class work **Problems on the Cosine Rule in a triangle** *Interaction pattern:* whole class work, individual work
4	**Summary of the Sine and Cosine Rule** *Interaction pattern:* whole class work	**Revision of the Cosine Rule** (calculating the angle from this formula) *Interaction pattern:* whole class work

	Problem tasks on the Sine and Cosine Rule – Chobot (finishing), Praděd, Žižkov television tower *Interaction pattern:* group work	**Textbook problem tasks on the Sine and Cosine Rule** *Interaction pattern:* whole class work, individual work
5	**Textbook problem tasks on the Sine and Cosine Rule** *Interaction pattern:* group work, individual work	**Textbook problem tasks on the Sine and Cosine Rule** *Interaction pattern:* whole class work, individual work

Both groups were taught the same subject matter by the same teacher. The control group solved mostly textbook problems, the PBL group worked on textbook problems as well as few PBL tasks which I created and can be found in chapter 3. These tasks were created to activate students' intuition before doing calculations by asking students to make guesses. These tasks were also a bit more demanding since students had to ask for any additional information they needed.

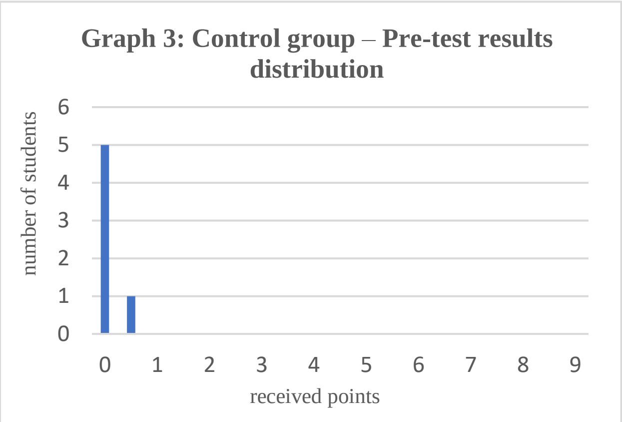
The table also includes types of interaction patterns which were used in the classroom and as can be observed, the two groups differed in the way the class interacted. The control group worked either as a whole class or students worked individually. Most frequently, new types of problems were first solved by the teacher on the blackboard and subsequently students worked on similar problems individually. Students were often asked to solve problems on the blackboard with teacher's help. In the PBL group the interaction patterns were more varied, sometimes students worked individually, sometimes new content was discussed with the whole class, and often they worked in groups. New content was usually first left to students to explore either individually or together with their classmates. Subsequently, the new learning matter was summarized with the help of the teacher. This order changed only once when the Cosine Rule was introduced. The reason was to make the lesson more efficient. However, in terms of solving problems, students were always given space to solve the tasks by themselves. Support from the teacher was offered whenever students needed. Moreover, stronger students were also frequently asked to explain the task or provide help to weaker students.

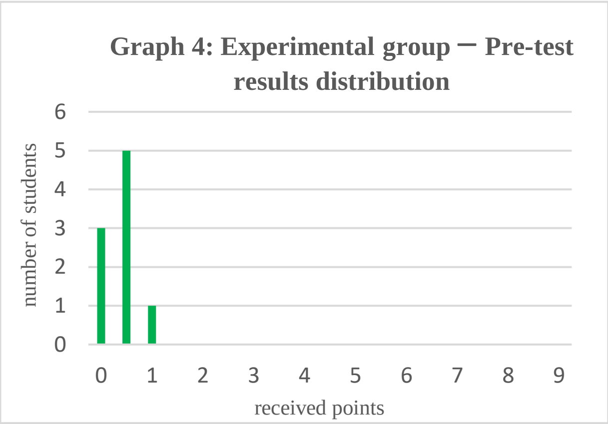
4. 2 Results and Discussion

4. 2. 1 Pre-test Results

In the beginning of this research, students from both groups wrote the same pre-test which was focused on the Sine and Cosine Rule. The pre-test consisted of three tasks and the highest possible score was 9 points (3 points for each task), students could receive half-points. Students were allowed to use calculators and a table which included all important formulas applicable for triangles including the Sine and Cosine Rule. In spite of that, the highest score was 1 point, and 8 out of 12 students were not able to solve any part of these tasks and received zero points. However, this is not surprising since up to this point the students have not heard of such rules. Only two students, both from the experimental group, found the needed formula. Nevertheless, they still were not able to complete their calculations. Many other students, 5 from the control group and 3 from the experimental group, used the Pythagorean theorem instead, even though this theorem can be applied only to right triangles.

The following graphs show the distribution of results from pre-tests of those students who returned the consent form, i.e., 6 from the control group and 9 from the experimental group.





Next, the two-tailed Mann-Whitney U test was conducted at a significance level of 0.05 using the following null and alternative hypotheses:

H_0: The pre-test results of the control and experimental group are not statistically different.
H_1: The pre-test results of the control and experimental group are statistically different.

The U values were calculated in Excel, the smaller U value was used as the test statistics and was compared to the corresponding critical value for the two-tailed Mann-Whitney U test using $n_1 = 6, n_2 = 9$ with a significance level of 0.05. The following table summarizes the results.

