PALACKÝ UNIVERSITY OLOMOUC
FACULTY OF SCIENCE
DEPARTMENT OF OPTICS

Symmetries of the Standard Model

BACHELOR'S THESIS

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UNIVERZITA PALACKÉHO V OLOMOUCI
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Symetrie Standardního modelu částic

BAKALÁŘSKÁ PRÁCE

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Abstract

This recherche thesis covers the topic of building up Lagrangian for the Standard model of the particle physics. It describes Schrödinger, Klein-Gordon and Dirac equations, their general solutions for a free particle and a particle in the electromagnetic field, and compilation of related continuity equations. The focus is on the part where various model Lagrangians are being constructed to be invariant under both Abelian and non-Abelian gauge calibrations.

Subsequently part of the Lagrangian of the Standard Model for electroweak unification and Higgs mechanism is discussed.

Keywords

Continuity equation, symmetries, Dirac equation, Standard model of the particle physics

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Declaration

I hereby declare that I - being supervised by Mgr. Jiří Kvita PhD. - am the only author of this bachelor's thesis and that all sources I have used are listed in the bibliography and identified as references. I agree with the further usage of this document according to the requirements of the Department of Optics.

In Olomouc ....

Doležalová Lenka

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Introduction

A thing is symmetrical if there is something we can do to it so that after we have done it, it looks the same as it did before. For example, a symmetrical vase is of such a kind that if we reflect or turn it, it will look the same as it did before. - H. Weyl [1]

If the Lagrangian is invariant under a continuous symmetry, there is a corresponding conserved quantity. - Emmy Noether [2]

It is increasingly clear that the symmetry group of nature is the deepest thing that we understand about nature today. - S. Weinberg [2]

Symmetries in nature give us a hope that one day we would find one equation, one law, that could describes every interaction. There are many predictions and theories attempting unification of all forces (string theory, quantum gravity and many more). The aim of this thesis is to build up the Lagrangian for the Standard model of particles focusing on the electroweak unification with the Higgs mechanism. It is the least step with which one could start with while longing for one single theory of physics.

Terminology

This thesis is written in natural units $\hbar = c = 1$, when needed, a footmark will be added to expose the SI expression. If not mentioned otherwise, we consider the Einstein sum rule. When talking about a Lagrangian, the L symbol is used, but while talking about the Lagrangian density, we use $\mathcal{L}$ and will call it Lagrangian as well. Where there the electric charge e is mentioned, we mean positron charge $e = 1.602 \cdot 10^{-19}$ C. We use hat symbol $\hat{}$ above operators only when expressly needed.

Metric tensor	$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$
Electromagnetic field tensor	$F_{\mu\nu} = \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & -B^z & B^y \\ -Ey & B^z & 0 & -B^x \\ -E^z & -B^y & B^x & 0 \end{pmatrix}$
Laplace operator	$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \Delta$
Momentum operator	$\hat{\boldsymbol{p}} = -i\boldsymbol{\nabla}$
D'Alembert operator	$\Box = \partial_\mu \partial^\mu = \partial_0^2 - \boldsymbol{\nabla}^2$
Space derivative	∂_j or ∇_j
Time derivative	$\frac{\partial}{\partial t} = \partial_0$
Space-time derivative	$\partial^\mu = (\partial^0, -\boldsymbol{\nabla})$
Covariant derivative	$D^\mu = \partial^\mu + igA^\mu$
Four-potential	$A^\mu = (\varphi, \boldsymbol{A})$

Chapter 1

Classical electrodynamics

We start with a short overview od classical electrodynamics.

1.1 Classical electromagnetism

The knowledge of electromagnetism can be put down in four Maxwell formulas for the electric (magnetic) intensity $\boldsymbol{E}$ ($\boldsymbol{H}$) and the electric (magnetic) induction $\boldsymbol{D}$ ($\boldsymbol{B}$)

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{H} - \partial_0 \boldsymbol{D} &= \boldsymbol{j} && (1.1) \\ \boldsymbol{\nabla} \cdot \boldsymbol{D} &= \rho && (1.2) \\ \boldsymbol{\nabla} \times \boldsymbol{E} + \partial_0 \boldsymbol{B} &= 0 && (1.3) \\ \boldsymbol{\nabla} \cdot \boldsymbol{B} &= 0, && (1.4) \end{aligned}$$

where $\boldsymbol{j}$ is the current density of free charge and ρ is the charge q density. and material definition

$$\begin{aligned} \boldsymbol{D} &= \epsilon_0 \boldsymbol{E} + \boldsymbol{P} && (1.5) \\ &= \epsilon_0 \boldsymbol{E}(1 + \kappa_e) && (1.6) \\ &= \epsilon_0 \epsilon_r \boldsymbol{E} && (1.7) \\ \boldsymbol{B} &= \mu_0 \boldsymbol{H} + \boldsymbol{M} && (1.8) \\ &= \mu_0 \boldsymbol{H}(1 + \kappa_m) && (1.9) \\ &= \mu_0 \mu_r \boldsymbol{H}. && (1.10) \end{aligned}$$

The $\boldsymbol{P}$ is electric polarisation and $\boldsymbol{M}$ is magnetization. Permittivity ϵ and permeability μ are constant in homogeneous isotropic medium or generally tensors in inhomogeneous and / or anisotropic medium. Permittivity of vacuum is $\epsilon_0 = 8.854 \cdot 10^{-12}$ Fm^{-1} and permeability of vacuum is $\mu_0 = 4\pi \cdot 10^{-7}$ Hm^{-1}.

There is formula for the speed of light in vacuum $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$ and in medium $c = \frac{1}{\sqrt{\epsilon\mu}}$.

1.2 Electrostatics

Important definition is the electric (scalar) potential as electrical work needs to be done to transfer unitary charge in electrostatic field along a trajectory l

$$\varphi = -\int_l \boldsymbol{E} \cdot d\boldsymbol{l}. \tag{1.11}$$

In case of an electrostatic field we write the third Maxwell's equation in the form of $\boldsymbol{\nabla} \times \boldsymbol{E} = 0$.

1.3 Electrodynamics

Let us consider a non-stationary non-conservative field. One can still use the scalar potential but must include magnetic field described by a vector potential $\boldsymbol{A}$ by defining

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}. \tag{1.12}$$

The fourth Maxwell equation is unchanged, because $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) = 0$. One can see that we can always add a gradient of any scalar function χ to the vector potential $\boldsymbol{A}$, and it will still satisfy the condition (1.12).
We now have definitions

$$\boldsymbol{E} = -\boldsymbol{\nabla}\varphi - \partial_0 \boldsymbol{A} \tag{1.13}$$
$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}. \tag{1.14}$$

1.4 Gauge calibration

As mentioned before, vector potential $\boldsymbol{A}$ can be redefined by adding gradient of an arbitrary scalar function χ. Let's call this potential $\boldsymbol{A'}$

$$\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\chi. \tag{1.15}$$

Scalar potential φ' can be redefined similarly

$$\varphi' = \varphi - \partial_0 \chi \tag{1.16}$$

or altogether as four-potential

$$A^\mu \equiv (\varphi, \boldsymbol{A}) \tag{1.17}$$
$$A'^\mu \equiv A^\mu - \partial^\mu \chi. \tag{1.18}$$

We put redefined potentials into (1.13) and (1.14) and we can see that the $\boldsymbol{E}$ and $\boldsymbol{B}$ remain unchanged when gauge calibrated in this way.

$$\begin{aligned}
\boldsymbol{E}' &= -\boldsymbol{\nabla}\varphi + \boldsymbol{\nabla}\frac{\partial\chi}{\partial t} - \frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{\nabla}\frac{\partial\chi}{\partial t} && (1.19)\\
&= -\boldsymbol{\nabla}\varphi - \frac{\partial \boldsymbol{A}}{\partial t} && (1.20)\\
&= \boldsymbol{E}, && (1.21)
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{B}' &= \boldsymbol{\nabla}\times\boldsymbol{A} + \boldsymbol{\nabla}\times\boldsymbol{\nabla}\chi && (1.22)\\
&= \boldsymbol{\nabla}\times\boldsymbol{A} && (1.23)\\
&= \boldsymbol{B}. && (1.24)
\end{aligned}$$

We have therefore a freedom to choose the calibration (gauge) function χ.

