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DIPLOMA THESIS

Emerging multipartite correlations

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DIPLOMOVÁ PRÁCE

Vynořující se multipartitní korelace

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Declaration

I declare that I worked on this thesis on my own and that I used only sources mentioned in the Bibliography section.

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Bibliographical identification

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Abstract	The aim of this thesis is theoretical design of two different counterparts of genuine multipartite entangled state whose entanglement can be verified only from its separable reduced density matrices. First, we use a mapping of quantum states onto probability distributions by a measurement to construct a classical cryptographic analogue of this concept. As a result, we obtain a collection of marginal distributions carrying no secret correlations, which are compatible only with distribution containing secret correlations. The analysis of secret correlations in the global probability distribution and its marginals was made using bound on the secret key rate. As a second counterpart, we construct several examples of multimode Gaussian states possessing the specified property using numerical calculation of a particular entanglement witness. Finally, we also designed a logical circuit for generation of our three-mode example.
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Abstrakt	Cílem této práce je teoretický návrh dvou různých protějšků skutečně multipartitně provázaného stavu, jehož provázanost může být ověřena pouze z jeho separabilních redukovaných matic hustoty. Nejprve jsme pomocí měření namapovali kvantový stav na rozdělení pravděpodobnosti, abychom vytvořili klasickou kryptografickou analogii tohoto konceptu. Výsledkem je, že získáme soubor marginálních rozdělení nenesoucích bezpečné korelace, které jsou slučitelné pouze s rozděleními nesoucí bezpečné korelace. Analýza bezpečných korelací v globálním rozdělení pravděpodobnosti a jeho marginálíích byla provedena pomocí meze na míru bezpečného klíče. Jako druhý protějšek konstruujeme několik příkladů mnohamódového Gaussovského stavu, které mají specifikovanou vlastnost, pomocí numerického výpočtu konkrétního svědka provázanosti. Nakonec navrhujeme logický obvod pro vytvoření našeho třímódového příkladu.
Klíčová slova	kvantová fyzika, kvantová provázanost, skutečná multipartitní provázanost, svědek provázanosti, rozdělení pravděpodobnosti
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Contents

Introduction

Today's civilization is based on informations and their processing. This causes the need for better understanding of it. One can look at information as a purely mathematical object and do not look at physical aspects of a medium that carries the information. This point of view is called classical information theory and it was studied from 1948 [1]. The other point of view is about physical aspects. The information is carried by physical systems which must obey physical laws. As there exist needs for faster and cheaper processing, we must use smaller systems until we use objects from microworld. Here we switch from physical laws for classical world to quantum physics where laws are different. Thus the information studied in microwolrd is known as quantum information theory. One of important things is to explore the relation between these two information theories and what can one theory give to the other. For example, one can explore the links between quantum entanglement being the central concept of quantum information theory and secret classical correlation, which is a crucial concept in classical cryptography.

Other important thing is to look at the relation between the whole system and its parts. In mathematical statistics it is known under the name of marginal problem whose origin dates back to 1940s [2]. In its basic form one wants to recognize a set of all joint probability distributions compatible with a given set of reduced probability distributions. In quantum physics we can meet its counterpart known as quantum marginal problem [3, 4]. Instead of probability distributions one works with global density matrices and reduced density matrices but the main question is still the same. The quantum marginal problem can be used to classification [5] or quantification [6] of quantum entanglement but also for detection of a global property of a composite system from its parts. The question is now of the form what can we say about properties of the global system using only the partial information contained in reduced density matrices. Moreover, if marginals do not carry the required property the task starts to be more interesting. We are looking for the so called "emerging property", i.e., property appearing only at a certain level of complexity of the system. This emerging property can be quantum entanglement or some special type of it. Is there a counterpart of this type of quantum marginal problem for classical probability distributions?

As interesting as the correspondence between quantum and classical marginal problem is the correspondece between properties of discrete and continuous variable systems. The problem of emerging property was constructed and solved in the discrete variables where an example of genuine multipartite entangled state was found whose entanglement can be concluded from its reduced separable density matrices [7]. In this work all possible reduced systems were used, but the same type of entanglement can be inferred only from a proper subset of nearest-neighbor marginals [8]. But can we find its counterpart in continuous variable systems?

In this work, we study these two questions. More precisely, we construct a classical analogue of a state from Ref.[9], which is genuine multipratite entangled and at the same time, its entanglement can be cocluded from its separable marginals. We also find its continuous variable counterpart where we use only a set of nearest-neighbor two-mode marginals.

This work is organized as follows. In Chapter 1 we give a brief introduction into the theory of quantum entanglement. In Chapter 2 we construct a classical analogue of emergent genuine multipartite entanglement. Finally, in Chapter 3 we give example of multimode Gaussian states whose genuine multipartite entanglement can be verified from the nearest-neighbour two-mode marginals.

The contest of Chapter 2 is based on our paper [10] and it follows experimental paper [11] dedicated to quantum problem from Ref. [7]. Chapter 3 is generalization of results from Ref. [11] to case, where one does not know all marginals and results of this chapter are preparing for publication.

Chapter 1

Quantum entanglement

This thesis is about a special property in the quantum physics called genuine multipartite entanglement. The concept of quantum entanglement was first mentioned in paper by Einstein, Podolsky and Rosen [12] where they discuss compelteness of the quantum mechanics. This concept was later implemented in quantum physics by Erwin Schrödinger [13]. To get to our observed property we need to explain basics of quantum mechanics.

A state of a physical system in quantum mechanics is described by a vector from separable complex Hilbert space which is a complete linear vector space with scalar product. The completeness is with respect to the distance induced by the scalar product and the linearity of the space is needed to guarantee the superposition principle. The space must be complex because we use a relative phase between vectors in the superposition. If the space is separable we can work with countable orthogonal base.

A state of a quantum system can be described by a state vector $|\psi\rangle$ in state space $\mathcal{H}$. The state vector contains all available information about the system. States described by state vectros are called pure states. A pure state is an idealization of a real state and often one cannot get a total information about the studied system. Then we talk about a mixed state which is characterized by a density matrix ρ. It is a positive semidefinite Hermitian operator, i.e., $\rho \geq 0$, $\rho = \rho^\dagger$ and $\mathrm{Tr}(\rho) = 1$. The density matrix can be written in the form

$$\rho = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|, \tag{1.1}$$

where p_i are probabilities that the system is in the state $|\psi_i\rangle$.

The state space of the system consisting of several subsystems with state spaces $\mathcal{H}_1$,$\mathcal{H}_2$,...,$\mathcal{H}_n$ is a direct product of state spaces of the particular subsystems

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_n. \tag{1.2}$$

The simplest system in which we can study the quantum entanglement is a bipartite system, which consists of two parts. For example, it can be made of two qubits, i.e., the simplest quantum systems with two-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^2$. A system consisting of two subsystems A and B is called separable if one can write it in the following form:

$$\rho_{A|B}^{sep} = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)}, \tag{1.3}$$

where $0 \leq p_i \leq 1$, $\sum_i p_i = 1$ are probabilities and $\rho_A^{(i)}$, $\rho_B^{(i)}$ are density matrices of the subsystems A and B. If a bipartite density matrix cannot be written in the form (1.3) it is called as entangled.

Systems consisting of more than two parts are called multipartite systems. With more parts of the system one have more possibilities how the system can be separable. We show this on the simplest multipartite system made of three subsystems A, B and C. For the tripartite system we have five classes of separability [14]. Major role there play partitions of the three systems into two sets. Altogether, we get three bipartite splitting $A|BC$, $B|AC$ and $C|AB$ and the following five separability classes [14]:

1. Fully inseparable states, i.e. states which are entangled with respect to all three splittings.
2. One-qubit biseparable states, i.e. states which are separable with respect to one splitting but they are entangled across the other two splittings.
3. Two-qubit biseparable states, i.e. states which are entangled with respect to one splitting but they are separable across the other two splittings.
4. Three-qubit biseparable states, i.e. states which are separable with respect to all three splittings but they cannot be written as the following convex matrix:
$$\rho_{ABC} \neq \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)} \otimes \rho_C^{(i)}, \tag{1.4}$$
where $\rho_j^{(i)}$, $j = A,B,C$ are states of individual subsystems.
5. Fully separable states are states which can be written in the form (1.4).

In this thesis we work with special states from the class of fully inseparable states called genuine multipartite entangled states. These states carry the strongest form of multipartite entanglement, which reguires a global operation on all three subsystems. Mathematically, a state ρ_{ABC} is genuine multipartite entangled if it cannot be written as the following convex mixture of states which are separable with respect to different bipartitons:
$$\rho_{ABC}^{\text{bisep}} \neq p_1\rho_{A|BC} + p_2\rho_{B|AC} + p_3\rho_{C|AB}, \tag{1.5}$$
where p_i, $i = 1,2,3$ are probabilities. States, which can be wirtten in this form are called biseparable and they cover not only states from classes $2-5$ but also some fully inseparable states.

Chapter 2

Classical analog of emergent multipartite entanglement

It is natural to wonder what can be said about the whole based only on the knowledge of its patrs. In mathematical statistics this problem is known as the marginal problem [2] where one wants to find all joint probability distributions compatible with a given set of reduced probability distributions. There is a quantum-mechanical analog of this problem known as the quantum marginal problem [3, 4]. Instead of the set of joint probability distributions one is given a set of marginal density matrices and seeks all global density matrices compatible with these.

One of the possible uses of the problem is the detection of a global property from parts. This task is more interesting if the marginals do not contain the property or any sign of it. Then we talk about an "emerging property", i.e., a property appearing only at a certain level of complexity of the system. So far, the marginal problem with emergent property has been investigated only in the context of multipartite entanglement in quantum systems with finite-dimensional Hilbert state space [7, 8, 9, 15, 16]. Below we use mapping of quantum states onto probability distributions to construct a classical analog of this marginal problem [10]. Content of this chapter is based on our paper [10].

2.1 Mapping entanglement onto secret correlations

Is there any example of the classical marginal problem with the emerging property? To answer this question one needs to map a quantum state on a probability distribution by a quantum mearurement [17]. A purification $|\psi\rangle_{ABE}$ of a density matrix ρ_{AB} of a two-level quantum system must meet the requirement $\rho_{AB} = \mathrm{Tr}_E(|\psi\rangle\langle\psi|_{ABE})$. From the purified state we can obtain a probability distribution by implementing local projective measurements P_i, $i = A,B,E$, on all subsystems of the purification. The outcomes of the measurement obey the following probability distribution:

$$P(A,B,E) = \mathrm{Tr}(P_A \otimes P_B \otimes P_E|\psi\rangle\langle\psi|). \tag{2.1}$$

By carrying out a suitable measurement of the original state ρ_{AB} carrying a specific quantum property one can get a probability distribution with a classical analog of this property.

A typical quantum property is some form of quantum entanglement. Its classical analog is the corresponding form of secret correlations. The concept of secret correlations originates from the classical secret-key agreement protocol [18]. Two honest parties, Alice and Bob, and an adversary Eve share independent realizations of three random variables A, B and E, which are distributed according to a probability distribution $P(A,B,E)$. Alice and Bob want to extract from their variables by local operations and public communication (LOPC) a secret key, i.e., a shared string of random bits about which Eve has no information. A necessary condition for this to be possible is that the distribution $P(A,B,E)$ contains secret correlations, i.e., the distribution cannot be created by LOPC. A convenient tool for detection of secret correlations is used the intrinsic information defined as [19]:

$$I(A;B \downarrow E) = \inf_{E\to\tilde{E}}[I(A;B|\tilde{E})]. \tag{2.2}$$

Here,

$$I(A;B|E) = H(A,E) + H(B,E) - H(A,B,E) - H(E), \tag{2.3}$$

where $H(X)$ is the Shanon enthropy, is the conditional mutual information, the minimalization is performed over all channels $E \to \tilde{E}$. A probability distribution contains secret correlations if, and only if, $I(A;B \downarrow E) > 0$ [20, 21]. To prove that the probability distribution does not carry any secret correlations one needs to show $I(A;B \downarrow E) = 0$, which can be done similarly as in [20, 22, 23]. It can be done by finding a suitable channel $E \to \tilde{E}$ which nullifies the conditional mutual information. Thus, the intrinsic information (2.2) vanishes and the investigated distribution does not carry secret correlations.

If one wants to show the presence of secret correlations in given probability distribution, one way is to prove the strict positivity of the intrinsic information (2.2). This way is very difficult because of minimization which must be done with respect to all possible channels $E \to \tilde{E}$. The intrinsic infromation is an upper bound on the secret key rate $S(A;B||E)$ [19] as thus the other approach may look at the lower bound on the secret key rate. The lower bound is given by [24]

$$S(A;B||E) \geq \max\left[I(A;B) - I(A;E), I(A;B) - I(B;E)\right], \tag{2.4}$$

where $I(X;Y) = H(X) + H(Y) - H(X,Y)$ is the mutual information. If the right-hand side of Eq. (2.4) is strictly positive, the secret key rate is strictly posivite and the probability distribution contains secret correlations.