Table 3: Pre-test Results

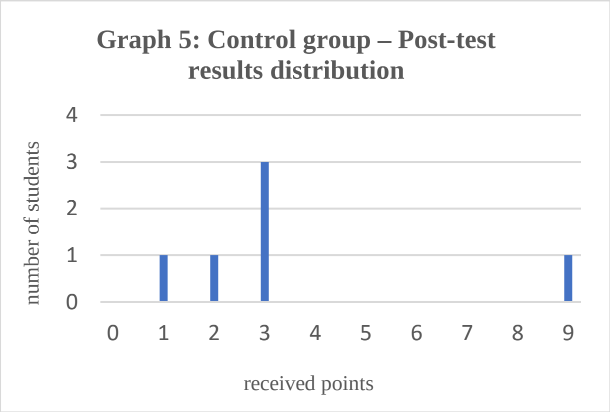
Group	Test	N	U value	Minimal U value	U critical value
Control	Pre-test	6	41	13	10
Experimental	Pre-test	9	13		

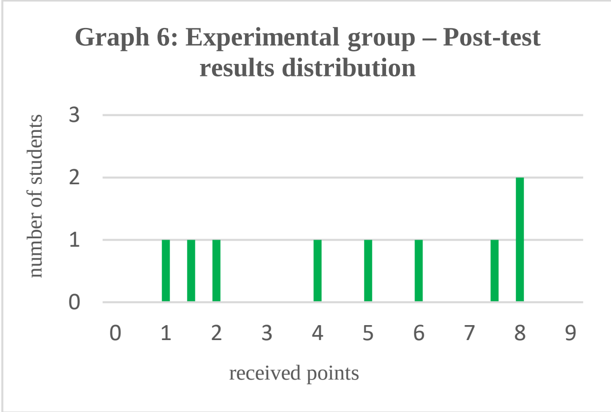
Our test statistics, i.e., minimal U value (13), is greater than our critical value (10), therefore, the null hypothesis is not rejected. Thus, the test showed that at the significance level of 0.05 there is not enough evidence to suggest that the pre-test results of the control and experimental group would be statistically different. Consequently, based on the results of the pre-test and their statistical analysis it will further be assumed that in the beginning of this study the two groups had similar knowledge of the Sine and Cosine Rule and their applications.

4. 2. 2 Post-test Results

The post-test was similar to the pre-test. There were also three problem tasks, the maximum achievable score was likewise 9 points, and students could receive half-points. Students were allowed to use the same aids as during the pre-test, i.e., a calculator and a table with needed formulas. Somewhat surpisingly, three students, one from the control group and two from the experimental group, in solving a general triangle used not only the Sine and Cosine Rule but also some formulas applicable only for right triangles, such as the Pythagorean Theorem or the trigonometric ratios. Another difference was observed between the two groups which is not reflected in the statistical analysis. The difference was in the way some students worked with the Sine Rule formula. Four students from the experimental group first calculated the ratio of the length of a side and the sine of the opposite angle, which equals the length of a diameter, and then worked with this ratio. This step presumably made the following calculations easier for them. Other students also used the Sine Rule, yet they first expressed the wanted variable from this formula and only then proceeded with calculations. This instance may be explained by the fact that students in the experimental group were supposed to solve problem tasks without studying example solutions. Thus, they had chance to develop their own strategies.

The following two graphs show the distribution of the post-test results.





Statistical analysis was conducted using the two-tailed Mann-Whitney U test at a significance level of 0.05 with the following null and alternative hypotheses:

H_0: The post-test results of the control and experimental group are not statistically different.
H_1: The post-test results of the control and experimental group are statistically different.

The test procedure was analogous to the one analysing the pre-test. Table 4 summarizes the results.

Table 4: Post-test Results

Group	**Test**	**N**	**U value**	**Minimal U value**	**U critical value**
Control	Post-test	6	33	21	10
Experimental	Post-test	9	21		

The test statistics (21) is greater than the critical value (10), therefore, the null hypothesis is not rejected at the significance level of 0.05. Thus, the hypothesis to the first research question was not confirmed since the statistical test which was performed did not show that there is a statistical difference between the control and experimental group.

4. 2. 3 Pre-test and Post-test Results Comparison

Next, the Wilcoxon signed-rank test was conducted comparing the pre-test and post-test results of the control group and subsequently of the experimental group. This test was first performed for the control group at a significance level of 0.05. The null and alternative hypotheses were as follows:

H_0: The control group results of the pre-test and post-test are not statistically different.
H_1: The control group results of the pre-test and post-test are statistically different.

Table 5 summarizes the results.

Table 5: Control group – Pre-test and Post-test Results

Group	Test	N	W	W critical value
Control	Pre-test, post-test	6	0	0

The test statistics W (0) is equal to the critical value for $n = 6$ (0); hence the null hypothesis is rejected. Therefore, we can conclude that at the significance level of 0.05 the results of the pre-test and post-test of the control group were statistically different. This means that students' test score increased significantly in the control group.

Second, the same test was performed for the experimental group with the null and alternative hypotheses were as follows:

H_0: The experimental group results of the pre-test and post-test are not statistically different.
H_1: The experimental group results of the pre-test and post-test are statistically different.

The results are summarized in the following table.

Table 6: Experimental group – Pre-test and Post-test Results

Group	Test	N	W	W critical value
Experimental	Pre-test, post-test	9	0	5

Here, the test statistics (0) is smaller than the critical value for $n = 9$ (5); thus, the null hypothesis is likewise rejected at the significance level of 0.05. Students' test scores, therefore, significantly increased also in the experimental group.

In summary, based on the statistical analysis the two groups did not differ significantly in their knowledge of the Sine and Cosine Rule and their applications in the beginning of the research nor at the end of the research. This means that my first hypothesis on the impact of PBL on students' achievement was not confirmed. Both groups, however, increased their test scores significantly.