1.5 Lagrangian and Hamiltonian for a classical particle in electromagnetic field

Classical Lagrangian for a charged particle in an electromagnetic field is

$$L(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \frac{m\dot{\boldsymbol{x}}^2}{2} - q\varphi + q\boldsymbol{A}\cdot\dot{\boldsymbol{x}}, \tag{1.25}$$

where $\varphi = \varphi(t, \boldsymbol{x})$ and $\boldsymbol{A} = \boldsymbol{A}(t, \boldsymbol{x})$ and the corresponding Hamiltonian can by formed as

$$\begin{aligned}
H(\boldsymbol{p}, \boldsymbol{x}) &= \sum_j \dot{x}_j \frac{\partial L}{\partial \dot{x}_j} - L && (1.26)\\
&= \frac{1}{2m}\Big(\boldsymbol{p} - q\boldsymbol{A}\Big)^2 + q\varphi. && (1.27)
\end{aligned}$$

By defining the canonical conjugate momentum, which is different from the kinematic momentum $m\dot{\boldsymbol{x}}$,

$$\begin{aligned}
\frac{\partial L}{\partial \dot{x}_j} &= m\dot{x}_j + qA_j && (1.28)\\
&\equiv \pi_j, && (1.29)
\end{aligned}$$

and computing

$$\begin{aligned}
\frac{\partial L}{\partial x_j} &= q\nabla_j(A_j\dot{x}_j) - q\nabla_j\varphi_j && (1.30)\\
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_j} &= m\ddot{x}_j + q\frac{d}{dt}A_j, && (1.31)
\end{aligned}$$

we can derive the Euler-Lagrange equations

$$\begin{aligned}
\frac{\partial L}{\partial x_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_j} &= 0 && (1.32)\\
-m\ddot{x}_j - q\frac{d}{dt}A_j &= 0. && (1.33)
\end{aligned}$$

From here we may derive the Lorentz force $\boldsymbol{F_L}$, while having (1.13) and (1.14) in mind

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \frac{\partial H}{\partial \boldsymbol{p}} && (1.34)\\
&= \frac{\boldsymbol{p} - q\boldsymbol{A}}{m} && (1.35)\\
\dot{\boldsymbol{p}} &= -\frac{\partial H}{\partial \boldsymbol{x}} && (1.36)\\
&= q\boldsymbol{\nabla}\frac{\boldsymbol{p} - q\boldsymbol{A}}{m}\boldsymbol{A} - q\boldsymbol{\nabla}\varphi && (1.37)\\
&= q\boldsymbol{\nabla}(\dot{\boldsymbol{x}}\boldsymbol{A}) - q\boldsymbol{\nabla}\varphi && (1.38)\\
\ddot{\boldsymbol{x}} &= \frac{1}{m}\dot{\boldsymbol{p}} - \frac{q}{m}\frac{d\boldsymbol{A}}{dt} && (1.39)
\end{aligned}$$

where we use $\dfrac{d\boldsymbol{A}}{dt} = \partial_0 A + (\dot{\boldsymbol{x}} \cdot \boldsymbol{\nabla})\boldsymbol{A}$

$$\begin{aligned}
&= \frac{q}{m}[\boldsymbol{\nabla}(\dot{\boldsymbol{x}}\boldsymbol{A} + \varphi)] - \frac{q}{m}\partial_0\boldsymbol{A} - \frac{q}{m}(\dot{\boldsymbol{x}} \cdot \boldsymbol{\nabla})\boldsymbol{A} && (1.40)\\
\boldsymbol{E} &= -\boldsymbol{\nabla}\varphi - \partial_0\boldsymbol{A} && (1.41)\\
\dot{\boldsymbol{x}} \times \boldsymbol{B} &= \boldsymbol{\nabla}(\dot{\boldsymbol{x}}\boldsymbol{A}) - (\dot{\boldsymbol{x}} \cdot \boldsymbol{\nabla})\boldsymbol{A} && (1.42)\\
m\ddot{\boldsymbol{x}} &= q(\boldsymbol{E} + \dot{\boldsymbol{x}} \times \boldsymbol{B}) && (1.43)\\
&= \boldsymbol{F_L}. && (1.44)
\end{aligned}$$

Chapter 2

Fundamental equations of quantum mechanics

2.1 Schrödinger equation

2.1.1 Lagrangian, Hamiltonian and the Euler-Lagrange equation

We are now moving from electrodynamics to quantum mechanic. The first difference is that we don't use any more momentum p, but operator $\hat{\boldsymbol{p}} = -i\boldsymbol{\nabla}$[1]. The Lagrangian density for the Schrödinger equation for a free particle can be derived in form [3]

$$\begin{aligned} \mathcal{L}_{\mathrm{Sch}} &= \psi^{\star}\Big(i\partial_0 - \hat{H}\Big)\psi & (2.1)\\ &= \psi^{\star}\Big(i\partial_0 - \frac{1}{2m}\Delta\Big)\psi & (2.2) \end{aligned}$$

where we used $\hat{H} = -\frac{1}{2m}\hat{p}^2$. One can also include potential energy V, but since we are dealing with a free particle, we consider it as zero.
The Euler-Lagrange equation is derived as

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\psi} &= 0 & (2.3)\\ \frac{\partial\mathcal{L}}{\partial\partial_0\psi} &= i\psi^{\star} & (2.4)\\ \frac{\partial\mathcal{L}}{\partial\psi} - \partial_0\frac{\partial\mathcal{L}}{\partial\partial_0\psi} &= 0 & (2.5)\\ &= -i\partial_0\psi^{\star}. & (2.6) \end{aligned}$$

[1] $\hat{\boldsymbol{p}} = -ih\boldsymbol{\nabla}$ in SI

Following section 3.1 in [4] one sees that from here it is possible to obtain the Schrödinger equation

$$i\partial_0\psi = \frac{1}{2m}\Delta\psi. \tag{2.7}$$

The Hamiltonian is then

$$\begin{aligned}
\mathcal{H} &= \frac{\partial\mathcal{L}}{\partial\partial_0\psi}\partial_0\psi - \mathcal{L} && (2.8)\\
&= i\psi^\star\partial_0\psi - \psi^\star i\partial_0\psi + \psi^\star\frac{1}{2m}\Delta\psi && (2.9)\\
&= \psi^\star\frac{1}{2m}\Delta\psi && (2.10)
\end{aligned}$$

2.1.2 Covariant derivative for particle in the electromagnetic field

Knowing the Hamiltonian (1.27) and using the canonical momentum operator $\hat{\boldsymbol{p}}$ we can write the Hamiltonian operator as

$$\hat{H} = \frac{1}{2m}\Big(-i\boldsymbol{\nabla} - q\boldsymbol{A}\Big)^2 + q\varphi. \tag{2.11}$$

Then it is possible to rewrite the Schrödinger equation as

$$\begin{aligned}
\hat{H}\psi &= i\partial_0\psi && (2.12)\\
\frac{1}{2m}\Big(-i\boldsymbol{\nabla} - q\boldsymbol{A}\Big)^2\psi &= i\partial_0\psi - q\varphi\psi && (2.13)\\
\frac{1}{2m}\Big[i\Big(-\boldsymbol{\nabla} + iq\boldsymbol{A}\Big)\Big]^2\psi &= i(\partial_0 + iq\varphi)\psi. && (2.14)
\end{aligned}$$

Denoting $A^\mu = (\varphi, \boldsymbol{A})$, where index μ represent both spatial or time component, it is possible to rewrite covariant derivative in form

$$\begin{aligned}
D_\mu &= \partial_\mu + iqA_\mu && (2.15)\\
&= \partial_0 + \boldsymbol{\nabla} + iq\varphi - iq\boldsymbol{A} && (2.16)\\
D^\mu &= \partial^\mu + iqA^\mu && (2.17)\\
&= \partial^0 - \boldsymbol{\nabla} + iq\varphi + iq\boldsymbol{A}. && (2.18)
\end{aligned}$$