2.2 Construction of the analog

Our task is to show if there exists a set of marginal probability distributions with no secret correlations which is compatible only with global distribution carrying secret correlations.

We start with the three-qubit state found in Ref. [9],

$$\rho = \frac{2}{3}|\xi\rangle\langle\xi| + \frac{1}{3}|111\rangle\langle 111| \tag{2.5}$$

where

$$|\xi\rangle = \frac{1}{2}|010\rangle + \frac{1}{2}|100\rangle + \frac{1}{\sqrt{2}}|001\rangle. \tag{2.6}$$

This state is genuine multipartite entangled but its three two-qubit reduced density matrices are separable. Additionally the reduced matrices are compatible only with the global state (2.5).

To obtain the global probability distribution we consturct a purification $|\psi\rangle_{ABCE}$ of the state (2.5). The purification reads as

$$|\psi\rangle_{ABCE} = \sqrt{\frac{2}{3}}|\xi\rangle|0\rangle + \sqrt{\frac{1}{3}}|111\rangle|1\rangle. \tag{2.7}$$

Next we perform suitable projective measurements P_i, $i = A,B,C,E$, on the purified state. The outcomes of this measurement are distributed according to the fourvariate probability distribution of the form:

$$P(A,B,C,E) = \mathrm{Tr}(P_A \otimes P_B \otimes P_C \otimes P_E|\psi\rangle\langle\psi|). \tag{2.8}$$

In case of the purified state (2.7) the probability distribution is obtained by a measurement in the computational basis. The nonzero probabilities are summarized in Tab. 2.1.

A	B	C	E	$P(A,B,C,E)$
0	0	1	0	1/3
0	1	0	0	1/6
1	0	0	0	1/6
1	1	1	1	1/3

Table 2.1: Probability distribution of outcomes of the measurement of the purification (2.7) in the computational basis.

Besides the global distribution we need to compute also marginal distributions $P(A,B,E)$, $P(A,C,E)$ and $P(B,C,E)$. Because the distributions $P(A,C,E)$ and $P(B,C,E)$ coincide, we use only distribution $P(A,C,E)$. Proofs for the distribution $P(B,C,E)$ will be exactly the same. Nonzero probabilities of both distributions are summarized in Tab. 2.2 and 2.3, respectively.

A	B	E	$P(A,B,E)$
0	0	0	1/3
0	1	0	1/6
1	0	0	1/6
1	1	1	1/3

Table 2.2: Marginal distribution $P(A,B,E)$

A	C	E	$P(A,C,E)$
0	0	0	1/6
0	1	0	1/3
1	0	0	1/6
1	1	1	1/3

Table 2.3: Marginal distribution $P(A,C,E)$. The table of the marginal distribution $P(B,C,E)$ is obtained from previous table by replacing A with B in the first row of the table.

In the next section we show, that the marginal distributions $P(A,B,E)$, $P(A,C,E)$ and $P(B,C,E)$ carry no secret correlations whereas all compatible distributions $P(A,B,C,E)$ carry secret correlations.

2.3 Proof of the properties of the analog

Now we show in few steps, that the probability distribution in Tab. 2.1 carries a classical analog of multipartite entanglement verifiable from its separable reductions. For this purpose we need to prove, that the marginal distributions in Tabs. 2.2 and 2.3 do not carry secret correlations and simultaneously we can deduce from them that all global distributions compatible with them carry secret correlations.

First, we need to find all global probability distributions compatible with the marginal distributions $P(A,B,E)$, $P(A,C,E)$ and $P(B,C,E)$. We use the marginal probabilities as known variables and the global probabilities as unknown variables. The marginal probability distribution $P\left(A,B,E\right)$ can be received from the global probability distribution by equation

$$P_{ABE}\left(i,j,k\right) = P_{ABCE}\left(i,j,0,k\right) + P_{ABCE}\left(i,j,1,k\right), \tag{2.9}$$

where $i,j,k = 0,1$. Other marginal distributions can be obtained analogously. Altogether it gives twenty-four equations for sixteen variables. Moreover, every variable is constrained by an inequality $0 \leq P_{ABCE}\left(i,j,k,l\right) \leq 1$. It seems that this system is overdetermined. However, many marginal probabilities are equal to zero so with inequalities it gives us eleven variables equal to zero. More precisely, $P\left(0,0,0,1\right) = P\left(0,0,1,1\right) = P\left(0,1,0,1\right) = P\left(0,1,1,0\right) = P\left(0,1,1,1\right) = P\left(1,0,0,1\right) = P\left(1,0,1,0\right) = P\left(1,0,1,1\right) = P\left(1,1,0,0\right) = P\left(1,1,0,1\right) = P\left(1,1,1,0\right) = 0$. It reduces number of equations to eight for five unknown variables. However, we get two pairs of identical equations and one is a linear combination of other equations. We are now left with only five equations for five unknown variables. From this set we directly recieve the values of last variables $P(0,0,0,0) = P(0,0,1,0) = P(1,1,1,1) = 1/3$ and $P(0,1,0,0) = P(1,0,0,0) = 1/6$. Therefore, the global distribution is unequivocally given. It is interesting that the uniqueness of global state ρ_{ABC} compatible with reductions ρ_{AB}, ρ_{AC} and ρ_{BC} was transferred to the probability distribution.

Next step is to show that the global distribution from the first step carries secret correlations across all bipartitions $A|BC$, $B|AC$ and $C|AB$. Because of the symmetry under exchange of bits A and B in the distribution we need to show it only for bipartitions $A|BC$ and $C|AB$. We use the formula (2.4) for the lower bound on the secret key rate but extended to the case of four variables. For bipartition $A|BC$ it is now in the form:

$$S\left(A;BC||E\right) \geq \max\left[I\left(A;BC\right) - I\left(A;E\right), I\left(A;BC\right) - I\left(BC;E\right)\right]. \tag{2.10}$$

The lower bound gives for the bipartition $A|BC$ approximately 0.541 and for the bipartition $C|AB$, respectively $B|AC$, it is equal to 2/3. Both bounds are posivite which means that the global distribution from Tab. 2.1 carries secret correlations across all bipartitions.

In the last step we need to prove that the marginal distributions do not contain secret correlations. It is the opposite of what we did in previous step. The probability distribution does not carry secret correlations if its instrinsic information (2.2) is zero [19]. We need to find a suitable channel $E \rightarrow \tilde{E}$ for both marginal distributions such, that the conditional mutual information (2.3) vanishes.

We start with the distribution $P(A,B,E)$. The suitable channel for this distribution can be charatcerized by the conditional probability distribution: $P_{E|\tilde{E}}(0,0) = 1$, $P_{E|\tilde{E}}(1,0) = 0$, $P_{E|\tilde{E}}(0,1) = 1/4$ and $P_{E|\tilde{E}}(1,1) = 3/4$. The marginal distribution with the new channel $P(A,B,\tilde{E})$ is shown in Table 2.4. If we calculate conditional mutual

A	B	$\tilde{E}$	$P(A,B,\tilde{E})$
0	0	0	1/3
0	1	0	1/6
1	0	0	1/6
1	1	0	1/12
1	1	1	1/4

Table 2.4: New marginal distribution $P(A,B,\tilde{E})$.

information (2.3) for the new distribution, we get $I(A;B|\tilde{E}) = 0$. It means that the instrinsic information $I(A;B \downarrow E) = 0$ and the marginal distribution does not carry secret correlations. Moreover, the distribution can be prepared by LOPC.

Moving to the other marginal distribution, one can use new channel described by the conditional distribution: $P_{E|\tilde{E}}(0,0) = 0$, $P_{E|\tilde{E}}(1,0) = 1$, $P_{E|\tilde{E}}(0,1) = 0$ and $P_{E|\tilde{E}}(1,1) = 1$. The obtained marginal distribution $P(A,C,\tilde{E})$ is diplayed in Tab. 2.5. The conditional mutual information (2.3) for the new distribution is $I(A;C|\tilde{E}) = 0$

A	C	$\tilde{E}$	$P(A,C,\tilde{E})$
0	0	1	1/6
0	1	1	1/3
1	0	1	1/6
1	1	1	1/3

Table 2.5: New marginal distribution $P(A,C,\tilde{E})$. The marginal distribution $P(B,C,\tilde{E})$ can be obtained by replacing A with B in the first column.

which implies that the distribution does not contain secret correlations as we wanted to prove. Recall finally, that the last marginal distribution $P(B,C,E)$ is obtained from $P(A,C,E)$ by replacing A with B. Thus by applying the latter channel to the distribution $P(B,C,E)$ one gets new distribution $P(B,C,\tilde{E})$ for which $I(B,C|\tilde{E}) = 0$ and therefore also the last distribution $P(B,C,E)$ carries no secret correlations as required.

2.4 Discussion and conclussions

We have obtained the global probability distribution carrying secret correlations which is determined by its marginal probability distributions without secret correlations. This shows that one can detect a global correlation property from marginals which do not have this property not only in quantum mechanics. Secret correlations carried by our global probability distribution appear across all three bipartite splittings and thus it can be viewed as a classical analog of a fully inseparable state.

The quantum state from Ref. [9], which we originally used to construct the global probability distribution is a state with stronger form of multipartite entanglement known as genuine multipartite entanglement. One can now ask if there exists a classical analog of the genuine multipartite entanglement. In analogy with biseparable state this probability distribution cannot be obtained as a convex mixture of distributions without secrete correlations across bipartite splittings. Moreover, to prove genuine multipartite entanglement of a quantum state it is needed to use a quantum operator called entanglement witness. Thus there arrise new question if one can find a classical analog of the entanglement witness and how it would look like.

The interesting thing is that in contrast with a genuine tripartite entangled state [9] where one needs to use all two-qubit reduced density matrices we worked only with three out of four marginal probability distributions. We do not use marginal distribution $P(A,B,C)$ because it does not contain variable E and thus for this marginal distribution the concept of secret correlations cannot be introduced.

We have demonstrated the possibility to map the quantum marginal problem with the constraints on separability of reduced states onto the classical marginal problem with the constraints on the secrecy content of the marginal distributions. We believe that there can be other distributions with the same properties. We tried to map a state from Ref. [7] in the same way like the presented state. The marginal probability distributions are not compatible with only one global distribution but with one-parametric set of the global distributions. When we calculated the lower bound on secret key rate (2.4) for the set of the global distributions, we discovered that for some values of the parameter the lower bound is not positive. It means that the proof of existence of the secret correlations in these global distributions will be more intricate and it needs further research which is beyond the scope of the present thesis.

Chapter 3

Gaussian genuine multipartite entanglement verifiable from separable marginals

So far, studies on emergent genuine multipartite entanglement focused on systems of qubits, At the outset, examples of multiqubit states have been constructed [7, 9] and experimentally demonstrated [11] whose genuine multipartite entanglement can be detected from all possible two-qubit marginals. Later, also examples of such states have been found, for which one can infer the entanglement from the so called minimal set of two-qubit marginals [8]. The minimal set of bipartite marginals contains every part of the system and the marginals between nearest neighbours. This guarantees that the global entanglement can be deduce from the set.

In this chapter we extend the concept of emergent genuine multipartite entanglement to the realm of Gaussian states of systems with infinite-dimensional systems called as modes in what follows. As for Gaussian states the knowledge of all two-mode marginals in equivalent with the knowledge of the entire state, here we consider influence of genuine multipartite entanglement from the minimal sets of two-mode marginals. The latter sets can be conveniently characterized by a special kind of graphs, where vertices represent modes and edges the two mode marginals belonging to the set.

The minimal set of two-mode marginals can be defined already for three-mode systems. We can look at the mininal set graphicaly where the vertices represents subsystems of the global system and the edges its bipartite marginals. The respective graphs for three- and four-mode systems are shown in Fig. 3.1.

In the next section we construct three-mode and four-mode Gaussian genuine multipartite entangled states whose genuine multipartite entanglement can be inferred from the minimal sets of its marginals depicted in Fig. 3.1.

3.1 Gaussian states

The most frequently used states in systems with infinite-dimensional Hilbert state space are Gaussian states. Because of the infinite dimension their density matrices are infinite-dimensional and there are more simple ways how to describe them. One of them is through the Wigner function which is a quasiprobability distribution. More

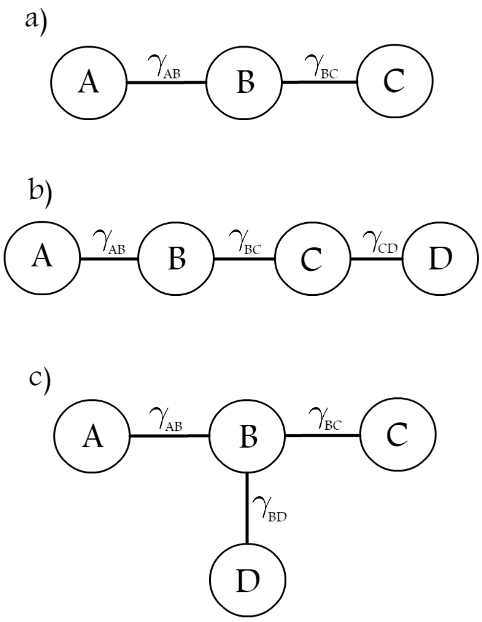

Figure 3.1: Graphical representation of the minimal set for the three-mode systems (a) and four-mode systems where one obtain two options: linear graph (b) and "t-shaped" graph (c). Symbols γ_{ij} stands for the covariance matrix of modes i and j.

precisely, it is a Weyl transformation of the density operator. The Wigner function of a Gaussian state is Gaussian-shaped. This allows to describe Gaussian states with first and second statistical moments of quadrature operators.