4. 2. 4 Limitations of this Research

There are some limitations of generalizability of this research. First, in terms of the sample size, there were only 15 participants (9 in the experimental group, 6 in the control group). Attaining a bigger sample size was not successful due to the following reasons. At the time of this research, the researcher was teaching mathematics only in one class consisting of 18 students, 15 of which participated in the research. There were also two parallel classes taught by different teachers. One option was to compare results of classes taught by different teachers. However, since the other two mathematics teachers were much more experienced in comparison with the researcher who was a newly trained teacher, this factor would make the results rather unreliable. Second option was to replace the other two teachers in their classes for the period of the research. Nevertheless, having a different teacher could influence the class learning environment, and consequently the research would not measure only the effects of using a different teaching method. Furthermore, there was a problem of conflicting schedules. Second, the two groups mainly consisted of male students, which makes the generalization to entire population also problematic.

4. 3 My Reflection on the Implementation of PBL

This section includes my reflection on the implementation of PBL as a newly trained teacher. First, I will shortly describe my impression of this class. The class where the research was conducted was used to mainly direct type of instruction. There were differences in the level of mathematical knowledge and motivation within the class. The two groups, as already mentioned, were divided randomly. However, there were rather big differences between the groups in both aspects, mathematical knowledge as well as motivation. One group had generally better test results throughout the school year before conducting the experiment. Moreover, students appeared to be more motivated though extrinsic motivation seemed to be common. My original plan was to adopt PBL in the other group which was not as motivated and often had worse test results. As Pagander and Read's literature review showed (2014), there seemed to be a lack of consensus on the question whether PBL is beneficial even for weaker students. Therefore, I thought that it would have been interesting to see whether even short implementation of PBL would improve the situation somehow. However, one student from this group asked for online lessons and since PBL would make it harder for the student to follow what was happening in the classroom, I decided to have this group as a control group.

Second, I will focus on some difficulties in implementing PBL for me as a newly trained teacher. The first obstacles were connected to my lack of understanding of what PBL looks like in the classroom and how effective this method is. I have not experienced PBL, therefore, I tried to familiarize myself with this approach by reading books, articles, and research on PBL and reflecting on PBL tasks. I have not found any practical course on PBL which I could attend. However, Pagander and Read (2014) pointed out that research agrees on the importance of having properly trained teachers for PBL since otherwise its implementation may not be successful. Hence, as I have not received any training in PBL, this factor may have influenced the successfulness of my PBL implementation. Moreover, as a newly trained teacher I still need more experience with class management in general.

Other hindrances were practical ones. I found it difficult to choose the right subject matter for PBL implementation. There were quite many topics to cover in the second-year mathematics in the school where I taught. In the beginning of the school year linear and quadratic functions and equations were revised and then other types of functions and equations were discussed, such as power functions, linear fractional functions, exponential, logarithmic and goniometric functions, and equations, these were followed by the Sine and Cosine Rule, and the final topic was stereometry and the volume and surface of certain objects. Hence, each topic was discussed quite briefly and even though I prepared PBL tasks also on exponential functions, we were able to focus on them very shortly not to fall behind the plan. There was a bit more time granted for the Sine and Cosine Rule, so I decided to conduct the research on this subject matter. Moreover, it was easier to create PBL tasks on this topic than on goniometric functions, for example, since I was able to think of problems which are connected to the world around us. These difficulties are in fact in agreement with Pagander and Read's conclusion (2014) that an important factor in implementing PBL is that the school curriculum is in line with the PBL approach and its objectives.

Lastly, I was quite fortunate that I was able to divide the class for the time of the research. It demanded some adjustments in the lesson schedule, yet for such a short time period it was possible. Having a PBL group consisting of only 10 students allowed me to help the students who needed some support or wanted to consult their results. Nevertheless, it would be harder with the entire class especially since many students were not used to independent study nor group work and as there was a significant number of underachieving students who needed more support.

In summary, some difficulties, i.e., the lack of training for PBL and the school curriculum, are in line with my hypothesis to the second research question. The possible difficulties of teaching a bigger class were also mentioned; thus, only the problem of reconciliating summative assessment and PBL was not reflected upon. However, I assume, that the main reason for the lack of discontentment about the assessment format was the length of PBL implementation since the research took only two weeks. Overall, my second hypothesis about the hindrances of PBL implementation for newly trained teachers, which was based on Pagander and Read's literature review of PBL (2014), was confirmed. However, this section consisted of my reflections on this topic, and thus, cannot be taken as a proof of any kind.

4. 4 Conclusion of the Research

This research focused on the effect of PBL method on students' mathematical achievement and on the possible problems with PBL implementation. The result of statistical analysis showed that even though the test scores of the experimental group increased significantly, there was not a statistically significant difference between the post-test results of the experimental and the control group. This means that the first hypothesis about the predominance of PBL over traditional method in enhancing students' mathematical achievement was not confirmed. However, due to the small sample size, the groups' composition, and the lack of teacher's training in PBL, there may be some limitations of generalizability of this research. The second hypothesis, which made conjectures about the possible hindrances for newly trained teachers from implementing PBL in their classrooms, was to a great extent confirmed. Researcher's reflection supported the claim about the importance of teacher's training in PBL, a proper size of the class and the school curriculum which is compatible with PBL.

Conclusion

The goal of this thesis was to investigate the problem-based and project-based methods. The review of literature and research on problem-based and project-based learning suggested that both methods indeed positively impact students' cognitive as well as affective domain. However, the question whether these methods are more effective than traditional teaching seems to be still unresolved. Though some researchers believe that there is already enough evidence to establish their effectiveness, others point out that due to the many limitations of existing studies on this topic and the lack of consensus on the exact definitions of these methods, generalization is often problematic.