If we gauge-calibrate potential, we rewrite covariant derivative

$$D'_\mu = \partial_\mu + iqA'_\mu. \tag{2.19}$$

One can easily prove that such change won't spoil mean values of coordinates and of the kinetic momentum

$$
\begin{aligned}
\langle x_j \rangle &= \langle \psi | \hat{x}_j | \psi \rangle && (2.20)\\
&= \int_{-\infty}^{\infty} \bar{\psi} x_j \psi \, dx_j && (2.21)\\
\langle x'_j \rangle &= \langle \psi' | \hat{x}'_j | \psi' \rangle && (2.22)\\
&= \int_{-\infty}^{\infty} \bar{\psi} e^{-i\chi} x_j e^{i\chi} \psi \, dx_j && (2.23)\\
&= \int_{-\infty}^{\infty} \bar{\psi} x_j \psi \, dx_j && (2.24)\\
\langle x_j \rangle &= \langle x'_j \rangle && (2.25)\\
\langle \pi_j \rangle &= \langle \psi | \hat{p}_j - qA_j | \psi \rangle && (2.26)\\
&= \int_{-\infty}^{\infty} \bar{\psi} (-i\partial_j - qA_j) \psi \, dx_j && (2.27)\\
&= -\int_{-\infty}^{\infty} (i\bar{\psi}\partial_j \psi + qA_j \bar{\psi}\psi) \, dx_j && (2.28)\\
\langle \pi'_j \rangle &= \langle \psi' | \hat{p}'_j - qA'_j | \psi' \rangle && (2.29)\\
&= -\int_{-\infty}^{\infty} \bar{\psi} e^{-i\chi} (i\partial_j + qA_j + \partial_j \chi) e^{i\chi} \psi \, dx_j && (2.30)\\
&= -\int_{-\infty}^{\infty} (i\bar{\psi}\partial_j \psi - \bar{\psi}\partial_j \chi \psi + qA_j \bar{\psi}\psi + \bar{\psi}\partial_j \chi \psi) \, dx_j && (2.31)\\
&= -\int_{-\infty}^{\infty} (i\bar{\psi}\partial_j \psi + qA_j \bar{\psi}\psi) \, dx_j && (2.32)\\
\langle \pi_j \rangle &= \langle \pi'_j \rangle . && (2.33)
\end{aligned}
$$

The Schrödinger equation with covariant derivatives is then

$$
-\frac{1}{2m} (\boldsymbol{D})^2 \psi = i D_0 \psi \qquad (2.34)
$$

One may verify that such Schrödinger equation will not hold the form when we gauge re-calibrate electromagnetic potentials.

$$
\begin{aligned}
i D'_0 \psi &= -\frac{1}{2m} (\boldsymbol{D'})^2 \psi && (2.35)\\
i(\partial_0 + iq\varphi')\psi &= -\frac{i}{2m} \Big(\Delta - iq \boldsymbol{\nabla} \cdot \boldsymbol{A'} - 2iq \boldsymbol{A'} \cdot \boldsymbol{\nabla} - q^2 \boldsymbol{A'}^2 \Big) \psi . && (2.36)
\end{aligned}
$$

This can be fixed by introducing

$$
\psi' = \psi e^{iq\chi}, \qquad (2.37)
$$

where $\chi = \chi(t, \boldsymbol{x})$ is an arbitrary continuous function. That means that if we change potentials

$$A_\mu \rightarrow A'_\mu \tag{2.38}$$
$$A'_\mu = A_\mu + \partial_\mu \chi \tag{2.39}$$

the wave function locally changes its phase by factor $e^{iq\chi}$,
There is another way, how to introduce calibration and transformation of ψ

$$\psi' = \psi e^{i\chi} \tag{2.40}$$
$$A'_\mu = A_\mu + \frac{1}{q}\partial_\mu \chi. \tag{2.41}$$

Disadvantage of this concept is that here is charge q in denominator and so it cannot be zero. Such attitude will be discussed in chapter 6.
Schrödinger equation with gauge-calibrated potential is then

$$-\frac{1}{2m}\boldsymbol{D}'^2\psi' = iD'_0\psi'. \tag{2.42}$$

It can be easily proved that

$$-\boldsymbol{D'}^2\psi' = -e^{iq\chi}\boldsymbol{D}^2\psi \tag{2.43}$$
$$= -\boldsymbol{D}^2\psi e^{iq\chi} \tag{2.44}$$

and

$$D'_0\psi' = e^{iq\chi}D_0\psi \tag{2.45}$$
$$= (D_0\psi)e^{iq\chi}. \tag{2.46}$$

Now it's easy to see that

$$iD'_0\psi' = iD_0\psi \tag{2.47}$$
$$-\frac{1}{2m}\boldsymbol{D}'^2\psi' = -\frac{1}{2m}\boldsymbol{D}^2\psi \tag{2.48}$$

which is the same equation as (2.34). When dealing with the electromagnetic field by using D'_j and D'_0, the Schrödinger equation will stay the same as long as we substitute ψ by ψ', knowing that ψ' give us only phase, yet local, change, and probability density is still the same: $|\psi|^2 = |\psi'|^2$. This will be shown later in section 2.1.4.
There is another way how to introduce covariant derivative D_μ (2.15)

$$D_\mu = \partial_\mu - iqA_\mu, \tag{2.49}$$

but now the function ψ (2.37) must be transformed as

$$\psi' = \psi e^{-iq\chi}. \tag{2.50}$$

Under this calibration the Schrödinger equation will stays invariant.

2.1.3 Solution to the three-dimensional Schrödinger equation

Three-dimensional Schrödinger equation is

$$-\frac{1}{2m}\Delta\psi \;=\; i\partial_0\psi. \tag{2.51}$$

We are looking for solution in form

$$\psi \;\sim\; e^{\boldsymbol{\lambda}\cdot\boldsymbol{x}}e^{-iEt}. \tag{2.52}$$

We put (2.52) into (2.51) and solve it

$$-\frac{e^{-iEt}}{2m}\Delta e^{\boldsymbol{\lambda}\cdot\boldsymbol{x}} \;=\; ie^{\boldsymbol{\lambda}\cdot\boldsymbol{x}}\partial_0 e^{-iEt} \tag{2.53}$$

$$-\frac{e^{-iEt}}{2m}\boldsymbol{\lambda}\cdot\boldsymbol{\lambda}e^{\boldsymbol{\lambda}\cdot\boldsymbol{x}} \;=\; -Ee^{-iEt}e^{\boldsymbol{\lambda}\cdot\boldsymbol{x}} \tag{2.54}$$

$$-\frac{\boldsymbol{\lambda}\cdot\boldsymbol{\lambda}}{2m} \;=\; E \tag{2.55}$$

$$\boldsymbol{\lambda}\cdot\boldsymbol{\lambda} \;=\; -2Em \tag{2.56}$$

$$\boldsymbol{\lambda} \;=\; \pm i\boldsymbol{p} \tag{2.57}$$

$$\|\boldsymbol{\lambda}\| \;=\; \pm i\sqrt{2Em}. \tag{2.58}$$

Only positive energy are needed to solve such equation and so the solution to the three-dimensional Schrödinger equation is

$$\psi_{\boldsymbol{p}} \;=\; K\Big(e^{i\boldsymbol{p}\cdot\boldsymbol{x}}e^{-iEt} + e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}e^{-iEt}\Big), \tag{2.59}$$

where $\boldsymbol{p} \in \mathbb{R}^3$ and $\int\limits_{-\infty}^{\infty} \psi^{\star}_{\boldsymbol{p'}}\psi_{\boldsymbol{p}}\, d^3x = \delta^3(\boldsymbol{p'} - \boldsymbol{p})$ so that $K = \dfrac{1}{(2\pi)^{\frac{3}{2}}}$.
Some exact solution of the Schrödinger equation can be found in [5]. Important solutions are for free particle, for particle in potential well (both finite and infinite) and the Schrödinger equation for the hydrogen atom.