Consider a system consisting of N modes A_j, $j = 1,2,\ldots,N$, where each mode is characterized by a position and momentum quadrature operator x_{A_j} and p_{A_j}, respectively. These operators are summarized in a $2N \times 1$ vector $\xi = (x_{A_1}, p_{A_1}, \ldots, x_{A_N}, p_{A_N})^T$. Components of this vector must satisfy the commutation relations $[\xi_k, \xi_l] = i(\Omega_N)_{kl}$ where

$$\Omega_N = \bigoplus_{i=1}^{N} J, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.1}$$

The first moments of the system are characterized by a $2N \times 1$ vector $\langle \xi \rangle = \mathrm{Tr}\,[\xi \rho]$. The second moments are summarized in a $2N \times 2N$ matrix known as covariance matrix (CM) γ. Elements of the CM are obtained as $\gamma_{ij} = \langle \xi_i \xi_j + \xi_j \xi_i \rangle - 2\langle \xi_i \rangle \langle \xi_j \rangle$. The first moments can be turn to zero by local displacements, which means that they are irrelevant for studies of the correlations. From now on the first moments are therefore set to zero. Every CM γ must obey the Heisenberg uncertainty principle which can be written in the form [25]

$$\gamma + i\Omega_N \geq 0, \tag{3.2}$$

which gives us a necessary and sufficient condition for a real symmetric $2N \times 2N$ matrix γ to be a CM of a quantum state.

Gaussian states can be transform by linear optics to which we count phase shifts, squeezers and beam-splitters. This transformations are described by the Hamiltonians which are quadratic functions of the quadrature operators. They induce linear transformations of the quadrature operators, i.e., $\xi' = S\xi$ and preserve Gaussian character of the state. The matrix S is the so called symplectic matrix which is a $2N \times 2N$ real matrix satisfying

$$S\Omega_N S^T = \Omega_N. \tag{3.3}$$

On the CM level a symplectic transformation transforms the CM γ to $\gamma' = S\gamma S^T$.

The CM carries complete information about the quantum correlations in the considered system. A quantum state ρ_{jk} of two subsystems j and k (the state can be generally multimode) is separable if it can be written as a convex mixture (1.3). For a two-mode Gaussian state one can use s partial transposition criterion [25, 26, 27] to certify the separability. The partial transposition operation T_j with respect to mode j of a two-mode Gaussian state ρ_{jk} transforms its CM γ_{jk} as

$$\gamma_{jk}^{T_j} = (\sigma_z \oplus \mathbb{1}_2)\gamma_{jk}(\sigma_z \oplus \mathbb{1}_2), \tag{3.4}$$

where $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the Pauli-z matrix and $\mathbb{1}_m$ is $m \times m$ identity matrix. The partial transposition criterion [25] then says that a two-mode Gaussian state ρ_{jk} is separable if and only if the matrix $\gamma_{jk}^{T_j}$ is a physical CM, i.e.

$$\gamma_{jk}^{T_j} + i\Omega_2 \geq 0. \tag{3.5}$$

This criterion is a sufficient condition for separability also for $1 \times M$-mode Gaussian states [28]. For more complicated systems, one needs a more powerfull criterion [28]. According to this criterion an N-mode Gaussian state with CM γ, which consists of an l-mode subsystem A and $(N-l)$-mode subsystem B is separable, if there exist CMs γ_A and γ_B of subsystems A and B such that

$$\gamma - \gamma_A \otimes \gamma_B \geq 0. \tag{3.6}$$

3.2 Witnessing Gaussian entanglement via SDP

Except the use on more complex systems the criterion (3.6) has another advantage. It can be formulated as the following semi-definite programme (SDP) [29]

$$\begin{aligned} \underset{\gamma_A,\gamma_B,x_e}{\text{minimize}} \quad & (-x_e) \\ \text{subject to} \quad & \gamma - \gamma_A \oplus \gamma_B \geq 0, \\ & \gamma_A \oplus \gamma_B + (1+x_e)i\Omega_N \geq 0. \end{aligned} \tag{3.7}$$

If the optimal solution x_e is nonnegative, i.e., $x_e \geq 0$, the CM γ describes a separable state. This is because there are CMs γ_A and γ_B such that the separability criterion (3.6) is satisfied. If the optimal solutions is negative, then the CM γ describes an entangled state.

The dual problem corresponding to the primal problem (3.7) can be written in a form [29]:

$$\begin{aligned} \underset{X_1,X_2}{\text{minimize}} \quad & \mathrm{Tr}[\gamma X_1^{\mathrm{re}}] - 1, \\ \text{subject to} \quad & X_1^{\mathrm{bd,re}} = X_2^{\mathrm{bd,re}}, \quad X_1 \geq 0, \quad X_2 \geq 0, \\ & \mathrm{Tr}[i\Omega_N X_2] = -1, \end{aligned} \tag{3.8}$$

where $X_{1,2}$ are $2N \times 2N$ Hermitian matrices, X_1^{re} is the real part of matrix X_1. If one expresses the matrices $X_{1,2}$ in the block form with respect to bipartition $A|B$

$$X_{1,2} = \begin{pmatrix} A_{1,2} & B_{1,2} \\ C_{1,2} & D_{1,2} \end{pmatrix}, \tag{3.9}$$

then $X_{1,2}^{bd}$ means $X_{1,2}^{bd} = A_{1,2} \oplus D_{1,2}$. For every possible solution $X_1 \oplus X_2$ the matrix X_1^{re} satisfies

$$\mathrm{Tr}\left[\gamma X_1^{re}\right] \geq 1 \tag{3.10}$$

for every CM γ of a separable state [29]. For entangled states one receives

$$\mathrm{Tr}\left[\gamma X_1^{re}\right] < 1, \tag{3.11}$$

which means that the matrix X_1^{re} is an entanglement witness.

In a multipartite scenario with Gaussian states, one can say that an N-mode state characterized by CM γ is biseparable if there exist a bipartitions $\pi(k)$ of the N modes into $M_k < N$ and $N - M_k$ modes, $k = 1,2,\ldots,2^{N-1} - 1$, the CMs $\gamma_{\pi(k)}$, being block diagonal with respect to the bipartitions and probabilities λ_k such that

$$\gamma - \sum_{k=1}^{K} \lambda_k \gamma_{\pi(k)} \geq 0. \tag{3.12}$$

The number $K \equiv 2^{N-1} - 1$ represents the number of all different inequivalent bipartitions of N modes [29]. The states which are not biseparable, are known as genuine multipartite entangled. As for separability, also biseparability can be decided by solving the following SDP [29]:

$$\begin{aligned}
\underset{\{\gamma_{\pi(k)},\lambda_k\},x_e}{\text{minimize}} \quad & (-x_e) \\
\text{subject to} \quad & \gamma - \sum_{k=1}^{K} \gamma_{\pi(k)} \geq 0, \\
& \gamma_{\pi(k)} + \lambda_k i\Omega_N \geq 0, \quad \text{for all} \quad k, \\
& \sum_{k=1}^{K} \lambda_k = 1 + x_e, \\
& \lambda_k \geq 0, \quad \text{for all} \quad k
\end{aligned} \tag{3.13}$$

The SDP (3.13) can be further also rewritten as [29]:

$$\begin{aligned}
\underset{\{x_{ij}^{\pi(k)},\lambda_k\},x_e}{\text{minimize}} \quad & (-x_e) \\
\text{subject to} \quad & \gamma + \sum_{i,j,k}^{\text{bd,re},\pi(k)} (-F_{ij}) x_{ij}^{\pi(k)} \geq 0, \\
& \sum_{i,j}^{\text{bd,re},\pi(k)} F_{ij} x_{ij}^{\pi(k)} + \lambda_k i\Omega_N \geq 0, \quad \text{for all} \quad k, \\
& \sum_{k=1}^{K} \lambda_k - x_e - 1 \geq 0, \\
& -\left(\sum_{k=1}^{K} \lambda_k\right) + x_e + 1 \geq 0, \\
& \lambda_k \geq 0, \quad \text{for all} \quad k,
\end{aligned} \tag{3.14}$$

where 'bd,re,$\pi(k)$' in summation refers to 'block-diagonal and real with respect to partition $\pi(k)$'.

The dual problem to (3.14) is of the form [29]:

$$\begin{aligned}
&\underset{X}{\text{maximize}} && -\operatorname{Tr}\{[\gamma \oplus \mathbb{0}_{2NK} \oplus (-\mathbb{1}) \oplus \mathbb{1} \oplus \mathbb{0}_K]X\} \\
&\text{subject to} && X_1^{\text{re,bd},\pi(\text{k})} = X_{k+1}^{\text{re,bd},\pi(\text{k})} \quad \text{for all} \quad k = 1,\dots,K, \\
& && \operatorname{Tr}[i\Omega_N X_{k+1}] + X_{K+2} - X_{K+3} + X_{K+3+k} = 0, \quad \text{for all} \quad k = 1,\dots,K, \\
& && X_{K+2} - X_{K+3} = 1,
\end{aligned} \tag{3.15}$$

where the $\mathbb{1}_m$ is the $m \times m$ identity matrix and $\mathbb{0}_m$ is the $m \times m$ zero matrix. The matrices X over which the maximation in (3.15) are performed are Hermitian posivite-semidefinite $[2N(K+1)+K+2]$-dimensional matrices. It can be without loss of generality written in a block-diagonal form

$$X = \bigoplus_{j=1}^{2K+3} X_j, \tag{3.16}$$

where X_j for $j = 1,2,\dots,K+1$ are $2N \times 2N$ Hermitian matrices and X_j, $j = K + 2,\dots,2K+3$ are 1×1 Hermitian matrices, i.e., real numbers.

In the first constraint $X_1^{\text{re,bd},\pi(\text{k})} = X_{k+1}^{\text{re,bd},\pi(\text{k})}$ the included matrices $X_1,X_2,\dots,X_{K+1}$ are $2N \times 2N$ blocks of the Hermitian matrices X. Moreover, this constraint is only on the elements $(X_k)_{ij}$ of the matrices corresponding to nonzero variables $x_{ij}^{\pi(k)}$, i.e., it is only on the diagonal blocks of the matrices X_k corresponding with diagonal blocks of the block-diagonal matrices $\gamma_{\pi(k)}$. If one writes the matrix X_j in the block form with respect to partitions $\pi(k)$

$$X_j = \begin{pmatrix} A_j^{\pi(k)} & B_j^{\pi(k)} \\ (B_j^{\pi(k)})^\dagger & D_j^{\pi(k)} \end{pmatrix}, \tag{3.17}$$

with the Hermitian block $A_j^{\pi(k)}$ corresponding to the set of modes in the first part of the bipartition $\pi(k)$ and the Hermitian block $D_j^{\pi(k)}$ to the set of modes in the second part of the bipartition. Then the matrix $X_j^{bd,\pi(k)}$ is a projection onto the block-diagonal form of the matrix $\gamma_{\pi(k)}$, i.e., $X_j^{bd,\pi(k)} = A_j^{\pi(k)} \oplus D_j^{\pi(k)}$. At last $X_j^{re,bd,\pi(k)}$ is real part of the matrix $X_j^{bd,\pi(k)}$, i.e., $X_j^{re,bd,\pi(k)} = \operatorname{Re} X_j^{bd,\pi(k)}$.

For a real symmetric matrix γ and a Hermitian matrix X_1 it holds $\operatorname{Tr}[\gamma X_1] = \operatorname{Tr}[\gamma X_1^{re}]$ and using the last constraint $X_{K+2} - X_{K+3} = 1$ one obtains reduced objective function of SDP (3.15) in the form [29]:

$$\operatorname{Tr}\{[\gamma \oplus \mathbb{0}_{2NK} \oplus (-\mathbb{1}) \oplus \mathbb{1} \oplus \mathbb{0}_K]X\} = \operatorname{Tr}[\gamma X_1^{re}] - 1. \tag{3.18}$$

As was shown for detection of separability (3.10) and (3.11) similar things happen here [29]. For every feasible solution X of the dual program (3.15) the matrix X_1^{re} is an optimal entanglement witness, i.e., it satisfies conditions

$$\begin{aligned}
&\text{(i)} && \operatorname{Tr}[\gamma X_1^{\text{re}}] \geq 1, && \text{for all biseparable} \quad \gamma, \\
&\text{(ii)} && \operatorname{Tr}[\gamma X_1^{\text{re}}] < 1, && \text{for some entangled} \quad \gamma.
\end{aligned} \tag{3.19}$$

The optimality of the witness is in the sense that set of all possible witnesses the value of $\operatorname{Tr}[\gamma X_1^{re}]$ is minimal.