This thesis also includes problem-based tasks on exponential functions, the Sine and Cosine Rule, and volume, which I created and can be used in mathematics classes. The tasks on the Sine and Cosine Rule were also used during the research on the effectiveness of problem-based learning. In the research, my hypothesis about the predominance of problem-based learning over traditional method was not confirmed, yet due to several limitations of the research, there may be problems with generalizability. The second hypothesis on the possible hindrances of PBL implementation for newly trained teachers was confirmed. The reasons seem to be connected with the size of the class, teacher's training in PBL and school curriculum. These findings may be helpful for mathematics educators or student teachers in their attempt to incorporate problem-based or project-based learning in their own teaching.

Bibliography

Andreescu, T., Cordeiro, K., & Andreescu, A. (2020). *Awesome Math: Teaching Mathematics with Problem Based Learning*. Jossey-Bass.

Anon. (2007). *Key Competences for Lifelong Learning: European Reference Framework*. Luxembourg: Publications Office of the European Union.

Arbo, J. B., & Ching, D. A. (2022). Problem-based learning approach in developing mathematical skills. *International Journal of Science, Technology, Engineering and Mathematics*, *2*(1). https://doi.org/10.53378/352873

Balada, J., et al. (2007). *Rámcový vzdělávací program pro gymnázia: RVP G*. Praha: Výzkumný ústav pedagogický v Praze.

Barton, C. (2018). *How I Wish I'd Taught Maths: Lessons learned from research, conversations with experts, and 12 years of mistakes.* John Catt Educational Ltd.

Boaler, J. (2023, March). *Crossing the Line: When Academic Disagreement becomes Harassment and Abuse.* Professor Jo Boaler. Retrieved April 8, 2023, from https://joboaler.people.stanford.edu/

Boaler, J. (2002). *Experiencing School Mathematics: Traditional and Reform Approaches To Teaching and Their Impact on Student Learning, Revised and Expanded Edition* (1st ed.). Routledge.

Borich, G. D. (2017). *Effective Teaching Methods: Research-Based Practice* (9th ed.). The University of Texas at Austin.

Cazzola, M. (2008 a). Problem-Based Learning and Mathematics: Possible Synergical Actions. *ICERI2008 Proceeding, IATED.*

Cazzola, M. (2008 b). *Problem-Based Learning and Teacher Training in Mathematics*. https://doi.org/10.48550/arXiv.0805.4477

Chen, C.-H., & Yang, Y.-C. (2019). Revisiting the effects of project-based learning on students' academic achievement: A meta-analysis investigating moderators. *Educational Research Review*, *26*, 71–81. https://doi.org/10.1016/j.edurev.2018.11.001

Condliffe, B., Quint, J., Visher, M. G., Bangser, M. R., Drohojowska, S., Saco, L., Nelson E. (2017). Project-Based Learning: A Literature Review. Working Paper. MDRC.

Čapek, R. (2015). *Moderní Didaktika: Lexikon Výukových a Hodnoticích Metod*. Grada.

Dewey, J. (1916). *Democracy and Education, An Introduction to the Philosophy of Education*. Macmillan.

Direct instruction definition. The Glossary of Education Reform. (2013, December 20). https://www.edglossary.org/direct-instruction/

European Commission. (2018). *COMMISSION STAFF WORKING DOCUMENT Accompanying the document Proposal for a COUNCIL RECOMMENDATION on Key Competences for LifeLong Learning*. Brusel: European Commission.

Experiencing school mathematics. (n.d.). Retrieved April 6, 2023, from https://www.taylorfrancis.com/books/mono/10.4324/9781410606365/experiencing-school-mathematics-jo-boaler

Gold Standard PBL: Essential Project Design elements. PBLWorks. (n.d.). Retrieved November 8, 2022, from https://www.pblworks.org/what-is-pbl/gold-standard-project-design

Hafeez, M. (2022). Challenges and research gap in Project-Based Learning – A Review. *International Journal of New Trends in Social Sciences*, *6*(1), 01–14. https://doi.org/10.18844/ijss.v6i1.5633

Helle, L., Tynjälä, P., & Olkinuora, E. (2006). Project-based learning in Post-Secondary Education – theory, practice and rubber sling shots. *Higher Education*, *51*(2), 287–314. https://doi.org/10.1007/s10734-004-6386-5

Hendl, J. (2004). *Přehled statistických Metod zpracování dat: Analýza a metaanalýza dat*. Portál.

Hilgard, E. R. (1996). Perspectives on Educational Psychology. *Educational Psychology Review*, *8*(4), 419–431. https://doi.org/10.1007/bf01463942

Hmelo-Silver, C. E., Duncan, R. G., & Chinn, C. A. (2007). Scaffolding and achievement in problem-based and inquiry learning: A response to Kirschner, Sweller, and Clark (2006). *Educational Psychologist*, *42*(2), 99–107. https://doi.org/10.1080/00461520701263368

Holmes, V.-L., & Hwang, Y. (2016). Exploring the effects of project-based learning in Secondary Mathematics Education. *The Journal of Educational Research*, *109*(5), 449–463. https://doi.org/10.1080/00220671.2014.979911

HQPBL. (n.d.). Retrieved November 15, 2022, from https://hqpbl.org/resources/

Hron, K., Kunderová, P., & Vencálek, O. (2018). *Základy Počtu pravděpodobnosti a Metod Matematické statistiky*. Univerzita Palackého v Olomouci, Přírodovědecká fakulta.

Indrawan, E. & Jalinus, N. & Syahril, S. (2019). Review Project Based Learning. *International Journal of Science and Research (IJSR)*, *8*(4), 1014 - 1018.

Jacques, L. (2017). What does Project-based Learning (PBL) Look like in the Mathematics Classroom? *American Journal of Educational Research*, *5*(4), 428-433.