2.1.4 Continuity equation

Let us introduce the probability density function $\rho = |\psi|^2$ that gives us information about what the probability to find particle in state $\psi(t, \boldsymbol{x})$ is. The integral over the whole space must be

$$\int\limits_{-\infty}^{\infty} \rho\, d^3x = 1 \tag{2.60}$$

so the particle *must* be somewhere!
The probability density function is thus

$$\rho = |\psi|^2 \quad = \quad \psi\psi^\star \tag{2.61}$$

and its time derivative is

$$\partial_0\rho \quad = \quad \psi^\star\partial_0\psi + \psi\partial_0\psi^\star. \tag{2.62}$$

The continuity equation can be written in form of

$$\boldsymbol{\nabla}\cdot\boldsymbol{J} + \partial_0\rho \quad = \quad 0. \tag{2.63}$$

It represents a local conservation law. In this case the probability that we find particle at some place in every moment of time is conserved. It is found by following instruction:

- Schrödinger equation is multiplied by $\psi^\star$ form left, where * means complex conjugation.
- Complex conjugated Schrödinger equation is multiplied by ψ from left.
- Those two equations are subtracted and continuity equation is what we are left with.

We will follow this rule every time while dealing with equation of motion to obtain a continuity equation.
It is good to realize that

$$\partial_\mu J^\mu \quad \equiv \quad \partial_0 J^0 + \boldsymbol{\nabla}\cdot\boldsymbol{J}. \tag{2.64}$$

In the ongoing process, we presume real functions

$$A \quad = \quad A^\star \tag{2.65}$$
$$\varphi \quad = \quad \varphi^\star \tag{2.66}$$
$$\chi \quad = \quad \chi^\star. \tag{2.67}$$

Continuity equation for the Schrödinger equation without the electromagnetic field

Schrödinger equation without the electromagnetic field can be generally written as

$$\Big(-\frac{1}{2m}\Delta + V\Big)\psi = i\partial_0\psi \tag{2.68}$$
$$\tag{2.69}$$

where $V = V(\boldsymbol{x}, t)$ is potential energy. According to the instructions above we write the Schrödinger equation and multiply it by $\psi^\star$:

$$-\frac{1}{2m}\psi^\star\Delta\psi + \psi^\star V\psi \;=\; i\psi^\star\partial_0\psi \tag{2.70}$$

and complex conjugated equation by ψ:

$$-\frac{1}{2m}\psi\Delta\psi^\star + \psi V\psi^\star \;=\; -i\psi\partial_0\psi^\star. \tag{2.71}$$

Now we subtract equation (2.71) from (2.70) and obtain:

$$\begin{aligned}
-\frac{1}{2m}\psi^\star\Delta\psi + \frac{1}{2m}\psi\Delta\psi^\star \;&=\; i\psi^\star\partial_0\psi + i\psi\partial_0\psi^\star & (2.72)\\
-\frac{1}{2m}[\psi^\star\Delta\psi - \psi\Delta\psi^\star] \;&=\; i\partial_0(\psi\psi^\star). & (2.73)
\end{aligned}$$

Right-hand side of last equation may be according to (2.61) denoted as $i\partial_0\rho$, while the left-hand side can be expressed as

$$\begin{aligned}
-\frac{1}{2m}[\psi^\star\Delta\psi - \psi\Delta\psi^\star] \;&=\; i\partial_0\rho & (2.74)\\
-\frac{1}{2m}\boldsymbol{\nabla}\cdot[\psi^\star\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^\star] \;&=\; i\partial_0\rho & (2.75)\\
\frac{i}{2m}\boldsymbol{\nabla}\cdot[\psi^\star\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^\star] \;&=\; \partial_0\rho & (2.76)
\end{aligned}$$

Therefore we see that density flux is

$$\begin{aligned}
\boldsymbol{J} \;&\propto\; \psi^\star\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^\star & (2.77)\\
\boldsymbol{J} \;&=\; \frac{i}{2m}(\psi^\star\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^\star) & (2.78)
\end{aligned}$$

and probability density ρ is

$$\rho \;=\; \psi\psi^\star = |\psi|^2. \tag{2.79}$$

Continuity equation for the Schrödinger equation for a particle in an electromagnetic field

We expand the equation (2.34)

$$\begin{aligned}
i(\partial_0 + ig\varphi)\psi \;=\; &-\frac{1}{2m}[\Delta - iq\boldsymbol{\nabla}\cdot\boldsymbol{A} - 2iq\boldsymbol{A}\cdot\boldsymbol{\nabla} - \\
&-q^2\boldsymbol{A}^2]\psi + V\psi & (2.80)
\end{aligned}$$

which we multiply by $\psi^\star$

$$
\begin{aligned}
i\psi^\star(\partial_0 + iq\varphi)\psi &= -\psi^\star\frac{1}{2m}[\Delta - iq\boldsymbol{\nabla}\cdot\boldsymbol{A} - 2iq\boldsymbol{A}\cdot\boldsymbol{\nabla} - \\
&\quad -q^2\boldsymbol{A}^2]\psi + \psi^\star V\psi. \qquad (2.81)
\end{aligned}
$$

Then we complex conjugate the same equation (**??**) and multiply it by ψ

$$
\begin{aligned}
-i\psi(\partial_0 - iq\varphi^\star)\psi^\star &= -\psi\frac{1}{2m}[\Delta + iq\boldsymbol{\nabla}\cdot\boldsymbol{A}^\star + 2iq\boldsymbol{A}^\star\cdot\boldsymbol{\nabla} - \\
&\quad -q^2\boldsymbol{A}^{\star 2}]\psi^\star + \psi V\psi^\star. \qquad (2.82)
\end{aligned}
$$

We subtract these two equations and get the continuity equation

$$
\begin{aligned}
i\partial_0(\psi\psi^\star) &= -\frac{1}{2m}[\psi^\star\Delta\psi - \psi\Delta\psi^\star - 2iq\boldsymbol{\nabla}\cdot(\boldsymbol{A}\psi\psi^\star)] \qquad (2.83)\\
i\partial_0\rho &= -\frac{1}{2m}[\psi^\star\Delta\psi - \psi\Delta\psi^\star - 2iq\boldsymbol{\nabla}\cdot(\boldsymbol{A}\rho)] \qquad (2.84)\\
\partial_0\rho &= \frac{i}{2m}[\psi^\star\Delta\psi - \psi\Delta\psi^\star - 2iq\boldsymbol{\nabla}\cdot(\boldsymbol{A}\rho)] \qquad (2.85)
\end{aligned}
$$

which can be also written in the form:

$$
\partial_0\rho = \frac{i}{2m}\boldsymbol{\nabla}\cdot[\psi^\star\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^\star - 2iq(\boldsymbol{A}\psi\psi^\star)] \qquad (2.86)
$$

and thus we see that probability density is again

$$
\rho = \psi\psi^\star \qquad (2.87)
$$

while the flux density is

$$
\begin{aligned}
\boldsymbol{J} &\propto \psi^\star\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^\star - 2iq\boldsymbol{A}\rho \qquad (2.88)\\
\boldsymbol{J} &= \frac{i}{2m}(\psi^\star\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^\star - 2iq\boldsymbol{A}\rho). \qquad (2.89)
\end{aligned}
$$

One can see that flux density is now modified by $-2iq\boldsymbol{A}\rho$. Such term is here because the canonical momentum $\boldsymbol{p}$ is not equal to kinetic momentum $\boldsymbol{\pi}$.