The witness from SDP (3.15) acts on entire CM γ but we need a witness acting only on the minimal set of two-mode marginal CMs. We are looking for the witness which does not act on the 2×2 off-diagonal blocks of the CM γ. These blocks correspond to the missing edges from the graph of the minimal set of marginals shown on Fig. 3.1. More precisely, the witness X_1^{re} we want should have zero 2×2 off-diagonal blocks corresponding to missing edges, i.e., edges from the complementary graph. In the following section we show the additional constrain on the witness it explicitly for a three- and four-mode cases.

3.2.1 Three-mode example

The three-partite state is the simplest state where one can study genuine multipartite entanglement. In discrete variable scenario it was shown [9] that a three-qubit state whose genuine multipartite entanglement is verifiable from separable marginals must be mixed. Moreover, examples of such states have been constructed in Ref. [7, 9] and experimentally demonstrated [11]. However, there is no example of genuine multipartite entangled state of three qubits certifiable only from the minimal set of marginals ρ_{AB} and ρ_{BC}. Examples with this property were found for four, five and six qubits in Ref. [8]. Despite that we try to find three-mode Gaussian state with this property, i.e., a three-mode Gaussian state with 6×6 CM γ_{ABC} with all two-mode marginals separable and whose genuine multipartite entanglement can be infered only from the minimal set of marginal CMs γ_{AB} and γ_{BC}.

For a gaussian state of three modes A, B and C the number of possible bipartitions is $K = 2^{3-1} - 1 = 3$, and they read explicitly as $\pi(1) = A|BC$, $\pi(2) = B|AC$ and $\pi(3) = C|AB$. The three corresponding CMs are $\gamma_{\pi(1)} = \gamma_A \oplus \gamma_{BC}$, $\gamma_{\pi(2)} = \gamma_B \oplus \gamma_{AC}$ and $\gamma_{\pi(3)} = \gamma_C \oplus \gamma_{AB}$, respectively. From the block-diagonal structure of the latter CMs we can see that some variables $x_{ij}^{\pi(k)}$ from SDP (3.14) are zero and so the number of variables over which the dual SDP (3.15) is done is lower. Specifically,

$$
\begin{aligned}
\{x_{ij}^{\pi(1)} &= 0,\, i = 1{,}2;\ j = 3{,}4{,}5{,}6\},\\
\{x_{ij}^{\pi(2)} &= 0,\, \{i = 1{,}2;\ j = 3{,}4\},\{i = 3{,}4;\ j = 5{,}6\}\},\\
\{x_{ij}^{\pi(3)} &= 0,\, i = 1{,}2{,}3{,}4;\ j = 5{,}6\}.
\end{aligned}
\tag{3.20}
$$

The first constraint from (3.15) restricts the structure of 6×6 matrices X_1, X_2, X_3 and X_4. For better overview, let us write the matrix X_j in the block form with respect to splitting $A|B|C$

$$
X_j = \begin{pmatrix} (\mathbf{X}_j)_{11} & (\mathbf{X}_j)_{12} & (\mathbf{X}_j)_{13} \\ (\mathbf{X}_j)_{12}^\dagger & (\mathbf{X}_j)_{22} & (\mathbf{X}_j)_{23} \\ (\mathbf{X}_j)_{13}^\dagger & (\mathbf{X}_j)_{23}^\dagger & (\mathbf{X}_j)_{33} \end{pmatrix}, \tag{3.21}
$$

where $(X_j)_{kl}$ are the 2×2 blocks, the diagonal blocks $(X_j)_{kk}$ are Hermitian. Projecting matrices X_j onto a block-diagonal form of matrices $\gamma_{\pi(1)}$, $\gamma_{\pi(2)}$ and $\gamma_{\pi(3)}$ we obtain

$$
X_j^{\mathrm{bd},\pi(1)} = \begin{pmatrix} (\mathbf{X}_j)_{11} & \mathbb{O}_2 & \mathbb{O}_2 \\ \mathbb{O}_2 & (\mathbf{X}_j)_{22} & (\mathbf{X}_j)_{23} \\ \mathbb{O}_2 & (\mathbf{X}_j)_{23}^\dagger & (\mathbf{X}_j)_{33} \end{pmatrix}, \tag{3.22}
$$

$$
X_j^{\mathrm{bd},\pi(2)} = \begin{pmatrix} (\mathbf{X}_j)_{11} & \mathbb{O}_2 & (\mathbf{X}_j)_{13} \\ \mathbb{O}_2 & (\mathbf{X}_j)_{22} & \mathbb{O}_2 \\ (\mathbf{X}_j)_{13}^\dagger & \mathbb{O}_2 & (\mathbf{X}_j)_{33} \end{pmatrix}, \tag{3.23}
$$

$$X_j^{\mathrm{bd},\pi(3)} = \begin{pmatrix} (\mathbf{X}_j)_{11} & (\mathbf{X}_j)_{12} & \mathbb{O}_2 \\ (\mathbf{X}_j)_{12}^\dagger & (\mathbf{X}_j)_{22} & \mathbb{O}_2 \\ \mathbb{O}_2 & \mathbb{O}_2 & (\mathbf{X}_j)_{33} \end{pmatrix}. \tag{3.24}$$

For the explicit form see Sec. 1 of Appendix A.

The first contraint in SDP (3.15) for three modes has the form:

$$\begin{aligned} X_1^{\mathrm{re,bd},\pi(1)} &= X_2^{\mathrm{re,bd},\pi(1)}, \\ X_1^{\mathrm{re,bd},\pi(2)} &= X_3^{\mathrm{re,bd},\pi(2)}, \\ X_1^{\mathrm{re,bd},\pi(3)} &= X_4^{\mathrm{re,bd},\pi(3)}. \end{aligned} \tag{3.25}$$

The minimal set of marginals consists of reduced CMs γ_{AB} and γ_{BC}. If we write the CM γ in block form with respect to splitting $A|B|C$

$$\gamma = \begin{pmatrix} \gamma_A & \omega_{AB} & \omega_{AC} \\ \omega_{AB}^T & \gamma_B & \omega_{BC} \\ \omega_{AC}^T & \omega_{BC}^T & \gamma_C \end{pmatrix}, \tag{3.26}$$

we see that the witness we are looking for has to be "blind" to the block ω_{AC} containing correlations between mode A and mode C. Thus the witness acting only on the respective part of CM is given by the 6×6 Hermitian matrix in block form:

$$X_1^{\mathrm{re}} = \begin{pmatrix} (\mathbf{X}_1^{\mathrm{re}})_{11} & (\mathbf{X}_1^{\mathrm{re}})_{12} & \mathbb{O}_2 \\ (\mathbf{X}_1^{\mathrm{re}})_{12}^T & (\mathbf{X}_1^{\mathrm{re}})_{22} & (\mathbf{X}_1^{\mathrm{re}})_{23} \\ \mathbb{O}_2 & (\mathbf{X}_1^{\mathrm{re}})_{23}^T & (\mathbf{X}_1^{\mathrm{re}})_{33} \end{pmatrix}. \tag{3.27}$$

We need to add a new constraint to the SDP (3.15) to find the witness of the form (3.27). The constraint can be written as

$$(\mathbf{X}_1^{\mathrm{re}})_{13} = \begin{pmatrix} (X_1^{\mathrm{re}})_{15} & (X_1^{\mathrm{re}})_{16} \\ (X_1^{\mathrm{re}})_{25} & (X_1^{\mathrm{re}})_{26} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.28}$$

and the SDP in now in the form

$$\begin{aligned} \underset{X_1,\dots,X_9}{\text{minimize}} \quad & \mathrm{Tr}[\gamma X_1^{\mathrm{re}}] - 1 \\ \text{subject to} \quad & X_1 \geq 0, \quad X_2 \geq 0, \dots, \quad X_9 \geq 0, \\ & X_1^{\mathrm{re,bd},\pi(1)} = X_2^{\mathrm{re,bd},\pi(1)}, \\ & X_1^{\mathrm{re,bd},\pi(2)} = X_3^{\mathrm{re,bd},\pi(2)}, \\ & X_1^{\mathrm{re,bd},\pi(3)} = X_4^{\mathrm{re,bd},\pi(3)}, \\ & \mathrm{Tr}[i\Omega_3 X_{k+1}] + X_5 - X_6 + X_{6+k} = 0, \quad \text{for all} \quad k = 1{,}2{,}3, \\ & X_5 - X_6 = 1, \\ & \begin{pmatrix} (X_1^{\mathrm{re}})_{15} & (X_1^{\mathrm{re}})_{16} \\ (X_1^{\mathrm{re}})_{25} & (X_1^{\mathrm{re}})_{26} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{3.29}$$

where we used the rewritten objective function.

3.2.2 Four-mode Gaussian state

For the four-mode state we have two possible minimal sets of marginals. Our goal is to add constraints into the SDP so as to integrate into it the requirement that the witness acts only on the marginal CMs from the minimal set. First we look at some features which are common to both the sets.

For four-mode state the number of possible bipartitions is $K = 2^{4-1} - 1 = 7$. Namely, it is $\pi(1) = A|BCD$, $\pi(2) = B|ACD$, $\pi(3) = C|ABD$, $\pi(4) = D|ABC$, $\pi(5) = AB|CD$, $\pi(6) = AC|BD$ and $\pi(7) = AD|BC$ with corresponding block-diagonal CMs $\gamma_{\pi(1)} = \gamma_A \oplus \gamma_{BCD}$, $\gamma_{\pi(2)} = \gamma_B \oplus \gamma_{ACD}$, $\gamma_{\pi(3)} = \gamma_C \oplus \gamma_{ABD}$, $\gamma_{\pi(4)} = \gamma_D \oplus \gamma_{ABC}$, $\gamma_{\pi(5)} = \gamma_{AB} \oplus \gamma_{CD}$, $\gamma_{\pi(6)} = \gamma_{AC} \oplus \gamma_{BD}$ and $\gamma_{\pi(7)} = \gamma_{AD} \oplus \gamma_{BC}$, respectively. From the block-diagonal structure of the matrices we can see that some variables $x_{ij}^{\pi(k)}$ from the SDP (3.14) are zero. Specifically,

$$\begin{aligned}
\{x_{ij}^{\pi(1)} &= 0,\ i = 1{,}2;\ j = 3{,}4{,}5{,}6{,}7{,}8\},\\
\{x_{ij}^{\pi(2)} &= 0,\ \{i = 1{,}2;\ j = 3{,}4\},\{i = 3{,}4;\ j = 5{,}6{,}7{,}8\}\},\\
\{x_{ij}^{\pi(3)} &= 0,\ i = 1{,}2{,}3{,}4{,}7{,}8;\ j = 5{,}6\},\\
\{x_{ij}^{\pi(4)} &= 0,\ i = 7{,}8;\ j = 1{,}2{,}3{,}4{,}5{,}6\},\\
\{x_{ij}^{\pi(5)} &= 0,\ i = 1{,}2{,}3{,}4;\ j = 5{,}6{,}7{,}8\},\\
\{x_{ij}^{\pi(6)} &= 0,\ \{i = 1{,}2;\ j = 3{,}4{,}7{,}8\},\{i = 3{,}4;\ j = 5{,}6\},\{i = 5{,}6;\ j = 7{,}8\}\},\\
\{x_{ij}^{\pi(7)} &= 0,\ \{i = 1{,}2;\ j = 3{,}4{,}5{,}6\},\{i = 3{,}4;\ j = 7{,}8\},\{i = 5{,}6;\ j = 7{,}8\}\}.
\end{aligned} \tag{3.30}$$