Jo Boaler. Jo Boaler's Profile | Stanford Profiles. (n.d.). Retrieved April 6, 2023, from https://profiles.stanford.edu/jo-boaler?tab=bio

Jurdak, M. (2016). *Learning and Teaching Real World Problem Solving in School Mathematics: A Multiple-Perspective Framework for Crossing the Boundary*. Springer International Publishing.

Kirschner, P. A., Sweller, J., & Clark, R. E. (2006). Why minimal guidance during instruction does not work: An analysis of the failure of constructivist, discovery, problem-based, experiential, and inquiry-based teaching. *Educational Psychologist*, *41*(2), 75–86. https://doi.org/10.1207/s15326985ep4102_1

Kokotsaki, D., Menzies, V., & Wiggins, A. (2016). Project-based learning: A review of the literature. *Improving Schools*, *19*(3), 267–277. https://doi.org/10.1177/1365480216659733

Krauss, J., & Boss, S. (2013). *Thinking Through Project-Based Learning: Guiding Deeper Inquiry*. Corwin.

Laine, C. E., & Mahmud, M. S. (2022). The influence of problem-based learning (PBL) on Mathematics learning: Systematic literature review. *International Journal of Academic Research in Progressive Education and Development, 11*(3). https://doi.org/10.6007/ijarped/v11-i3/15033

Mallibhat, K., & Joshi, G. (2020). A Systematic Review on Frameworks of Project-Based Learning. *Educate for the future: PBL, Sustainability and Digitalisation 2020*, 538-550. Aalborg University Press.

Meyer, D. (2016, September 17). *Starter Pack*. Dy/Dan. Retrieved March 14, 2023, from https://blog.mrmeyer.com/starter-pack/

Meyer, D. (2009, June 10). *What I Would Do with This: Glassware*. Dy/Dan. Retrieved March 14, 2023, from blog.mrmeyer.com/2009/what-i-would-do-with-this-glassware/

Mustaffa, N. B., Ismail, Z. B., Tasir, Z. B., & Said, M. N. (2014). Problem-Based Learning (PBL) in Mathematics: a Meta Analysis. *International Education Postgraduate Seminar 2014*, 301.

Mustaffa, N., Ismail, Z., Tasir, Z., & Said, M. N. (2016). The impacts of implementing problem-based learning (PBL) in Mathematics: A review of literature. *International Journal of Academic Research in Business and Social Sciences*, *6*(12). https://doi.org/10.6007/ijarbss/v6-i12/2513

Nezvalová, D. (2008). *Moduly pro profesní přípravu učitele přírodovědných předmětů a matematiky: Modulární přístup v počáteční přípravě učitelů přírodovědných předmětů pro střední školy: Učební text*. Univerzita Palackého v Olomouci.

Nurbavliyev, O. K., Kaymak, S., & Almas, A. (2020). The impact of project-based learning on students' achievement in Mathematics. *Bulletin of Toraighyrov University. Pedagogics Series,* (3.2020), 367–377. https://doi.org/10.48081/KXBI5168.

Pagander, L. & Read, J. (2014). *Is Problem-Based Learning (PBL) An Effective Teaching Method?* Linköping University.

PBLWorks. (n.d.) Retrieved November 15, 2022, from https://www.pblworks.org/

Pecore, J. L. (2015). From Kilpatrick’s project method to project-based learning. *International Handbook of Progressive Education 1*(25), 155–171.

Petty, G., & Foltýn, J. (2013). *Moderní Vyučování*. Portál.

Podlahová, L. (2012). *Didaktika pro Vysokoškolské Učitele: Vybrané Kapitoly*. Grada.

Sammamish Collaborative. (2015). *7 Key Elements of Problem-Based Learning (PBL) Classrooms.* Retrieved July 22, 2022 from https://bsd405.org/wp-content/uploads/sites/33/2015/08/Key-Elements-SHS-Final-July2015.pdf

Sani, F., & Todman, J. (2006). Appendix 1: Statistical Tables. In *Experimental Design and Statistics for Psychology: A First Course*, 183–196. Blackwell.

Savery, J. R. (2006). Overview of problem-based learning: Definitions and distinctions. *Interdisciplinary Journal of Problem-Based Learning*, *1*(1). https://doi.org/10.7771/1541-5015.1002

Schunk, D. H. (2012). *Learning theories: An educational perspective* (6th ed.). Pearson.

Skalková, J. (2007). *Obecná Didaktika: Vyučovací Proces, Učivo a Jeho Výběr, Metody, Organizační Formy Vyučování*. Grada.

Turek, I. (2014). *Didaktika*. Bratislava Edícia Škola.

Trávníček, S., Calábek, P., & Švrček J. (2014). *Matematická analýza I: (pro učitelské obory)*. Univerzita Palackého v Olomouci.

Ullah, O., & Iqbal, M. (2020). Comparison of impact of traditional and modern teaching methods on students’ performance at elementary school level. *Global Regional Review*, *V*(I), 386–395. https://doi.org/10.31703/grr.2020(v-i).42

Wolpert-Gawron, H. (2016). *DIY Project Based Learning for Math and Science*. Routledge.