Continuity equation for the Schrödinger equation for a particle in an electromagnetic field with a gauge-calibrated potential

Since equations (2.34) and (2.42) are formally the same (according to (2.43) and (2.45)), the continuity equation is also the same, which leads to interesting observation that $\boldsymbol{J}$ stay the same as well for both, without gauge-calibrated potentials for wave function ψ' and also for ordinary ψ without gauge-calibrated potentials. The same is for ρ and ρ', so we can write

$$\boldsymbol{J}' = \boldsymbol{J} \tag{2.90}$$
$$\rho' = \rho \tag{2.91}$$
$$J'^{\mu} = J^{\mu}. \tag{2.92}$$

If we take the Schrödinger equation with gauge-calibrated potentials but only with wave function ψ instead of ψ', the flux density will have form

$$\boldsymbol{J}' \propto \psi^{\star}\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^{\star} - 2iq\boldsymbol{A}\rho \tag{2.93}$$
$$\boldsymbol{J}' = \frac{i}{2m}(\psi^{\star}\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^{\star} - 2iq\boldsymbol{A}\rho). \tag{2.94}$$

Schrödinger equation is great starting point for describing particles in electromagnetic field while using quantum mechanics, but it fails as soon as we try to add relativity. In relativity we have one covariant derivative D^{μ} that contains both spatial and time derivative, but Schrödinger equation separates them and even more-spatial derivative is here in second order but time derivative is only first order. To fix this, the idea of Klein-Gordon equation is needed.

2.2 Klein-Gordon equation

The real Klein-Gordon equation describes a neutral spin zero scalar particle (a Higgs boson, for example) and the complex Klein-Gordon equation describe charged spin zero particle (such particle hasn't been detected yet).

2.2.1 Lagrangian, Hamiltonian and Euler-Lagrange equation

Having Lagrangian density for Klein-Gordon equation in form of [6]

$$\mathcal{L} = \frac{1}{2}\partial_{\mu}\psi\partial^{\mu}\psi - \frac{1}{2}m^2\psi^2 \tag{2.95}$$

The corresponding Hamiltonian can be written as

$$\mathcal{H} \equiv \partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\psi)} - L \tag{2.96}$$
$$\mathcal{H} = \frac{1}{2}\partial_{\mu}\psi\partial^{\mu}\psi + \frac{1}{2}m^2\psi^2. \tag{2.97}$$

The kinetic momentum is

$$\begin{aligned} \boldsymbol{\pi} &= \frac{\partial \mathcal{L}}{\partial \partial_0 \psi} && (2.98) \\ &= \partial_0 \psi. && (2.99) \end{aligned}$$

We must take into account that $\psi = \psi(x)$ and thus we are dealing with infinite amount of Euler-Lagrange equations while having infinite number of degrees of freedom. Deriving the Euler-Lagrange equation one obtains the Klein-Gordon equation

$$\begin{aligned} \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} - \frac{\partial \mathcal{L}}{\partial \psi} &= 0 && (2.100) \\ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} &= \partial^\mu \psi && (2.101) \\ \frac{\partial \mathcal{L}}{\partial \psi} &= -\psi m^2 && (2.102) \\ (\partial_\mu \partial^\mu + m^2)\psi &= 0 && (2.103) \end{aligned}$$

or shortened as

$$(\Box + m^2)\psi = 0. \qquad (2.104)$$

Defining the D'Alembertian we obtain an operator that doesn't distinguish between space and time in order of derivative. This is the main difference between Schrödinger (non-relativistic) and Klein-Gordon equation.
One would like to see what happens to the Lagrangian (2.95) if there is interaction between two (complex) scalar fields

$$\begin{aligned} \psi &= \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2) && (2.105) \\ \psi^\dagger &= \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2), && (2.106) \end{aligned}$$

where ψ_1, ψ_2 are real components. The Lagrangian will now consist of the kinetic term and the mass term, as it was so far, but there will be an interacting term that describes interaction between fields ψ and $\psi^\dagger$

$$\begin{aligned} \mathcal{L} &= \partial^\mu \psi^\dagger \partial_\mu \psi - m^2 \psi^\dagger \psi - g(\psi^\dagger \psi)^2, && (2.107) \\ &= \partial^\mu \psi^\dagger \partial_\mu \psi - m^2 \psi^\dagger \psi - \frac{g}{4}(\psi_1^2 + \psi_2^2)^2 && (2.108) \end{aligned}$$

where g corresponds to the strength of the interaction [6].

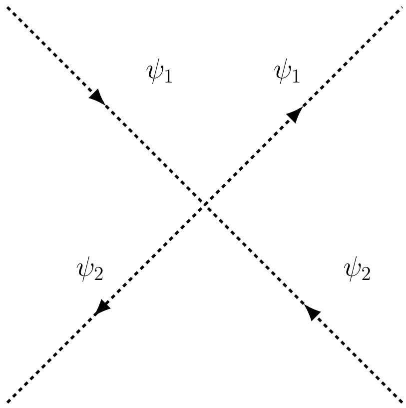


Figure 2.1: Interaction term in 2.107, drawn in [7]

2.2.2 Klein-Gordon equation with the electromagnetic field

In order to introduce the minimal electromagnetic interaction we substitute $\partial^\mu \;\to\; D^\mu = \partial^\mu + iqA^\mu$ in (2.103).
The Klein-Gordon equation with the electromagnetic field with can by then rewrite as

$$\begin{aligned}
0 &= (D_\mu D^\mu + m^2)\psi && (2.109)\\
0 &= (\Box + iq(\partial_0\varphi + \boldsymbol{\nabla}\cdot\boldsymbol{A} + 2\varphi\partial_0 + 2\boldsymbol{A}\cdot\boldsymbol{\nabla}) - \\
&\quad -q^2(\varphi^2 - \boldsymbol{A}^2) + m^2)\psi && (2.110)\\
0 &= (\Box - q^2 A^\mu A_\mu + iq\partial^\mu A_\mu + 2iqA_\mu\partial^\mu + m^2)\psi. && (2.111)
\end{aligned}$$

and the Klein-Gordon equation with the electromagnetic field with a gauge-calibrated potential as

$$\begin{aligned}
0 &= (D_0'^2 - \boldsymbol{D'}\cdot\boldsymbol{D'} + m^2)\psi' && (2.112)\\
0 &= [\Box + iq(\partial_0\varphi' + \boldsymbol{\nabla}\cdot\boldsymbol{A'} + 2\varphi'\partial_0 + 2\boldsymbol{A'}\cdot\boldsymbol{\nabla}) - \\
&\quad -q^2(\varphi'^2 - \boldsymbol{A'}^2) + m^2]\psi' && (2.113)\\
0 &= (\Box - q^2 A'^\mu A'_\mu + iq\partial^\mu A'_\mu + 2iqA'_\mu\partial^\mu + m^2)\psi' && (2.114)
\end{aligned}$$

It can be proven that this form of Klein-Gordon equation may be obtained if we change wave function $\psi \to \psi' = \psi e^{iq\chi}$, $\chi = \chi(s)$ and simultaneously $D^\mu \to D'^\mu$,$A^\mu \to A'^\mu$ in (2.109).

2.2.3 Solution to the Klein-Gordon equation

The Klein-Gordon equation for a free particle is

$$(\partial_0^2 - \Delta + m^2)\psi = 0. \tag{2.115}$$

We expect solution in form

$$\psi \sim e^{\boldsymbol{\lambda}\cdot\boldsymbol{x}} e^{i\omega t}. \tag{2.116}$$

We put (2.116) into (2.115) and solve it to see what $\boldsymbol{\lambda}$ is, with respect to relativistic property that energy ω^2 is always bigger that mass m^2

$$\begin{aligned}
-\omega^2 e^{\boldsymbol{\lambda}\cdot\boldsymbol{x}} e^{i\omega t} - \boldsymbol{\lambda}\cdot\boldsymbol{\lambda} e^{\boldsymbol{\lambda}\cdot\boldsymbol{x}} e^{i\omega t} + m^2 e^{\boldsymbol{\lambda}\cdot\boldsymbol{x}} e^{i\omega t} &= 0 & (2.117)\\
\boldsymbol{\lambda}\cdot\boldsymbol{\lambda} &= m^2 - \omega^2 & (2.118)\\
&= -(\omega^2 - m^2) & (2.119)\\
&= -\boldsymbol{p}\cdot\boldsymbol{p} & (2.120)\\
\boldsymbol{\lambda} &= \pm\boldsymbol{p}, & (2.121)
\end{aligned}$$

$\boldsymbol{p} \in \mathbb{R}^3$. For energy ω we obtain

$$\begin{aligned}
\omega^2 &= -\boldsymbol{\lambda}\cdot\boldsymbol{\lambda} + m^2 & (2.122)\\
&= \boldsymbol{p}\cdot\boldsymbol{p} + m^2 & (2.123)\\
E \equiv \omega = \pm\sqrt{\boldsymbol{p}\cdot\boldsymbol{p} + m^2}. & & (2.124)
\end{aligned}$$

The solution of the three-dimensional Klein-Gordon equation is

$$\psi = C_1 e^{i(\omega t + \boldsymbol{p}\cdot\boldsymbol{x})} + C_2 e^{i(\omega t - \boldsymbol{p}\cdot\boldsymbol{x})}, \tag{2.125}$$

where C_1 and $C_2 \in \mathbb{R}$ that satisfy $\int\limits_{-\infty}^{\infty} \psi^\star_{\boldsymbol{p'}}\psi_{\boldsymbol{p}}\, d^3x = \delta^3(\boldsymbol{p'} - \boldsymbol{p})$.