The first constraint from (3.15) projects the matrices X_j, $j = 1,2,\ldots,7$, onto the block-diagonal form of the matrices $\gamma_{\pi(j)}$ and thus we obtain:

$$X_j^{\mathrm{bd},\pi(1)} = \begin{pmatrix} (\mathbf{X}_j)_{11} & \mathbb{O}_2 & \mathbb{O}_2 & \mathbb{O}_2 \\ \mathbb{O}_2 & (\mathbf{X}_j)_{22} & (\mathbf{X}_j)_{23} & (\mathbf{X}_j)_{24} \\ \mathbb{O}_2 & (\mathbf{X}_j)_{23}^{\dagger} & (\mathbf{X}_j)_{33} & (\mathbf{X}_j)_{34} \\ \mathbb{O}_2 & (\mathbf{X}_j)_{24}^{\dagger} & (\mathbf{X}_j)_{34}^{\dagger} & (\mathbf{X}_j)_{44} \end{pmatrix}, \tag{3.31}$$

$$X_j^{\mathrm{bd},\pi(2)} = \begin{pmatrix} (\mathbf{X}_j)_{11} & \mathbb{O}_2 & (\mathbf{X}_j)_{13} & (\mathbf{X}_j)_{14} \\ \mathbb{O}_2 & (\mathbf{X}_j)_{22} & \mathbb{O}_2 & \mathbb{O}_2 \\ (\mathbf{X}_j)_{13}^{\dagger} & \mathbb{O}_2 & (\mathbf{X}_j)_{33} & (\mathbf{X}_j)_{34} \\ (\mathbf{X}_j)_{14}^{\dagger} & \mathbb{O}_2 & (\mathbf{X}_j)_{34}^{\dagger} & (\mathbf{X}_j)_{44} \end{pmatrix}, \tag{3.32}$$

$$X_j^{\mathrm{bd},\pi(3)} = \begin{pmatrix} (\mathbf{X}_j)_{11} & (\mathbf{X}_j)_{12} & \mathbb{O}_2 & (\mathbf{X}_j)_{14} \\ (\mathbf{X}_j)_{12}^{\dagger} & (\mathbf{X}_j)_{22} & \mathbb{O}_2 & (\mathbf{X}_j)_{24} \\ \mathbb{O}_2 & \mathbb{O}_2 & (\mathbf{X}_j)_{33} & \mathbb{O}_2 \\ (\mathbf{X}_j)_{14}^{\dagger} & (\mathbf{X}_j)_{24}^{\dagger} & \mathbb{O}_2 & (\mathbf{X}_j)_{44} \end{pmatrix}, \tag{3.33}$$

$$X_j^{\mathrm{bd},\pi(4)} = \begin{pmatrix} (\mathbf{X}_j)_{11} & (\mathbf{X}_j)_{12} & (\mathbf{X}_j)_{13} & \mathbb{O}_2 \\ (\mathbf{X}_j)_{12}^{\dagger} & (\mathbf{X}_j)_{22} & (\mathbf{X}_j)_{23} & \mathbb{O}_2 \\ (\mathbf{X}_j)_{13}^{\dagger} & (\mathbf{X}_j)_{23}^{\dagger} & (\mathbf{X}_j)_{33} & \mathbb{O}_2 \\ \mathbb{O}_2 & \mathbb{O}_2 & \mathbb{O}_2 & (\mathbf{X}_j)_{44} \end{pmatrix}, \tag{3.34}$$

$$X_j^{\mathrm{bd},\pi(5)} = \begin{pmatrix} (\mathbf{X}_j)_{11} & (\mathbf{X}_j)_{12} & \mathbb{O}_2 & \mathbb{O}_2 \\ (\mathbf{X}_j)_{12}^{\dagger} & (\mathbf{X}_j)_{22} & \mathbb{O}_2 & \mathbb{O}_2 \\ \mathbb{O}_2 & \mathbb{O}_2 & (\mathbf{X}_j)_{33} & (\mathbf{X}_j)_{34} \\ \mathbb{O}_2 & \mathbb{O}_2 & (\mathbf{X}_j)_{34}^{\dagger} & (\mathbf{X}_j)_{44} \end{pmatrix}, \tag{3.35}$$

$$X_j^{\mathrm{bd},\pi(6)} = \begin{pmatrix} (\mathbf{X}_j)_{11} & \mathbb{O}_2 & (\mathbf{X}_j)_{13} & \mathbb{O}_2 \\ \mathbb{O}_2 & (\mathbf{X}_j)_{22} & \mathbb{O}_2 & (\mathbf{X}_j)_{24} \\ (\mathbf{X}_j)_{13}^\dagger & \mathbb{O}_2 & (\mathbf{X}_j)_{33} & \mathbb{O}_2 \\ \mathbb{O}_2 & (\mathbf{X}_j)_{24}^\dagger & \mathbb{O}_2 & (\mathbf{X}_j)_{44} \end{pmatrix}, \tag{3.36}$$

$$X_j^{\mathrm{bd},\pi(7)} = \begin{pmatrix} (\mathbf{X}_j)_{11} & \mathbb{O}_2 & \mathbb{O}_2 & (\mathbf{X}_j)_{14} \\ \mathbb{O}_2 & (\mathbf{X}_j)_{22} & (\mathbf{X}_j)_{23} & \mathbb{O}_2 \\ \mathbb{O}_2 & (\mathbf{X}_j)_{23}^\dagger & (\mathbf{X}_j)_{33} & \mathbb{O}_2 \\ (\mathbf{X}_j)_{14}^\dagger & \mathbb{O}_2 & \mathbb{O}_2 & (\mathbf{X}_j)_{44} \end{pmatrix}. \tag{3.37}$$

For the explicit form of the matrices see Sec. 2 of Appendix A.

A. Linear graph

First type of the minimal set of marginals corresponds to the linear graph in Fig. 3.1 b). It does not contain marginal CMs γ_{AC}, γ_{AD} and γ_{BD} which implies that the witness will ignore the blocks ω_{AC}, ω_{AD} and ω_{BD} of the CM (3.43). The witness then shloud be in the form:

$$X_1^{\mathrm{re}} = \begin{pmatrix} (\mathbf{X}_1^{\mathrm{re}})_{11} & (\mathbf{X}_1^{\mathrm{re}})_{12} & \mathbb{O}_2 & \mathbb{O}_2 \\ (\mathbf{X}_1^{\mathrm{re}})_{12}^T & (\mathbf{X}_1^{\mathrm{re}})_{22} & (\mathbf{X}_1^{\mathrm{re}})_{23} & \mathbb{O}_2 \\ \mathbb{O}_2 & (\mathbf{X}_1^{\mathrm{re}})_{23}^T & (\mathbf{X}_1^{\mathrm{re}})_{33} & (\mathbf{X}_1^{\mathrm{re}})_{34} \\ \mathbb{O}_2 & \mathbb{O}_2 & (\mathbf{X}_1^{\mathrm{re}})_{34}^T & (\mathbf{X}_1^{\mathrm{re}})_{44} \end{pmatrix}, \tag{3.38}$$

so the new constraints which should be added to the SDP (3.15) are

$$(\mathbf{X}_1^{\mathrm{re}})_{13} = \begin{pmatrix} (X_1^{\mathrm{re}})_{15} & (X_1^{\mathrm{re}})_{16} \\ (X_1^{\mathrm{re}})_{25} & (X_1^{\mathrm{re}})_{26} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.39}$$

$$(\mathbf{X}_1^{\mathrm{re}})_{14} = \begin{pmatrix} (X_1^{\mathrm{re}})_{17} & (X_1^{\mathrm{re}})_{18} \\ (X_1^{\mathrm{re}})_{27} & (X_1^{\mathrm{re}})_{28} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.40}$$

$$(\mathbf{X}_1^{\mathrm{re}})_{24} = \begin{pmatrix} (X_1^{\mathrm{re}})_{37} & (X_1^{\mathrm{re}})_{38} \\ (X_1^{\mathrm{re}})_{47} & (X_1^{\mathrm{re}})_{48} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.41}$$

If we then rewrite the SDP as in previous case we obtain

$$\begin{aligned}
&\underset{X_1,\ldots,X_{17}}{\text{minimize}} \quad \mathrm{Tr}[\gamma X_1^{\mathrm{re}}] - 1 \\
&\text{subject to} \quad X_1 \geq 0, \quad X_2 \geq 0, \ldots, \quad X_{17} \geq 0, \\
&\qquad X_1^{\mathrm{re,bd},\pi(1)} = X_2^{\mathrm{re,bd},\pi(1)}, \\
&\qquad X_1^{\mathrm{re,bd},\pi(2)} = X_3^{\mathrm{re,bd},\pi(2)}, \\
&\qquad X_1^{\mathrm{re,bd},\pi(3)} = X_4^{\mathrm{re,bd},\pi(3)}, \\
&\qquad X_1^{\mathrm{re,bd},\pi(4)} = X_5^{\mathrm{re,bd},\pi(4)}, \\
&\qquad X_1^{\mathrm{re,bd},\pi(5)} = X_6^{\mathrm{re,bd},\pi(5)}, \\
&\qquad X_1^{\mathrm{re,bd},\pi(6)} = X_7^{\mathrm{re,bd},\pi(6)}, \\
&\qquad X_1^{\mathrm{re,bd},\pi(7)} = X_8^{\mathrm{re,bd},\pi(7)}, \\
&\qquad \mathrm{Tr}[i\Omega_4 X_{k+1}] + X_9 - X_{10} + X_{10+k} = 0, \quad \text{for all} \quad k = 1,\ldots,7, \\
&\qquad X_9 - X_{10} = 1, \\
&\qquad \begin{pmatrix} (X_{1^{\mathrm{re}}})_{15} & (X_1^{\mathrm{re}})_{16} \\ (X_1^{\mathrm{re}})_{25} & (X_1^{\mathrm{re}})_{26} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
&\qquad \begin{pmatrix} (X_1^{\mathrm{re}})_{17} & (X_1^{\mathrm{re}})_{18} \\ (X_1^{\mathrm{re}})_{27} & (X_1^{\mathrm{re}})_{28} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
&\qquad \begin{pmatrix} (X_1^{\mathrm{re}})_{37} & (X_1^{\mathrm{re}})_{38} \\ (X_1^{\mathrm{re}})_{47} & (X_1^{\mathrm{re}})_{48} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\end{aligned} \tag{3.42}$$

where matrices $X_j^{bd,\pi(k)}$ are given by Eq. (3.31)-(3.37).

B. 't-shaped' graph

Second type of the minimal set of marginals is captured by the 't-shaped' graph in Fig. 3.1 c). It reveals that in this case one knows marginal CMs γ_{AB}, γ_{BC} and γ_{BD}. From the full CM γ written in the block form

$$\gamma = \begin{pmatrix} \gamma_A & \omega_{AB} & \omega_{AC} & \omega_{AD} \\ \omega_{AB}^T & \gamma_B & \omega_{BC} & \omega_{BD} \\ \omega_{AC}^T & \omega_{BC}^T & \gamma_C & \omega_{CD} \\ \omega_{AD}^T & \omega_{BD}^T & \omega_{CD}^T & \gamma_D \end{pmatrix}, \tag{3.43}$$

we can see that for this minimal set the witness should ignore the blocks ω_{AC}, ω_{AD} and ω_{CD}, i.e., correlations between modes A and C, A and D and C and D. Thus the witness X_1^{re} which acts only on the corresponding part of the CM would read as:

$$X_1^{\mathrm{re}} = \begin{pmatrix} (\mathbf{X}_1^{\mathrm{re}})_{11} & (\mathbf{X}_1^{\mathrm{re}})_{12} & \mathbb{0}_2 & \mathbb{0}_2 \\ (\mathbf{X}_1^{\mathrm{re}})_{12}^T & (\mathbf{X}_1^{\mathrm{re}})_{22} & (\mathbf{X}_1^{\mathrm{re}})_{23} & (\mathbf{X}_1^{\mathrm{re}})_{24} \\ \mathbb{0}_2 & (\mathbf{X}_1^{\mathrm{re}})_{23}^T & (\mathbf{X}_1^{\mathrm{re}})_{33} & \mathbb{0}_2 \\ \mathbb{0}_2 & (\mathbf{X}_1^{\mathrm{re}})_{24}^T & \mathbb{0}_2 & (\mathbf{X}_1^{\mathrm{re}})_{44} \end{pmatrix}. \tag{3.44}$$

Thus we get three new constraints to the SDP (3.15) to find the witness in the form (3.44). Namely, they are given by (3.39), (3.40) and

$$(\mathbf{X}_1^{\mathrm{re}})_{34} = \begin{pmatrix} (X_1^{\mathrm{re}})_{57} & (X_1^{\mathrm{re}})_{58} \\ (X_1^{\mathrm{re}})_{67} & (X_1^{\mathrm{re}})_{68} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.45}$$

The SDP with all the new constraints and itemized first constraint has now form:

$$
\begin{aligned}
\underset{X_1,\ldots,X_{17}}{\text{minimize}} \quad & \text{Tr}[\gamma X_1^{\text{re}}] - 1 \\
\text{subject to} \quad & X_1 \geq 0, \quad X_2 \geq 0, \ldots, \quad X_{17} \geq 0, \\
& X_1^{\text{re,bd},\pi(1)} = X_2^{\text{re,bd},\pi(1)}, \\
& X_1^{\text{re,bd},\pi(2)} = X_3^{\text{re,bd},\pi(2)}, \\
& X_1^{\text{re,bd},\pi(3)} = X_4^{\text{re,bd},\pi(3)}, \\
& X_1^{\text{re,bd},\pi(4)} = X_5^{\text{re,bd},\pi(4)}, \\
& X_1^{\text{re,bd},\pi(5)} = X_6^{\text{re,bd},\pi(5)}, \\
& X_1^{\text{re,bd},\pi(6)} = X_7^{\text{re,bd},\pi(6)}, \\
& X_1^{\text{re,bd},\pi(7)} = X_8^{\text{re,bd},\pi(7)}, \\
& \text{Tr}[i\Omega_4 X_{k+1}] + X_9 - X_{10} + X_{10+k} = 0, \quad \text{for all} \quad k = 1, \ldots, 7, \\
& X_9 - X_{10} = 1, \\
& \begin{pmatrix} (X_1^{\text{re}})_{15} & (X_1^{\text{re}})_{16} \\ (X_1^{\text{re}})_{25} & (X_1^{\text{re}})_{26} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
& \begin{pmatrix} (X_1^{\text{re}})_{17} & (X_1^{\text{re,}})_{18} \\ (X_1^{\text{re}})_{27} & (X_1^{\text{re}})_{28} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
& \begin{pmatrix} (X_1^{\text{re}})_{57} & (X_1^{\text{re}})_{58} \\ (X_1^{\text{re}})_{67} & (X_1^{\text{re}})_{68} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\end{aligned} \tag{3.46}
$$

where matrices $X_j^{bd,\pi(k)}$ are given by Eqs. (3.31)-(3.37).