Reference for real-world tasks

Davidson, K. (2021, September 2). *How quickly can bacterial contamination occur?* Healthline. Retrieved July 22, 2022, from https://www.healthline.com/nutrition/how-quickly-can-bacterial-contamination-occur

Dean, B. (2022). *Netflix Subscriber and Growth Statistics: How Many People Watch Netflix in 2022?* Retrieved July 23, 2022, from: https://backlinko.com/netflix-users

Delhi, India Population 1950-2019. Macro Trends. (n.d.). Retrieved July 24, 2022, from www.macrotrends.net/cities/21228/delhi/population

Depreciation Calculator. Claims Pages. (n.d.). Retrieved July 24, 2022, from www.claimspages.com/tools/depreciation/

Directions and Bearings. Maths Teacher. (n.d.). Retrieved March 14, 2023, from https://mathsteacher.com.au/year10/ch15_trigonometry/11_directions/23dir.htm

Food processing entrepreneurs – food technology & processing food technology & processing. Food Processing Entrepreneurs – Food Technology & Processing. (n.d.). Retrieved July 22, 2022, from https://aggie-horticulture.tamu.edu/food-technology/food-processing-entrepreneurs/

Google. (n.d.-a) [Google Maps Chobot] Retrieved March 14, 2023, from https://www.google.com/maps/@49.9615614,16.2075548,14.92z

Google (n.d.-b) [Google Maps Praděd] Retrieved March 14, 2023, from https://www.google.com/maps/@50.0837903,17.2281423,402m/data=!3m1!1e3

Google. (n.d.-c) [Google Maps Žižkovská televizní věž] Retrieved March 14, 2023, from https://www.google.com/maps/@50.080949,14.4511225,129m/data=!3m1!1e3

Hoel, S., Vadstein, O., & Jakobsen, A. N. (n.d.). *Effects of storage temperature on bacterial growth rates and community structure in Fresh Retail Sushi*. Journal of applied microbiology. Retrieved July 22, 2022, from https://pubmed.ncbi.nlm.nih.gov/28654203/

Param Visions (2021, March 17). *How to calculate the height of any buildings? (type-1)/ how to calculate the height of an object using a theodolite?/height of building by trigonometric levelling.* How to calculate the height of any buildings? (Type-1)/ How to calculate the height of an object using a theodolite?/Height of building by trigonometric levelling. Retrieved March 15, 2023, from https://www.paramvisions.com/2021/03/how-to-calculate-height-of-any.html

Rajniš, M. (n.d.). *ROZHLEDNA BÁRA II*. Huť Architektury Martin Rajniš. Retrieved March 11, 2023, from https://hutarchitektury.cz/blog/projekty/rozhledna-bara/

Online resources for mathematics teachers – inspiration for engaging mathematical tasks

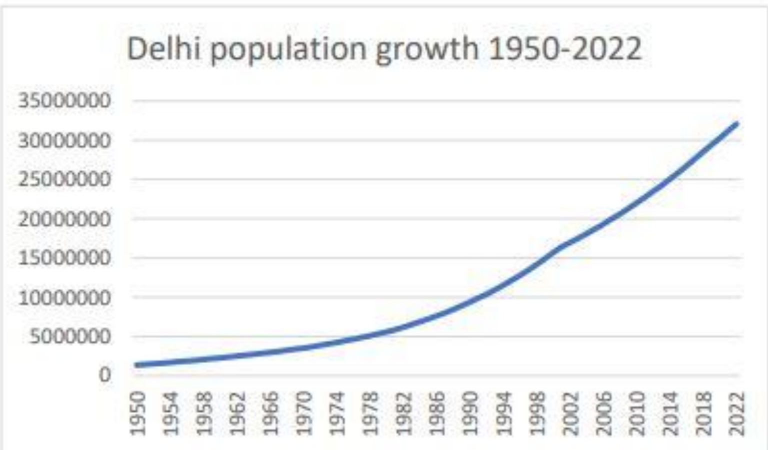
1. https://blog.mrmeyer.com/
2. https://www.desmos.com/
3. https://www.nytimes.com/spotlight/learning-article-of-the-day
4. https://smileprogram.info/
5. https://nrich.maths.org/teacher-secondary
6. https://learn.makemathmoments.com/tasks/
7. https://www.comap.com/resources/free-materials
8. https://www.nacloweb.org/practice.php
9. https://m3challenge.siam.org/resources/sample-problems
10. https://purplecomet.org/?action=resource/oldcontests
11. https://www.mathsteachercircles.org/maths/
12. https://www.gapminder.org/teaching/
13. https://www.stem.org.uk/secondary/resources/collections/maths/secondary-maths
14. https://www.youcubed.org/tasks/
15. https://my.pblworks.org/projects
16. https://www.hightechhigh.org/student-work/projects/
17. https://pblessentials.org/

List of Appendices

Appendix A

Data for the Delhi Population Growth Task

Year	Population	Year	Population	Year	Population
1950	1369000	1976	4646000	2002	16956000
1951	1461000	1977	4865000	2003	17516000
1952	1536000	1978	5095000	2004	18095000
1953	1613000	1979	5335000	2005	18691000
1954	1695000	1980	5587000	2006	19309000
1955	1782000	1981	5862000	2007	19946000
1956	1872000	1982	6177000	2008	20606000
1957	1967000	1983	6508000	2009	21285000
1958	2067000	1984	6858000	2010	21988000
1959	2172000	1985	7226000	2011	22714000
1960	2283000	1986	7614000	2012	23464000
1961	2394000	1987	8022000	2013	24239000
1962	2499000	1988	8453000	2014	25039000
1963	2610000	1989	8906000	2015	25866000
1964	2725000	1990	9384000	2016	26720000
1965	2845000	1991	9885000	2017	27602000
1966	2971000	1992	10406000	2018	28514000
1967	3102000	1993	10954000	2019	29399000
1968	3239000	1994	11531000	2020	30291000
1969	3381000	1995	12138000	2021	31181000
1970	3531000	1996	12779000	2022	32066000
1971	3689000	1997	13451000		
1972	3864000	1998	14160000		
1973	4046000	1999	14906000		
1974	4236000	2000	15692000		
1975	4436000	2001	16414000		



Picture: Screenshot of a PDF of an Excel file Delhi population growth.xlsx

Appendix B
Pre-test and Post-test

Students received both tests in the Czech language, however, here I included these tests also in the English version.