Introducing four-vector $P^\mu_\pm = (E, \pm\boldsymbol{p})$ and knowing $x^\mu = (t, \boldsymbol{x})$, we can rewrite solution of the three-dimensional Klein-Gordon equation as

$$\begin{aligned}
\psi_\pm(x) &= C e^{\pm i P^\mu_\pm x_\mu} & (2.126)\\
\psi_{\boldsymbol{p}}(x) &= C e^{\pm i P_\mu x^\mu}. & (2.127)
\end{aligned}$$

According to this, if ψ is interpreted as single particle wave function, there is a possibility to have negative energy, which may be problem when particles start to interact. Its energy may fall to minus infinity while emitting infinite amount of energy. That is clearly problem, because nothing like that happens in nature. The problem may by solve by interpreting ψ not as single particle, but as quantum field, which solve the problem with non-positive-definite ρ as well [8]. R. Feynman and E. Stuckelberg, independently, considered negative energy states as particle with opposite charge moving backward in time while having negative momentum for consistency (CPT symmetry). Such particles are call *antiparticles* and in Feynman diagrams are noted as particles going backward in time. According to this the solution may be interpeted as incoming particle and outgouing antiparticle with phase factor $\pm i(Et - px)$ [6].

If we denote linear operator $\hat{K} = \Box + m^2 = \partial_\mu \partial^\mu + m^2$, it is possible to rewrite and verify the free-particle solution to the Klein-Gordon equation

$$\hat{K}\psi_{\boldsymbol{p}}(x) = (\partial_\mu \partial^\mu + m^2)e^{\pm i P_\mu x^\mu} = 0. \tag{2.128}$$

2.2.4 Continuity equation

Continuity equation for Klein-Gordon equation without the electromagnetic field

We deal with continuity equation the same way as we did for the Schrödinger equation, so we take the Klein-Gordon equation without the electromagnetic field and multiply it by $\psi^\star$

$$\psi^\star \Box \psi + m^2 \psi^\star \psi = 0 \tag{2.129}$$

$$\psi^\star \partial_0^2 \psi - \psi^\star \Delta \psi + m^2 \psi^\star \psi = 0, \tag{2.130}$$

then we take complex-conjugated equation and multiply is by ψ

$$(\Box \psi^\star)\psi + m^2 = 0 \tag{2.131}$$

$$(\partial_0^2 \psi^\star)\psi - (\Delta \psi^\star)\psi + m^2 \psi^\star \psi = 0 \tag{2.132}$$

and finally we subtract these two equation

$$\psi^\star \partial_0^2 \psi - \psi \partial_0^2 \psi^\star + \psi \Delta \psi^\star - \psi^\star \Delta \psi = 0 \tag{2.133}$$

$$\partial_0(\psi^\star \partial_0 \psi - \psi \partial_0 \psi^\star) - \boldsymbol{\nabla} \cdot (\psi^\star \boldsymbol{\nabla} \psi - \psi \boldsymbol{\nabla} \psi^\star) = 0. \tag{2.134}$$

One can see some similarities with flux density for Schrödinger equation without electromagnetic field (2.77), but density probability ρ has now a different form

$$\boldsymbol{J} = -\Big[\psi^\star \boldsymbol{\nabla} \psi - \psi \boldsymbol{\nabla} \psi^\star\Big] \tag{2.135}$$

$$= \psi \boldsymbol{\nabla} \psi^\star - \psi^\star \boldsymbol{\nabla} \psi \tag{2.136}$$

$$\rho = \rho_s - \psi \partial_0 \psi^\star \tag{2.137}$$

$$= \psi^\star \partial_0 \psi - \psi \partial_0 \psi^\star. \tag{2.138}$$

It may be shown [6] that ρ now depends on value of energy and since it may be both positive and negative, there comes a problem. One can no longer consider ψ as a single particle wave function, but it is now scalar field, as mentioned above in 2.2.3.
Denoting $J^\mu = (\rho, \boldsymbol{J})$ we can rewrite flux density as four-vector

$$J^\mu = \psi^\star \partial^\mu \psi - \psi^\star \partial^\mu \tag{2.139}$$

and its four-divergence as

$$\begin{aligned} \partial_\mu J^\mu &= \psi^\star \Box \psi - \psi \Box \psi^\star && (2.140)\\ &= 0. && (2.141) \end{aligned}$$

Continuity equation for Klein-Gordon equation with the electromagnetic field

The Klein-Gordon equation with the electromagnetic field multiplied by $\psi^\star$ is

$$\begin{aligned} 0 &= \psi^\star[\Box + iq(\partial_0\varphi + 2\varphi\partial_0 + \boldsymbol{\nabla}\cdot\boldsymbol{A} + 2\boldsymbol{A}\cdot\boldsymbol{\nabla}) - \\ &\quad -q^2(\varphi^2 - \boldsymbol{A}^2) + m^2]\psi && (2.142) \end{aligned}$$

and complex-conjugated Klein-Gordon equation with the electromagnetic field multiplied by ψ is

$$\begin{aligned} 0 &= \psi[\Box - iq(\partial_0\varphi^\star + 2\varphi^\star\partial_0 + \boldsymbol{\nabla}\cdot\boldsymbol{A}^\star + 2\boldsymbol{A}^\star\cdot\boldsymbol{\nabla}) - \\ &\quad -q^2(\varphi^{\star 2} - \boldsymbol{A}^{\star 2}) + m^2]\psi^\star. && (2.143) \end{aligned}$$

Then we subtract these two equations. For shortening length of equation we now denote $\psi\psi^\star = \rho_s$ according to density probability of continuity equation for the Schrödinger equation (2.79).

$$\begin{aligned} 0 &= \psi^\star\Box\psi - \psi\Box\psi^\star + 2iq[\partial_0(\varphi\rho_s) + \boldsymbol{\nabla}\cdot(\boldsymbol{A}\rho_s)] && (2.144)\\ 0 &= \psi\partial_0^2\psi^\star - \psi^\star\partial_0^2\psi + \psi^\star\Delta\psi - \psi\Delta\psi^\star - \\ &\quad -2iq[\partial_0(\varphi\rho_s) + \boldsymbol{\nabla}\cdot(\boldsymbol{A}\rho_s)] && (2.145)\\ 0 &= \partial_0(\psi\partial_0\psi^\star - \psi^\star\partial_0\psi - 2iq\varphi\rho_s) + \\ &\quad \boldsymbol{\nabla}\cdot\left[\psi^\star\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^\star + 2iq\boldsymbol{A}\rho_s\right] && (2.146)\\ 0 &= \partial_0(\psi\partial_0\psi^\star - \psi^\star\partial_0\psi - 2iq\varphi\rho_s) - \\ &\quad -\boldsymbol{\nabla}\cdot\left[\psi\boldsymbol{\nabla}\psi^\star - \psi^\star\boldsymbol{\nabla}\psi - 2iq\boldsymbol{A}\rho_s\right] && (2.147)\\ 0 &= \partial_0\rho + \boldsymbol{\nabla}\cdot\boldsymbol{J} && (2.148) \end{aligned}$$

We deal with electromagnetic field, so we have this $-2iq\boldsymbol{A}\psi\psi^\star$ part in flux density. This member also appears in density probability $-2iq\varphi\psi\psi^\star$.

$$\begin{aligned}
\boldsymbol{J} &= -\Big[\psi\boldsymbol{\nabla}\psi^\star - \psi^\star\boldsymbol{\nabla}\psi - 2iq\boldsymbol{A}\psi\psi^\star\Big] && (2.149)\\
&= \psi^\star\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^\star + 2iq\boldsymbol{A}\psi\psi^\star && (2.150)\\
\rho &= \psi\partial_0\psi^\star - \psi^\star\partial_0\psi - 2iq\varphi\psi\psi^\star. && (2.151)
\end{aligned}$$

One can here rewrite flux density as four-vector as well

$$\begin{aligned}
J^\mu &= \psi^\star\partial^\mu\psi - \psi^\star\partial^\mu\psi - 2iq(A^\mu\psi^\star\psi) && (2.152)\\
\partial_\mu J^\mu &= 0. && (2.153)
\end{aligned}$$

Those two last equation are important for us to see that four-divergence is still zero, in other words, that four-vector of flux density is conserved.