3.3 Results for three modes

Our taks is now to find a three-mode genuine multipartite entangled Gaussian state with all two-mode marginals separable and such that its entanglement can be verified only from marginal CMs γ_{AB} and γ_{BC}. It can be done by iterations consisting of two steps. First step of the iteration is to find an optimal entanglement witness for given CM which is provided by SDP (3.29). Second step is to find an optimal CM to the received witness. This step is provided by the following SDP:

$$
\begin{aligned}
\underset{\gamma}{\text{minimize}} \quad & \text{Tr}[\gamma \text{Re} X_1] \\
\text{subject to} \quad & \gamma + i\Omega_N \geq 0, \\
& \gamma_{jk}^{(T_j)} + i\Omega_2 \geq 0, \quad \text{for all} \quad j \neq k = 1, \ldots, N, \\
& \gamma_{2j-12k} = \gamma_{2j2k-1} = 0, \quad j,k = 1, \ldots, N,
\end{aligned} \tag{3.47}
$$

where the minimization is performed over all real symmetric $2N \times 2N$ matrices γ. In the first constraint, we guarentee that the matrix is CM of a physical state. The second constraint assures that all two-mode marginals are separable and the third constraint causes that there are no $x - p$ correlations in the obtained CM.

To start the iteration process we need to generate some initial CM. Similarly as it was done for qubits in Ref. [7] we choose to start with a randomly generated CM of a pure Gaussian state which possesses no $x - p$ correlations.

3.3.1 Found state

We have carried out the iterative search numerically with the help of Mgr. Jan Provazník from Palacký University, and from Viktor Nordgren and prof. Natalia Korolkova from University of St. Andrews. we found many examples of the sought state the best one being described by the following CM:

$$\gamma_3 = \begin{pmatrix} 18.69 & 0 & -12.09 & 0 & 7.07 & 0 \\ 0 & 0.26 & 0 & -0.01 & 0 & -0.32 \\ -12.09 & 0 & 10.11 & 0 & -5.79 & 0 \\ 0 & -0.01 & 0 & 4.93 & 0 & 4.92 \\ 7.07 & 0 & -5.79 & 0 & 9.41 & 0 \\ 0 & -0.32 & 0 & 4.92 & 0 & 5.43 \end{pmatrix}. \tag{3.48}$$

The original solution was rounded to two decimal places. The corresponding optimal witness for the rounded CM (3.48) is of the form:

$$X_1^{re} = \begin{pmatrix} 7.38 & 0 & 10.39 & 0 & 0 & 0 \\ 0 & 47.81 & 0 & 2.17 & 0 & 0 \\ 10.39 & 0 & 15.6 & 0 & 1.14 & 0 \\ 0 & 2.17 & 0 & 65.26 & 0 & -60.18 \\ 0 & 0 & 1.14 & 0 & 1.31 & 0 \\ 0 & 0 & 0 & -60.18 & 0 & 55.59 \end{pmatrix} \cdot 10^{-2} \tag{3.49}$$

and it gives $\mathrm{Tr}\,[\gamma X_1^{re}] - 1 \doteq -0.125$. This tells us based on the condition for the witness (3.11) that the state (3.48) is genuine multipartite entangled. One can also see, that the upper right and lower left blocks of the witness are zero and thus it is "blind" to the block ω_{AC}. Moreover, this state has stronger effect than the best four-qubit example found in Ref.[8], where the mean value of the witness is $\mathrm{Tr}\,[\rho W] \doteq -3.15 \cdot 10^{-3}$.

The separability of the marginals can be verified by the partial transposition criterion (3.5). The minimal eigenvalues of the matrix on the right hand side of the criterion are shown in Tab. 3.1. They are all positive which means that the reduced CMs describe separable states. The closeness of two eigenvalues to zero indicates that the respective marginals are close to the set of entangled states.

ij	AB	AC	BC
α_{ij}	0.006	0.156	0.002

Table 3.1: The minimal eigenvalues α_{ij} of the matrix $\gamma_{ij}^{T_i} + i\Omega_2$ for the CM (3.48)

In the possible experimental demonstration the closeness of the marginals to the set entangled states can be a problem but we can add a moderated amount of white noise to the CM γ_3. This would shift the marginals more far from the set of entangled states while keeping the mean value of the witness negative and far enough below zero. The white noise can be added as $\gamma_3' = \gamma + p\mathbb{1}$ with $0 \leq p \leq p_{max}$, where $p_{max} \doteq 0.066$ is the amount of the white noise when the quantity $\mathrm{Tr}\,[\gamma_3' X_1^{re}] - 1$ vanishes. Thus if one adds whereas, for instance, half of the tolerable noise $p = p_{max}/2$, one obtains $\mathrm{Tr}\,[\gamma_3' X_1^{re}] - 1 \doteq -0.063$, $\alpha_{AB} \doteq 0.039$ and $\alpha_{BC} \doteq 0.035$.

3.3.2 Logical circuit

Thanks to a high quality of the found three-mode state (3.48), it appears to be a good candidate for an experimental demonstration and thus we want to design a linear-optical scheme for preparation of the state. For thus purpose we use the Williamson's symplectic diagonalization of a CM [31], the Bloch-Messiah decomposition of a symplectic matrix [32] and the decomposition of an orthogonal symplectic matrix into an array of beam-splitters and phase-shifters [33, 34].

According to Williamson's theorem [31] here is a symplectic transformation S such that any CM can be transformed to

$$S\gamma S^T = \mathrm{diag}(\nu_1,\nu_1,\nu_2,\nu_2,\nu_3,\nu_3) \equiv W, \tag{3.50}$$

where ν_1,ν_2,ν_3 are the so called symplectic eigenvalues. For the CM γ_3, Eq. (3.48) the eigenvalues read explicitly as $\nu_1 = 6.508$, $\nu_2 = 1.083$ and $\nu_3 = 1.005$. The symplectic matrix S can be found by using a method in Ref. [35] or a method in Ref. [36]. It can be further decomposed into a passive and active linear-optical elements using the Bloch-Messiah decomposition [32] as $S = URV^T$ where U and V are orthogonal and symplectic matrices correasponding to passive elements and R is a diagonal matrix containing squeezing parametrs as entries. Matrices U and V can be decomposed to an array of beam-splitters and phase-shifters but without $x-p$ correlations in the final state we need only phase-free beam-splitters. Matrices U and V are in the form [34]

$$U = B^U_{23}(T_{23})B^U_{13}(T_{13})B^U_{12}(T_{12}), \tag{3.51}$$

$$V = B^V_{23}(\tau_{23})B^V_{13}(\tau_{13})B^V_{12}(\tau_{12}), \tag{3.52}$$

where T_{jk} and τ_{jk}, $jk = 12,13,23$ are beam-splitter transmission coefficients. For the matrix U the beam-splitter transmission coefficients T_{jk} are shown in Tab. 3.2 and the corresponding matrices are in the form:

$$B^U_{23} = \begin{pmatrix} \mathbb{1} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \sqrt{1-T_{23}^2}\mathbb{1} & T_{23}\mathbb{1} \\ \mathbb{0} & T_{23}\mathbb{1} & -\sqrt{1-T_{23}^2}\mathbb{1} \end{pmatrix}, \tag{3.53}$$

$$B^U_{13} = \begin{pmatrix} \sqrt{1-T_{13}^2}\mathbb{1} & \mathbb{0} & T_{13}\mathbb{1} \\ \mathbb{0} & \mathbb{1} & \mathbb{0} \\ T_{13}\mathbb{1} & \mathbb{0} & \sqrt{1-T_{13}^2}\mathbb{1} \end{pmatrix}, \tag{3.54}$$

$$B^U_{12} = \begin{pmatrix} \sqrt{1-T_{12}^2}\mathbb{1} & T_{12}\mathbb{1} & \mathbb{0} \\ -T_{12}\mathbb{1} & \sqrt{1-T_{12}^2}\mathbb{1} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & -\mathbb{1} \end{pmatrix}, \tag{3.55}$$

where $\mathbb{1}$ and $\mathbb{0}$ are 2×2 identity and zero matrices. Beam-splitters that comprise the matrix V are given by

$$B^V_{23} = \begin{pmatrix} \mathbb{1} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & -\sqrt{1-\tau_{23}^2}\mathbb{1} & -\tau_{23}\mathbb{1} \\ \mathbb{0} & \tau_{23}\mathbb{1} & -\sqrt{1-\tau_{23}^2}\mathbb{1} \end{pmatrix}, \tag{3.56}$$

$$B^V_{13} = \begin{pmatrix} \sqrt{1-\tau_{13}^2}\mathbb{1} & \mathbb{0} & \tau_{13}\mathbb{1} \\ \mathbb{0} & \mathbb{1} & \mathbb{0} \\ \tau_{13}\mathbb{1} & \mathbb{0} & \sqrt{1-\tau_{13}^2}\mathbb{1} \end{pmatrix}, \tag{3.57}$$

$$B^V_{12} = \begin{pmatrix} \sqrt{1-\tau_{12}^2}\mathbb{1} & \tau_{12}\mathbb{1} & \mathbb{0} \\ \tau_{12}\mathbb{1} & -\sqrt{1-\tau_{12}^2}\mathbb{1} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{1} \end{pmatrix} \tag{3.58}$$

jk	12	13	23
T_{jk}	0.093	0.374	0.999
τ_{jk}	0.991	0.641	0.774

Table 3.2: Transmision coeficients for beam-splitter matrices (3.53)-(3.58).

with transmission coefficients in Tab. 3.2. Finally the matrix R is of the form

$$R = \mathrm{diag}(1.314,0.761,0.650,1.539,0.193,5.178) \tag{3.59}$$

(see Appendix B for more details of the decomposiiton). If we give together all parts we obtain the linear-optical scheme for preparation the state desrcibed by the CM (3.48), which is shown in Fig. 3.2.

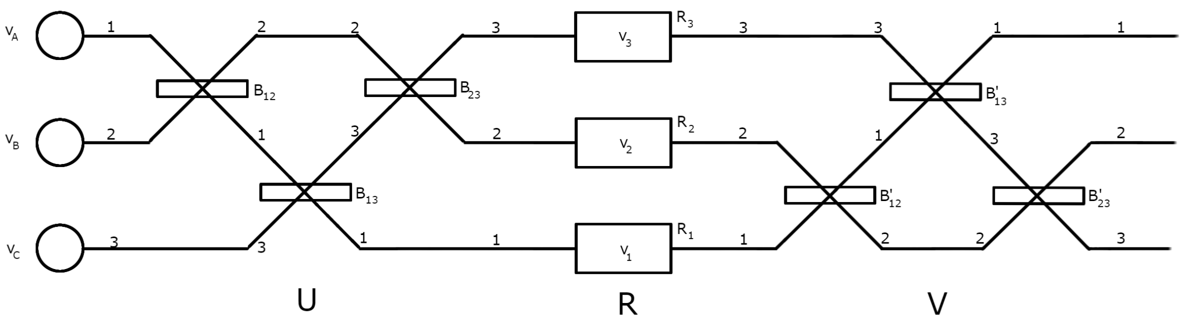

Figure 3.2: Linear-optical scheme for the CM γ_3; ν_j - thermal states with mean number of thermal photons $(\nu_j - 1)/2$, $j = A,B,C$; U - passive transformation consisting of beam splitters B_{jk}, $jk = 12,23,13$; V - passive transformation consisting of beam splitters B'_{jk}; R - squeezing transformation consisting of one squeezer in momentum quadrature, R_1, and two squeezers in position quadrature, R_2 and R_3.

3.4 Results for four modes

Next, we extended the search also to the four-mode case.

A. Linear graph

We also looked at the four-mode state with minimal set given by a linear graph in Fig. 3.1 b). We obtain the following CM

$$\gamma_4^{(l)} = \begin{pmatrix} 5.95 & 0 & -0.22 & 0 & -2.63 & 0 & 0.55 & 0 \\ 0 & 1.19 & 0 & -2.15 & 0 & 1.83 & 0 & -0.52 \\ -0.22 & 0 & 7.50 & 0 & 6.59 & 0 & -1.38 & 0 \\ 0 & -2.15 & 0 & 4.70 & 0 & -3.88 & 0 & 1.10 \\ -2.63 & 0 & 6.59 & 0 & 7.51 & 0 & -0.02 & 0 \\ 0 & 1.83 & 0 & -3.88 & 0 & 5.27 & 0 & -1.46 \\ 0.55 & 0 & -1.38 & 0 & -0.02 & 0 & 5.49 & 0 \\ 0 & -0.52 & 0 & 1.10 & 0 & -1.46 & 0 & 0.60 \end{pmatrix}. \tag{3.60}$$

For the CM, which was again rounded to two decimal places, we get $\gamma_4^{(l)}$ Tr $\left[\gamma_4^{(l)} X_1^{re}\right] - 1 \doteq -0.0485$. The separability is once again verified by the partial transposition criterion (3.5) and the minimal eigenvalues for all marginals are shown in Tab. 3.3.

ij	AB	AC	AD	BC	BD	CD
$\alpha_{ij}^{(t)}$	0.011	0.234	0.098	0.01	0.114	0.003

Table 3.3: The minimal eigenvalues $\alpha_{ij}^{(t)}$ of the matrix $(\gamma_4^{(l)})_{ij}^{T_i} + i\Omega_2$ for the CM (3.60)
.