English version:

Pre-test

1) A triangle ABC has sides *a*, *b* of lengths *a* = 65 cm, *b* = 46 cm, and the angle $\alpha = 42°35'$. Calculate the length of the side *c* and the size of angles β a γ.

2) A person observes an object that is 12 metres long. He is located 15 metres from one end and 12 metres from the other end. Calculate the object's angular size.

3) The target C is observed from two artillery observation posts A and B, which are 975 m apart and $|\sphericalangle \mathrm{BAC}| = 63°, |\sphericalangle \mathrm{ABC}| = 48°$. Calculate the distance |AC|.

Post-test

1) A triangle ABC has sides *a*, *b* of lengths a = 65 cm, b = 46 cm, and the angle $\alpha = 42°35'$. Calculate the length of the side *c* and the size of angles β a γ.

2) Calculate the length of a tunnel if the distance between the two ends of the tunnel from a selected place is 619,8 m and 437,8 m and the angular size of the tunnel is 97°45`.

3) Two boats are observed from a sighting device, which is located 150 m above the surface of the lake. The angles of depression from the sighting device to the boats are 57° and 39°. The sighting device and both boats are in a plane perpendicular to the surface of the lake. Calculate the distance between the two boats.

Czech version:

Pre-test

1) V trojúhelníku *ABC* jsou dány strany a, b a úhel α: $a = 65\ cm, b = 46\ cm, \alpha = 42°35'$. Dopočítejte velikost strany c a úhly $\beta\ a\ \gamma$.

2) Určete velikost zorného úhlu, pod nímž vidí pozorovatel předmět 12 m dlouhý, je-li od jednoho konce vzdálen 15 m a od druhého 24 m.

3) Cíl C je pozorován ze dvou dělostřeleckých pozorovatelen A, B, které jsou od sebe vzdáleny 975 m, přitom je $|\sphericalangle BAC| = 63°, |\sphericalangle ABC| = 48°$. Vypočítejte vzdálenost $|AC|$.

Post-test

1) V trojúhelníku *ABC* jsou dány strany a, b a úhel α: $a = 65\ cm, b = 46\ cm, \alpha = 42°35'$. Dopočítejte velikost strany c a úhly β a γ.

2) Vypočtěte délku tunelu, jestliže vzdálenost konců tunelu od zvoleného místa je 619,8 m a 437,8 m a úhel, pod kterým vidíme oba konce tunelu má velikost 97°45`.

3) Dvě loďky jsou zaměřeny z výšky 150 m nad hladinou jezera pod hloubkovými úhly o velikostech 57° a 39°. Vypočítejte vzdálenost obou loděk, jestliže zaměřovací přístroj a obě loďky jsou v rovině kolmé k hladině jezera.

These tasks were taken from the following two sources:

Jirásek, F., Braniš, K., Horák, S., & Vacek, M. (1988). *Sbírka úloh z matematiky pro SOŠ a pro studijní obory SOU*. Státní pedagogické nakladatelství.

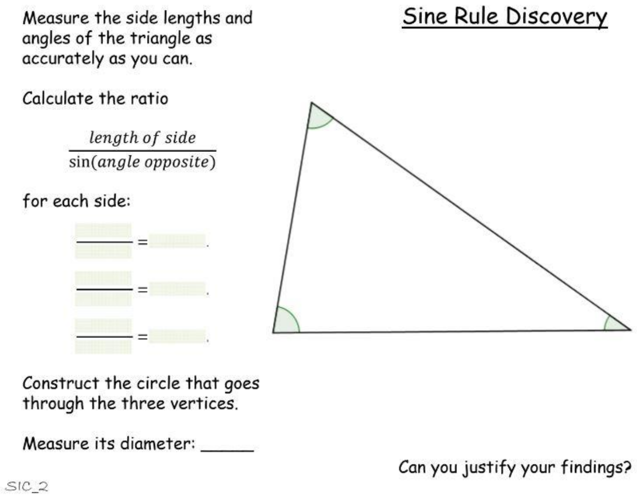
Slovní úlohy Sinová a Kosinová Věta. Střední průmyslová škola stavební Valašské Meziříčí. (n.d.). Retrieved May 14, 2023, from http://spsstavvm.cz/cs/pro-studenty/studijni-materialy/matematika/mgr-dvorak/slovni-ulohy-sinova-a-kosinova-veta.html

Appendix C

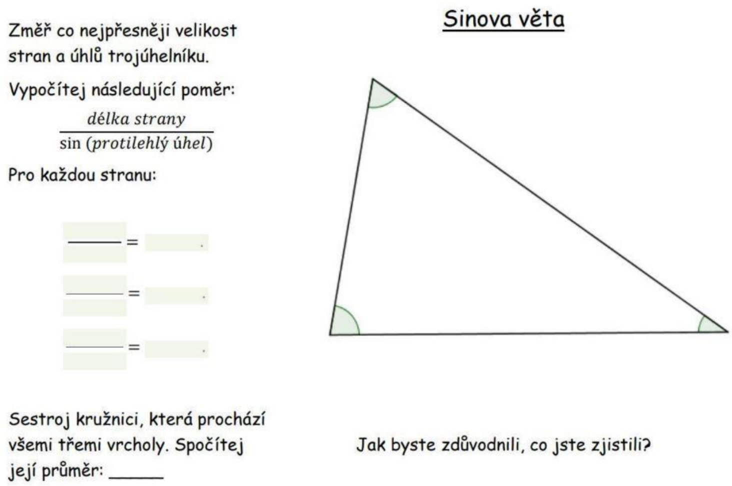
List of Tasks Used During the Research

The following tasks were used during my research.