Continuity equation for Klein-Gordon equation with the electromagnetic field with gauge-calibrated potential

One can use formulas (2.43) and (2.45) we derive before and rewrite the Klein-Gordon equation with gauge-calibrated potential in the same form as the one without calibrated potential, only multiplied by phase factor $e^{-iq\chi}$ and $e^{iq\chi}$ and those two cancel each other out. We are then left with equation same as in the section 2.2.4.
Flux density for the Klein-Gordon equation is thus the same with or without calibrated potential.

$$\begin{aligned}
\boldsymbol{J'} &= -\Big[\psi\boldsymbol{\nabla}\psi^\star - \psi^\star\boldsymbol{\nabla}\psi - 2iq\boldsymbol{A}\psi\psi^\star\Big] && (2.154)\\
&= \psi^\star\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^\star + 2iq(\boldsymbol{A}\rho) && (2.155)\\
\rho' &= \psi\partial_0\psi^\star - \psi^\star\partial_0\psi - 2iq(\varphi\rho) && (2.156)
\end{aligned}$$

Four-vector J'^μ now has the same form as the one without gauge-calibrated potential

$$\begin{aligned}
J'^\mu &= \psi^\star\partial^\mu\psi - \psi^\star\partial^\mu\psi - 2iq(A^\mu\psi^\star\psi) && (2.157)\\
\partial_\mu J'^\mu &= 0. && (2.158)
\end{aligned}$$

Advantage of the Klein-Gordon equation is that it has unified time and spatial derivative. Its problems are that such derivative is there in second order and so to solve it, one must know also the initial condition, and that it suggest negative energies for ψ interpreted as single-particle.

2.3 Pauli equation and matrices

We won't be dealing with the Pauli equation as widely as we did with Schrödinger and Klein-Gordon equations, the aim of this section is to introduce Pauli matrices and some of its properties. The Pauli equation describes a charged particle with spin $\frac{1}{2}$ and is non-relativistic approximation of the Dirac equation.
The Pauli equation is [6]

$$\left[\frac{1}{2m}[\boldsymbol{\sigma}\cdot\boldsymbol{p}]^2 + g\varphi\right]\psi \quad = \quad i\partial_0\psi, \tag{2.159}$$

where the wave function has two components $\psi = \begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix}$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are Pauli matrices. These are 2×2 complex unitary matrices

$$\sigma_1 \quad = \quad \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix} \tag{2.160}$$

$$\sigma_2 \quad = \quad \begin{pmatrix}0 & -i \\ i & 0\end{pmatrix} \tag{2.161}$$

$$\sigma_3 \quad = \quad \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}. \tag{2.162}$$

2.3.1 Properties of Pauli matrices

Unitarity and Hermitian conjugate

Hermitian conjugated matrix $\sigma_i^\dagger$ means $\sigma_i^\dagger \equiv (\sigma_i^\star)^T$. It is easy to see that

$$\sigma_i \cdot \sigma_i^\dagger \quad = \quad \mathbb{I}, \tag{2.163}$$

$$\sigma_i^2 \quad = \quad \mathbb{I}, \tag{2.164}$$

$$\sigma_i \quad = \quad \sigma_i^\dagger, \tag{2.165}$$

where $\mathbb{I}$ is the identity matrix

$$\mathbb{I} = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}. \tag{2.166}$$

It also applies that $|\det(\sigma_i)| = 1$ and that

$$\det(\sigma_i) \quad = \quad -1. \tag{2.167}$$

Those three matrices form a Lie algebra which generate a Lie group $SU(2)$. Let us note that $e^{i\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{n}} \in SU(2)$, where $\|\boldsymbol{n}\| = 1$ and $\frac{\boldsymbol{\sigma}}{2} = \boldsymbol{J}$. The $S_{\boldsymbol{n},\varphi} = e^{i\boldsymbol{J}\cdot\boldsymbol{n}\varphi}$ is generator of rotation.

Commutation and anti-commutation

Commutation relations are

$$\left[\sigma_i, \sigma_j\right] \equiv \sigma_i\sigma_j - \sigma_j\sigma_i = 2i\epsilon_{ijk}\sigma_k. \tag{2.168}$$

Anti-commutation relations are

$$\{\sigma_i, \sigma_j\} \equiv \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\mathbb{I}. \tag{2.169}$$

It is therefore possible to write

$$\begin{aligned} \sigma_i\sigma_j &= \frac{1}{2}[\sigma_i, \sigma_j] + \frac{1}{2}\{\sigma_i, \sigma_j\} && (2.170)\\ &= i\epsilon_{ijk}\sigma_k + \delta_{ij}\mathbb{I}. && (2.171) \end{aligned}$$

2.4 Dirac equation

It turned out that the Klein-Gordon is not a good equation when one wants to examine a single particle and that it has some crucial issues with negative energy and probability. The main issue lies in second-order time derivative and our aim thus is to obtain first order relativistic equation - The Dirac equation which serves well for spin-$\frac{1}{2}$ particles-both charged and neutral.

2.4.1 Building up the Hamiltonian for the Dirac equation

Assumptions

This section will be lead by chapter 3 in [9].
For the Hamiltonian we require

1. to respect the relativistic energy relation

$$E = \sqrt{\boldsymbol{p}\cdot\boldsymbol{p} + m^2}. \tag{2.172}$$

Once again we emphasize that we work with natural units. Otherwise, this equation would have form $E = \sqrt{\boldsymbol{p}\cdot\boldsymbol{p}c^2 + (mc^2)^2}$. That is to respect

$$H^2 = \hat{\boldsymbol{p}}^2 + m^2.^2 \tag{2.173}$$

2. to be linear in $\hat{\boldsymbol{p}} = -i\boldsymbol{\nabla}$. and fot the equation of motion to have a form

$$H\psi = i\partial_0\psi.^3 \tag{2.174}$$

We know that there is formula $(a+b)^2 = a^2 + b^2 + 2ab$, but let us presume that we can write our Hamiltonian as

$$H = \boldsymbol{\alpha}\hat{\boldsymbol{p}} + \beta m,^4 \tag{2.175}$$

to be linear in space derivative as $\hat{\boldsymbol{p}} \sim \boldsymbol{\nabla}$. In the equation above $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and β are $n \times n$ matrices and similarly to Pauli equation, ψ is a multicomponent wave function $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix}$.

The main points we need to find out are

1. what is n?

2. what are $\alpha_1, \alpha_2, \alpha_3$ and β?