B. 't-shaped' graph

Finally, for a 't-shaped' graph we obtain the following CM rounded to two decimal places

$$\gamma_4^{(t)} = \begin{pmatrix} 3.02 & 0 & -1.76 & 0 & -0.53 & 0 & 0.69 & 0 \\ 0 & 5.65 & 0 & 3.22 & 0 & 0.83 & 0 & -4.94 \\ -1.76 & 0 & 6.89 & 0 & 5.38 & 0 & 3.79 & 0 \\ 0 & 3.22 & 0 & 2.28 & 0 & 0.10 & 0 & -2.82 \\ -0.53 & 0 & 5.38 & 0 & 6.30 & 0 & 4.43 & 0 \\ 0 & 0.83 & 0 & 0.10 & 0 & 1.13 & 0 & -1.08 \\ 0.69 & 0 & 3.79 & 0 & 4.43 & 0 & 5.04 & 0 \\ 0 & -4.94 & 0 & -2.82 & 0 & -1.08 & 0 & 5.03 \end{pmatrix}. \tag{3.61}$$

The rounded CM yields Tr $\left[\gamma_4^{(t)} X_1^{re}\right] - 1 \doteq -0.0156$ which evidenced the presence of genuine multipartite entanglement. For the verification of the separability of the marginals we used again the partial transposition criterion (3.5). The obtained eigenvalues of the right-hand side are shown in Tab. 3.4.

ij	AB	AC	AD	BC	BD	CD
$\alpha_{ij}^{(t)}$	0.010	0.772	0.067	0.341	0.062	0.142

Table 3.4: The minimal eigenvalues $\alpha_{ij}^{(t)}$ of the matrix $(\gamma_4^{(t)})_{ij}^{T_i} + i\Omega_2$ for the CM (3.61)
.

Thanks to the weaker effect, the possible experimental demonstration will be more challenging.

Conclusion

In this thesis we extended analysis of a phenomen of emergent genuine multipartite entanglement to the realm of classical discrete random variables and Gaussian continuous variables.

In the first part of this thesis we used the mapping between quantum states and probability distributions to get a cryptographic analog of the investigated phenomenon. More precisely, we found a set of marginal distributions which carry no secret correlations yet the global distribution with which they are compatible carries the correlations. This demonstrates, that the investigated effect does not exist only in quantum world but it can be found also in the context of classical random variables.

In the second part of this thesis we found Gaussian states, which exhibit the studied effect. Interstingly, we found the effect already for three-mode case, whereas a three-qubit examples is not known. The effect in our three-mode example is relatively strong and thus it is attractive from the point of view of experiment.

Appendix A

Explicit forms of matrices in (3.8)

In this appendix we show the explicit form of the matrices $X_j^{bd,\pi(k)}$ which appear in SDP (3.8) for $N=3$ and $N=4$.

1. $N = 3$

For $N=3$ we have 3 bipartitions $\pi(1)=A|BC$, $\pi(2)=B|AC$ and $\pi(3)=C|AB$. The first constraint in the SDP (3.8) is on certain elements of real parts of the 6×6 Hermitian matrices X_j, $j=1$,2,3,4. Projecting matrices X_j onto a block-diagonal form of matrices $\gamma_{\pi(1)}$, $\gamma_{\pi(2)}$ and $\gamma_{\pi(3)}$ we obtain

$$X_j^{\mathrm{bd},\pi(1)} = \begin{pmatrix} (\mathbf{X}_j)_{11} & \mathbb{0}_2 & \mathbb{0}_2 \\ \mathbb{0}_2 & (\mathbf{X}_j)_{22} & (\mathbf{X}_j)_{23} \\ \mathbb{0}_2 & (\mathbf{X}_j)_{23}^\dagger & (\mathbf{X}_j)_{33} \end{pmatrix}, \tag{A.1}$$

$$X_j^{\mathrm{bd},\pi(2)} = \begin{pmatrix} (\mathbf{X}_j)_{11} & \mathbb{0}_2 & (\mathbf{X}_j)_{13} \\ \mathbb{0}_2 & (\mathbf{X}_j)_{22} & \mathbb{0}_2 \\ (\mathbf{X}_j)_{13}^\dagger & \mathbb{0}_2 & (\mathbf{X}_j)_{33} \end{pmatrix}, \tag{A.2}$$

$$X_j^{\mathrm{bd},\pi(3)} = \begin{pmatrix} (\mathbf{X}_j)_{11} & (\mathbf{X}_j)_{12} & \mathbb{0}_2 \\ (\mathbf{X}_j)_{12}^\dagger & (\mathbf{X}_j)_{22} & \mathbb{0}_2 \\ \mathbb{0}_2 & \mathbb{0}_2 & (\mathbf{X}_j)_{33} \end{pmatrix}. \tag{A.3}$$

If we itemize it in terms of elements $(X_j)_{kl}$ we get

$$X_j^{\mathrm{bd},\pi(1)} = \begin{pmatrix} (X_j)_{11} & (X_j)_{12} \\ (X_j)_{12}^* & (X_j)_{22} \end{pmatrix} \oplus \begin{pmatrix} (X_j)_{33} & (X_j)_{34} & (X_j)_{35} & (X_j)_{36} \\ (X_j)_{34}^* & (X_j)_{44} & (X_j)_{45} & (X_j)_{46} \\ (X_j)_{35}^* & (X_j)_{45}^* & (X_j)_{55} & (X_j)_{56} \\ (X_j)_{36}^* & (X_j)_{46}^* & (X_j)_{56}^* & (X_j)_{66} \end{pmatrix}, \tag{A.4}$$

$$X_j^{\mathrm{bd},\pi(2)} = \begin{pmatrix} (X_j)_{11} & (X_j)_{12} & 0 & 0 & (X_j)_{15} & (X_j)_{16} \\ (X_j)_{12}^* & (X_j)_{22} & 0 & 0 & (X_j)_{25} & (X_j)_{26} \\ 0 & 0 & (X_j)_{33} & (X_j)_{34} & 0 & 0 \\ 0 & 0 & (X_j)_{34}^* & (X_j)_{44} & 0 & 0 \\ (X_j)_{15}^* & (X_j)_{25}^* & 0 & 0 & (X_j)_{55} & (X_j)_{56} \\ (X_j)_{16}^* & (X_j)_{26}^* & 0 & 0 & (X_j)_{56}^* & (X_j)_{66} \end{pmatrix}, \tag{A.5}$$

$$X_j^{\mathrm{bd},\pi(3)} = \begin{pmatrix} (X_j)_{11} & (X_j)_{12} & (X_j)_{13} & (X_j)_{14} \\ (X_j)_{12}^* & (X_j)_{22} & (X_j)_{23} & (X_j)_{24} \\ (X_j)_{13}^* & (X_j)_{23}^* & (X_j)_{33} & (X_j)_{34} \\ (X_j)_{14}^* & (X_j)_{24}^* & (X_j)_{34}^* & (X_j)_{44} \end{pmatrix} \oplus \begin{pmatrix} (X_j)_{55} & (X_j)_{56} \\ (X_j)_{56}^* & (X_j)_{66} \end{pmatrix}. \tag{A.6}$$

2. $N = 4$

For $N = 4$ we have 7 bipartitions $\pi(1) = A|BCD$, $\pi(2) = B|ACD$, $\pi(3) = C|ABD$, $\pi(4) = D|ABC$, $\pi(5) = AB|CD$, $\pi(6) = AC|BD$ and $\pi(7) = AD|BC$. Thus the matrices $X_j^{bd,\pi(k)}$, $k = 1, \ldots, 7$ are

$$X_j^{\mathrm{bd},\pi(1)} = \begin{pmatrix} (\mathbf{X}_j)_{11} & \mathbb{O}_2 & \mathbb{O}_2 & \mathbb{O}_2 \\ \mathbb{O}_2 & (\mathbf{X}_j)_{22} & (\mathbf{X}_j)_{23} & (\mathbf{X}_j)_{24} \\ \mathbb{O}_2 & (\mathbf{X}_j)_{23}^\dagger & (\mathbf{X}_j)_{33} & (\mathbf{X}_j)_{34} \\ \mathbb{O}_2 & (\mathbf{X}_j)_{24}^\dagger & (\mathbf{X}_j)_{34}^\dagger & (\mathbf{X}_j)_{44} \end{pmatrix}, \tag{A.7}$$

$$X_j^{\mathrm{bd},\pi(2)} = \begin{pmatrix} (\mathbf{X}_j)_{11} & \mathbb{O}_2 & (\mathbf{X}_j)_{13} & (\mathbf{X}_j)_{14} \\ \mathbb{O}_2 & (\mathbf{X}_j)_{22} & \mathbb{O}_2 & \mathbb{O}_2 \\ (\mathbf{X}_j)_{13}^\dagger & \mathbb{O}_2 & (\mathbf{X}_j)_{33} & (\mathbf{X}_j)_{34} \\ (\mathbf{X}_j)_{14}^\dagger & \mathbb{O}_2 & (\mathbf{X}_j)_{34}^\dagger & (\mathbf{X}_j)_{44} \end{pmatrix}, \tag{A.8}$$

$$X_j^{\mathrm{bd},\pi(3)} = \begin{pmatrix} (\mathbf{X}_j)_{11} & (\mathbf{X}_j)_{12} & \mathbb{O}_2 & (\mathbf{X}_j)_{14} \\ (\mathbf{X}_j)_{12}^\dagger & (\mathbf{X}_j)_{22} & \mathbb{O}_2 & (\mathbf{X}_j)_{24} \\ \mathbb{O}_2 & \mathbb{O}_2 & (\mathbf{X}_j)_{33} & \mathbb{O}_2 \\ (\mathbf{X}_j)_{14}^\dagger & (\mathbf{X}_j)_{24}^\dagger & \mathbb{O}_2 & (\mathbf{X}_j)_{44} \end{pmatrix}, \tag{A.9}$$

$$X_j^{\mathrm{bd},\pi(4)} = \begin{pmatrix} (\mathbf{X}_j)_{11} & (\mathbf{X}_j)_{12} & (\mathbf{X}_j)_{13} & \mathbb{O}_2 \\ (\mathbf{X}_j)_{12}^\dagger & (\mathbf{X}_j)_{22} & (\mathbf{X}_j)_{23} & \mathbb{O}_2 \\ (\mathbf{X}_j)_{13}^\dagger & (\mathbf{X}_j)_{23}^\dagger & (\mathbf{X}_j)_{33} & \mathbb{O}_2 \\ \mathbb{O}_2 & \mathbb{O}_2 & \mathbb{O}_2 & (\mathbf{X}_j)_{44} \end{pmatrix}, \tag{A.10}$$

$$X_j^{\mathrm{bd},\pi(5)} = \begin{pmatrix} (\mathbf{X}_j)_{11} & (\mathbf{X}_j)_{12} & \mathbb{O}_2 & \mathbb{O}_2 \\ (\mathbf{X}_j)_{12}^\dagger & (\mathbf{X}_j)_{22} & \mathbb{O}_2 & \mathbb{O}_2 \\ \mathbb{O}_2 & \mathbb{O}_2 & (\mathbf{X}_j)_{33} & (\mathbf{X}_j)_{34} \\ \mathbb{O}_2 & \mathbb{O}_2 & (\mathbf{X}_j)_{34}^\dagger & (\mathbf{X}_j)_{44} \end{pmatrix}, \tag{A.11}$$

$$X_j^{\mathrm{bd},\pi(6)} = \begin{pmatrix} (\mathbf{X}_j)_{11} & \mathbb{O}_2 & (\mathbf{X}_j)_{13} & \mathbb{O}_2 \\ \mathbb{O}_2 & (\mathbf{X}_j)_{22} & \mathbb{O}_2 & (\mathbf{X}_j)_{24} \\ (\mathbf{X}_j)_{13}^\dagger & \mathbb{O}_2 & (\mathbf{X}_j)_{33} & \mathbb{O}_2 \\ \mathbb{O}_2 & (\mathbf{X}_j)_{24}^\dagger & \mathbb{O}_2 & (\mathbf{X}_j)_{44} \end{pmatrix}, \tag{A.12}$$

$$X_j^{\mathrm{bd},\pi(7)} = \begin{pmatrix} (\mathbf{X}_j)_{11} & \mathbb{O}_2 & \mathbb{O}_2 & (\mathbf{X}_j)_{14} \\ \mathbb{O}_2 & (\mathbf{X}_j)_{22} & (\mathbf{X}_j)_{23} & \mathbb{O}_2 \\ \mathbb{O}_2 & (\mathbf{X}_j)_{23}^\dagger & (\mathbf{X}_j)_{33} & \mathbb{O}_2 \\ (\mathbf{X}_j)_{14}^\dagger & \mathbb{O}_2 & \mathbb{O}_2 & (\mathbf{X}_j)_{44} \end{pmatrix}. \tag{A.13}$$