Sine Rule Discovery – English version



Students received a version which I translated into **Czech**:



This task was found on the STEM website:

Sine rule discovery. STEM. (n.d.). Retrieved May 13, 2023, from https://www.stem.org.uk/resources/elibrary/resource/422713/sine-rule-discovery

Other tasks which were used were taken from the following mathematics textbook:

Jirásek, F., Braniš, K., Horák, S., & Vacek, M. (1988). *Sbírka úloh z matematiky pro SOŠ a pro studijní obory SOU*. Státní pedagogické nakladatelství, 333-337.

Cvičení

9.82 Řešte trojúhelník ABC, je-li dáno:
a) $a = 65$, $b = 46$, $\alpha = 42°35'$
b) $b = 13{,}4$, $c = 16{,}3$, $\gamma = 70°12'$
c) $b = 14{,}5$, $c = 25{,}8$, $\beta = 54°28'$
d) $a = 5{,}2$, $c = 8{,}8$, $\gamma = 52°08'$

9.83 Řešte trojúhelník ABC, je-li dáno:
a) $c = 210$, $\alpha = 62°32'$, $\beta = 48°56'$
b) $a = 40$, $\alpha = 26°38'$, $\beta = 89°40'$
c) $b = 225$, $\alpha = 107°35'$, $\beta = 30°40'$
d) $b = 79{,}5$, $\beta = 65°20'$, $\gamma = 54°40'$

9.85 Vypočtěte ostatní prvky trojúhelníku ABC, ve kterém je dáno:
a) $a = 7$, $b = 4$, $\gamma = 38°$
b) $b = 32$, $c = 40$, $\alpha = 100°21'$
c) $a = 10{,}9$, $c = 15{,}2$, $\beta = 67°$
d) $b = 51{,}32$, $c = 34{,}76$, $\alpha = 89°57'$

9.86 V trojúhelníku ABC jsou dány velikosti stran. Vypočtěte velikosti vnitřních úhlů.
a) $a = 16$, $b = 25$, $c = 36$
b) $a = 4{,}2$, $b = 3{,}8$, $c = 5{,}5$
c) $a = 5$, $b = 6$, $c = 7$

9.100 Dvě obce A, B jsou odděleny lesem; obě jsou viditelné z obce C, která je s oběma spojena přímými cestami. Jak dlouhá je projektovaná cesta z A do B, je-li $|AC| = 2\,003$ m, $|BC| = 1\,593$ m a $|\sphericalangle ABC| = 63°23'$?
[2 122,2 m]

9.101 Cíl C je pozorován ze dvou dělostřeleckých pozorovatelen A, B, které jsou od sebe vzdáleny 975 m, přitom je $|\sphericalangle BAC| = 63°$, $|\sphericalangle ABC| = 48°$. Vypočítejte vzdálenost $|AC|$.
[776 m]

9.102 Dvě důlní štoly vycházejí ze stejného místa P v šachtě a svírají úhel o velikosti 51°45'. Délky štol jsou: $|PQ| = 479$ m, $|PR| = 796$ m. Vypočtěte délku spojovací štoly $|QR|$.
[625 m]

9.111 Sílu o velikosti $F = 2\,217{,}6$ N je třeba rozložit na dvě složky, které s ní svírají úhly o velikostech $\alpha = 46°32'$ a $\beta = 54°12'$. Vypočítejte velikosti složek $\boldsymbol{F}_1$ a $\boldsymbol{F}_2$.

[$F_1 \doteq 1\,830{,}6$ N, $F_2 \doteq 1\,638{,}1$ N]

9.112 Síly o velikostech $F_1 = 42$ N, $F_2 = 35$ N působí ve společném bodě a svírají úhel o velikosti $77°12'$. Jak veliká je výsledná síla $\boldsymbol{F}$?

[$F \doteq 62{,}35$ N]

9.113 Sílu o velikosti $F = 300$ N rozložte na složky $\boldsymbol{F}_1$ a $\boldsymbol{F}_2$. První složka svírá se silou $\boldsymbol{F}$ úhel o velikosti $47°14'$ a druhá úhel o velikosti $18°53'$. Určete velikosti sil $\boldsymbol{F}_1$ a $\boldsymbol{F}_2$.

[$F_1 \doteq 106{,}2$ N, $F_2 \doteq 240{,}9$ N]

9.114 Tři síly, jejichž velikosti jsou v poměru 9 : 10 : 17, působí v rovině v jednom bodě tak, že jsou v rovnováze. Určete velikosti úhlů, které svírají každé dvě síly.

[$53°08'$, $154°57'$, $151°55'$]

Appendix D
Informed Consent Form

Informovaný souhlas zákonného zástupce žáků třídy, kde výzkum probíhal

Vážení rodiče,

jmenuji se Klára Spilková a učím matematiku ve třídě, kam chodí Váš syn/dcera ______________________. Chtěla bych Vás požádat o souhlas se zpracováním informací z výzkumu v rámci mé diplomové práce. Výzkum bude zaměřen na efektivitu problémové a projektové výuky. V mé práci budu zpracovávat informace z výsledků dvou testů (na začátku a na konci výzkumu) a obecné informace o třídě a aktivitě žáků během daných hodin. Vše bude zpracováno anonymně.

Pokud se zpracováním výše uvedených informací souhlasíte, prosím o Váš podpis.

V _______________ dne ____________ ________________

Podpis

Klára Spilková