If we itemize it in terms $(X_j)_{kl}$ we obtain:

$$X_j^{\mathrm{bd},\pi(1)} = \begin{pmatrix} (X_j)_{11} & (X_j)_{12} \\ (X_j)_{12}^* & (X_j)_{22} \end{pmatrix} \oplus \begin{pmatrix} (X_j)_{33} & (X_j)_{34} & (X_j)_{35} & (X_j)_{36} & (X_j)_{37} & (X_j)_{38} \\ (X_j)_{34}^* & (X_j)_{44} & (X_j)_{45} & (X_j)_{46} & (X_j)_{47} & (X_j)_{48} \\ (X_j)_{35}^* & (X_j)_{45}^* & (X_j)_{55} & (X_j)_{56} & (X_j)_{57} & (X_j)_{58} \\ (X_j)_{36}^* & (X_j)_{46}^* & (X_j)_{56}^* & (X_j)_{66} & (X_j)_{67} & (X_j)_{68} \\ (X_j)_{37}^* & (X_j)_{47}^* & (X_j)_{57}^* & (X_j)_{67}^* & (X_j)_{77} & (X_j)_{78} \\ (X_j)_{38}^* & (X_j)_{48}^* & (X_j)_{58}^* & (X_j)_{68}^* & (X_j)_{78}^* & (X_j)_{88} \end{pmatrix}, \tag{A.14}$$

$$X_j^{\mathrm{bd},\pi(2)} = \begin{pmatrix} (X_j)_{11} & (X_j)_{12} & 0 & 0 & (X_j)_{15} & (X_j)_{16} & (X_j)_{17} & (X_j)_{18} \\ (X_j)_{12}^* & (X_j)_{22} & 0 & 0 & (X_j)_{25} & (X_j)_{26} & (X_j)_{27} & (X_j)_{28} \\ 0 & 0 & (X_j)_{33} & (X_j)_{34} & 0 & 0 & 0 & 0 \\ 0 & 0 & (X_j)_{34}^* & (X_j)_{44} & 0 & 0 & 0 & 0 \\ (X_j)_{15}^* & (X_j)_{25}^* & 0 & 0 & (X_j)_{55} & (X_j)_{56} & (X_j)_{57} & (X_j)_{58} \\ (X_j)_{16}^* & (X_j)_{26}^* & 0 & 0 & (X_j)_{56}^* & (X_j)_{66} & (X_j)_{67} & (X_j)_{68} \\ (X_j)_{17}^* & (X_j)_{27}^* & 0 & 0 & (X_j)_{57}^* & (X_j)_{67}^* & (X_j)_{77} & (X_j)_{78} \\ (X_j)_{18}^* & (X_j)_{28}^* & 0 & 0 & (X_j)_{58}^* & (X_j)_{68}^* & (X_j)_{78}^* & (X_j)_{88} \end{pmatrix}, \tag{A.15}$$

$$X_j^{\mathrm{bd},\pi(3)} = \begin{pmatrix} (X_j)_{11} & (X_j)_{12} & (X_j)_{13} & (X_j)_{14} & 0 & 0 & (X_j)_{17} & (X_j)_{18} \\ (X_j)_{12}^* & (X_j)_{22} & (X_j)_{23} & (X_j)_{24} & 0 & 0 & (X_j)_{27} & (X_j)_{28} \\ (X_j)_{13}^* & (X_j)_{23}^* & (X_j)_{33} & (X_j)_{34} & 0 & 0 & (X_j)_{37} & (X_j)_{38} \\ (X_j)_{14}^* & (X_j)_{24}^* & (X_j)_{34}^* & (X_j)_{44} & 0 & 0 & (X_j)_{47} & (X_j)_{48} \\ 0 & 0 & 0 & 0 & (X_j)_{55} & (X_j)_{56} & 0 & 0 \\ 0 & 0 & 0 & 0 & (X_j)_{56}^* & (X_j)_{66} & 0 & 0 \\ (X_j)_{17}^* & (X_j)_{27}^* & (X_j)_{37}^* & (X_j)_{47}^* & 0 & 0 & (X_j)_{77} & (X_j)_{78} \\ (X_j)_{18}^* & (X_j)_{28}^* & (X_j)_{38}^* & (X_j)_{48}^* & 0 & 0 & (X_j)_{78}^* & (X_j)_{88} \end{pmatrix}, \tag{A.16}$$

$$X_j^{\mathrm{bd},\pi(4)} = \begin{pmatrix} (X_j)_{11} & (X_j)_{12} & (X_j)_{13} & (X_j)_{14} & (X_j)_{15} & (X_j)_{16} \\ (X_j)_{12}^* & (X_j)_{22} & (X_j)_{23} & (X_j)_{24} & (X_j)_{25} & (X_j)_{26} \\ (X_j)_{13}^* & (X_j)_{23}^* & (X_j)_{33} & (X_j)_{34} & (X_j)_{35} & (X_j)_{36} \\ (X_j)_{14}^* & (X_j)_{24}^* & (X_j)_{34}^* & (X_j)_{44} & (X_j)_{45} & (X_j)_{46} \\ (X_j)_{15}^* & (X_j)_{25}^* & (X_j)_{35}^* & (X_j)_{45}^* & (X_j)_{55} & (X_j)_{56} \\ (X_j)_{16}^* & (X_j)_{26}^* & (X_j)_{36}^* & (X_j)_{46}^* & (X_j)_{56}^* & (X_j)_{66} \end{pmatrix} \oplus \begin{pmatrix} (X_j)_{77} & (X_j)_{78} \\ (X_j)_{78}^* & (X_j)_{88} \end{pmatrix}, \tag{A.17}$$

$$X_j^{\mathrm{bd},\pi(5)} = \begin{pmatrix} (X_j)_{11} & (X_j)_{12} & (X_j)_{13} & (X_j)_{14} \\ (X_j)_{12}^* & (X_j)_{22} & (X_j)_{23} & (X_j)_{24} \\ (X_j)_{13}^* & (X_j)_{23}^* & (X_j)_{33} & (X_j)_{34} \\ (X_j)_{14}^* & (X_j)_{24}^* & (X_j)_{34}^* & (X_j)_{44} \end{pmatrix} \oplus \begin{pmatrix} (X_j)_{55} & (X_j)_{56} & (X_j)_{57} & (X_j)_{58} \\ (X_j)_{56}^* & (X_j)_{66} & (X_j)_{67} & (X_j)_{68} \\ (X_j)_{57}^* & (X_j)_{67}^* & (X_j)_{77} & (X_j)_{78} \\ (X_j)_{58}^* & (X_j)_{68}^* & (X_j)_{78}^* & (X_j)_{88} \end{pmatrix}, \tag{A.18}$$

$$X_j^{\mathrm{bd},\pi(6)} = \begin{pmatrix} (X_j)_{11} & (X_j)_{12} & 0 & 0 & (X_j)_{15} & (X_j)_{16} & 0 & 0 \\ (X_j)_{12}^* & (X_j)_{22} & 0 & 0 & (X_j)_{25} & (X_j)_{26} & 0 & 0 \\ 0 & 0 & (X_j)_{33} & (X_j)_{34} & 0 & 0 & (X_j)_{37} & (X_j)_{38} \\ 0 & 0 & (X_j)_{34}^* & (X_j)_{44} & 0 & 0 & (X_j)_{47} & (X_j)_{48} \\ (X_j)_{15}^* & (X_j)_{25}^* & 0 & 0 & (X_j)_{55} & (X_j)_{56} & 0 & 0 \\ (X_j)_{16}^* & (X_j)_{26}^* & 0 & 0 & (X_j)_{56}^* & (X_j)_{66} & 0 & 0 \\ 0 & 0 & (X_j)_{37}^* & (X_j)_{47}^* & 0 & 0 & (X_j)_{77} & (X_j)_{78} \\ 0 & 0 & (X_j)_{38}^* & (X_j)_{48}^* & 0 & 0 & (X_j)_{78}^* & (X_j)_{88} \end{pmatrix}, \tag{A.19}$$

$$X_j^{\mathrm{bd},\pi(7)} = \begin{pmatrix} (X_j)_{11} & (X_j)_{12} & 0 & 0 & 0 & 0 & (X_j)_{17} & (X_j)_{18} \\ (X_j)_{12}^* & (X_j)_{22} & 0 & 0 & 0 & 0 & (X_j)_{27} & (X_j)_{28} \\ 0 & 0 & (X_j)_{33} & (X_j)_{34} & (X_j)_{35} & (X_j)_{36} & 0 & 0 \\ 0 & 0 & (X_j)_{34}^* & (X_j)_{44} & (X_j)_{45} & (X_j)_{46} & 0 & 0 \\ 0 & 0 & (X_j)_{35}^* & (X_j)_{45}^* & (X_j)_{55} & (X_j)_{56} & 0 & 0 \\ 0 & 0 & (X_j)_{36}^* & (X_j)_{46}^* & (X_j)_{56}^* & (X_j)_{66} & 0 & 0 \\ (X_j)_{17}^* & (X_j)_{27}^* & 0 & 0 & 0 & 0 & (X_j)_{77} & (X_j)_{78} \\ (X_j)_{18}^* & (X_j)_{28}^* & 0 & 0 & 0 & 0 & (X_j)_{78}^* & (X_j)_{88} \end{pmatrix}. \tag{A.20}$$

Appendix B

Bloch-Messiah decomposition of the symplectic matrix

To get the exact form of symplectic matrix realizing the Wiliamson's symplectic diagonalization of the CM γ_3 we use extended method from Ref. [37]. To easier manipulation, we transform the obtained symplectic matrix to the second ordering of the quadrature operators, where the vector of quadratures is in the form $\tau = (x_1,x_2,\ldots,x_N,p_1,p_2,\ldots,p_N)^T$. It is correlated with ξ by a tranformation $\tau = T\xi$, where T is an orthogonal matrix, i.e., $T^{-1} = T^T$. Nonzero elements of T are $T_{j2j-1} = 1$ and T_{N+j2j} for $j = 1,2,\ldots,N$. The transformation transforms symplectic matrix as $\sigma = TST^T$. Due to missing $x - p$ correlations is the sympletic matrix in the block diagonal form $\sigma = \sigma_x \oplus \sigma_p$. We can work only with the first block corresponding to the x-quadratures because the block correspoding to p-quadratures can be obtained as $\sigma_p = (\sigma_x^T)^{-1}$. Using single-value decomposition we receive exact forms of matrices u, v and r which are first blocks of matrices U, V and R, respectively. Elements of the matrix r give us squeezing parameters $s_1 = 1.314$, $s_2 = 0.650$ and $s_3 = 0.193$ which means that the first mode is squeezed in p quadrature and second and third modes are squeezed in x quadrature.

We need to decompose matrices u and v to an array of beam-splitters, i.e., in form [34]

$$u = b^u_{23}(T_{23})b^u_{13}(T_{13})b^u_{12}(T_{12}), \tag{B.1}$$

where T_{jk}, $jk = 12,13,23$ are beam-splitter transmision coeficients. The array of beam-splitters for the matrix u is of beam-splitters with transmision coeficients T_{jk} shown in Tab. 3.2. The matrices are in the form:

$$b^u_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{1-T_{23}^2} & T_{23} \\ 0 & T_{23} & -\sqrt{1-T_{23}^2} \end{pmatrix}, \tag{B.2}$$

$$b^u_{13} = \begin{pmatrix} \sqrt{1-T_{13}^2} & 0 & T_{13} \\ 0 & 1 & 0 \\ T_{13} & 0 & \sqrt{1-T_{13}^2} \end{pmatrix}, \tag{B.3}$$

$$b^u_{12} = \begin{pmatrix} \sqrt{1-T_{12}^2} & T_{12} & 0 \\ -T_{12} & \sqrt{1-T_{12}^2} & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{B.4}$$

Beam-splitters that are in the array in the matrix v are

$$b_{23}^{v} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sqrt{1-\tau_{23}^2} & -\tau_{23} \\ 0 & \tau_{23} & -\sqrt{1-\tau_{23}^2} \end{pmatrix}, \tag{B.5}$$

$$b_{13}^{v} = \begin{pmatrix} \sqrt{1-\tau_{13}^2} & 0 & \tau_{13} \\ 0 & 1 & 0 \\ \tau_{13} & 0 & \sqrt{1-\tau_{13}^2} \end{pmatrix}, \tag{B.6}$$

$$b_{12}^{v} = \begin{pmatrix} \sqrt{1-\tau_{12}^2} & \tau_{12} & 0 \\ \tau_{12} & -\sqrt{1-\tau_{12}^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{B.7}$$

Final matrices U and V can be obtained as $U = u \oplus u$ and $V = v \oplus v$ in the second ordering of quadrature operators.

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