We begin from

$$H^2 = (\alpha\hat{\boldsymbol{p}} + \beta m)^2 \tag{2.176}$$

and we want this to be equal

$$H^2 = \hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{p}} + m^2 \tag{2.177}$$

[3] $H\psi = i\hbar\partial_0\psi$ in SI units.
[3] $H^2 = \hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{p}}c^2 + m^2c^4$ in SI units.
[4] $H = \boldsymbol{\alpha}\hat{\boldsymbol{p}}c + \beta mc^2$ in SI units.

and to respect (2.172). We get

$$\begin{aligned}
\Big(\sum_j \alpha_j p_j &+ \beta m\Big)\Big(\sum_k \alpha_k p_k + \beta m\Big) = \\
&= \Big(-i\sum_j \alpha_j \partial_j + \beta m\Big)\Big(-i\sum_k \alpha_k \partial_k + \beta m\Big) && (2.178)\\
&= -\sum_{j,k} \alpha_j \alpha_k (\partial_j \partial_k) - i\sum_j \partial_j m \alpha_j \beta \\
&\quad -i\sum_k \partial_k m \beta \alpha_k + \beta^2 m^2 && (2.179)\\
&= -\sum_{jk} \alpha_j \alpha_k (\partial_j \partial_k) - \\
&\quad -im\sum_j \Big(\partial_j \alpha_j \beta + \partial_j \beta \alpha_j\Big) + \beta^2 m^2 && (2.180)
\end{aligned}$$

We compare corresponding terms of these equations.

The term containing $\sum_j \partial_j(\alpha_j\beta + \beta\alpha_j\Big)$

It is clearly seen that $-im\sum_j \partial_j\Big(\alpha_j\beta + \beta\alpha_j\Big)$ must be zero. This can by achieved if

$$\{\alpha_i, \beta\} = 0, \tag{2.181}$$

meaning that matrices α_i and β must anti-commute.

The term containing p^2

$$\begin{aligned}
\hat{p}^2\mathbb{I} &= -\sum_{j,k} \alpha_j \alpha_k (\partial_j \partial_k) && (2.182)\\
-\sum_j \partial_j \partial_j \mathbb{I} &= -\sum_{j,k} \alpha_j \alpha_k (\partial_j \partial_k) && (2.183)\\
&= \sum_{j,k}\Big(\frac{1}{2}[\alpha_j, \alpha_k] + \frac{1}{2}\{\alpha_j, \alpha_k\}\Big)\partial_j \partial_k && (2.184)
\end{aligned}$$

Knowing that $[\alpha_j, \alpha_k]$ is antisymmetric and $\partial_j \partial_k$ is symmetric makes the term with commutator equals zero (2.181). We are left with

$$= \sum_{j,k} \frac{1}{2}\{\alpha_j, \alpha_k\}\partial_j \partial_k \tag{2.185}$$

and from (2.189)

$$-\partial_j\partial_k\mathbb{I} = \sum_{j,k}\delta_{jk}\partial_j\partial_k\mathbb{I} \tag{2.186}$$

$$= \sum_{j,k}\delta_{jk}\partial_j^2\mathbb{I}. \tag{2.187}$$

This whole means that

$$\sum_{j,k}\delta_{jk}\partial_j^2\mathbb{I} = \sum_{j,k}\frac{1}{2}\{\alpha_j,\alpha_k\}\partial_j\partial_k \tag{2.188}$$

$$2\delta_{jk}\mathbb{I} = \{\alpha_j,\alpha_k\} \tag{2.189}$$

and so that

$$\alpha_j^2 = \frac{1}{2}\{\alpha_j,\alpha_j\} = \mathbb{I}. \tag{2.190}$$

The term containing m^2

We get prescription for β:

$$m^2\mathbb{I} = \beta^2 m^2 \tag{2.191}$$

$$\mathbb{I} = \beta^2 \tag{2.192}$$

From which it applies that the eigenvalues λ of the matrix β

$$\beta^2\boldsymbol{\lambda} = \lambda^2\boldsymbol{\lambda} \tag{2.193}$$

an that leads us to the eigenvalues of $\beta^2 = 1 = e^{i\delta}$, but we want $H = H^\dagger$, in other words, we want the Hamiltonian operator to have real eigenvalues and so $\beta = \beta^\dagger$ to also have real eigenvalues. It all means that $\beta = \pm 1$ and it must be in some basis a diagonal matrix with only ones and minus ones on its diagonal such that

$$\operatorname{Tr}\beta = \sum_i \lambda_i. \tag{2.194}$$

This applies from

$$\operatorname{Tr}\beta = \operatorname{Tr}(\beta\mathbb{I}) \tag{2.195}$$

$$= \operatorname{Tr}(\beta\alpha_i^2) \tag{2.196}$$

$$= \operatorname{Tr}(\alpha_i\beta\alpha_i) \tag{2.197}$$

$$= \operatorname{Tr}(-\beta\alpha_i\alpha_i) \tag{2.198}$$

$$\operatorname{Tr}(\alpha_i\beta\alpha_i) = -\operatorname{Tr}(\beta\alpha_i^2) \tag{2.199}$$

$$= -\operatorname{Tr}(\beta). \tag{2.200}$$

This can be achieved only by

$$\mathrm{Tr}(\beta) = \sum_i \lambda_i = 0. \tag{2.201}$$

This condition will be satisfied if there will be the same number of ones and minus ones on the diagonal of β. Since we have declared that β is a $n \times n$ matrix, we see that n must be even number. It will be shown later that the least possibility is $n = 4$.
Since we have (2.181), we can write

$$\begin{aligned}
\{\alpha_i, \beta\} &= \alpha_i\beta + \beta\alpha_i && (2.202)\\
\alpha_i\beta &= -\beta\alpha_i && (2.203)\\
\mathrm{Tr}\,\alpha_i &= \mathrm{Tr}(\mathbb{I}\alpha_i) && (2.204)\\
&= \mathrm{Tr}(\beta^2\alpha_i) && (2.205)\\
&= \mathrm{Tr}(\beta\alpha_i\beta) && (2.206)\\
&= \mathrm{Tr}(-\alpha_i\beta\beta) && (2.207)\\
\mathrm{Tr}(\beta\alpha_i\beta) &= -\mathrm{Tr}(\alpha_i\beta^2) && (2.208)\\
&= -\mathrm{Tr}(\alpha_i) && (2.209)
\end{aligned}$$

and this can be achieved only by

$$\mathrm{Tr}(\alpha_i) = \sum_i \alpha_i = 0. \tag{2.210}$$

There is a certain similarity between α_i and Pauli spin matrices. Let's try then to substitute $\alpha_i = \sigma_i$. Will it satisfy our Hamiltonian (2.176) and all following (2.181) and (2.189)? From (2.169) we see that the only condition that we must prove is $\{\sigma_i, \beta\} = 0$.
Let's write $\beta = C \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = C \cdot \sigma_3$. Then:

$$\{\sigma_i, \beta\} = \{\sigma_i, \sigma_3\} \tag{2.211}$$

which fails when $i = 3$

$$\begin{aligned}
\{\sigma_3, \beta\} &\neq \{\sigma_3, \sigma_3\} && (2.212)\\
\{\sigma_3, \beta\} &= 0 \text{ since } (2.181) && (2.213)\\
\{\sigma_3, \sigma_3\} &= 2\mathbb{I} \text{ since } (2.169) && (2.214)\\
0 &\neq 2\mathbb{I}. && (2.215)
\end{aligned}$$

Therefore we see that n cannot be 2 as pointed out before.

Solution Dirac has came with is to take $\beta = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$ and $\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$. We must understand 0 and $\mathbb{I}$ as 2×2 matrices.
It is now simple to see that

$$\{\alpha_j, \alpha_k\} = 2\delta_{jk}\mathbb{I}, \tag{2.216}$$

where $\mathbb{I}$ is the 4×4 unit matrix.

2.4.2 Covariant form of the Dirac equation

We begin from simple Dirac equation for a particles of mass m, now in SI units instead of natural

$$i\hbar\partial_0\psi = (-i\hbar c\boldsymbol{\nabla} \cdot \boldsymbol{\alpha} + \beta mc^2)\psi, \tag{2.217}$$

where we must not forget that $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix}$. We will multiply both sides of equation by $\frac{1}{\beta\hbar c}$ and get

$$\begin{aligned} i\beta\frac{\partial\psi}{\partial ct} &= (-i\boldsymbol{\nabla} \cdot \beta\boldsymbol{\alpha} + \frac{mc}{\hbar})\psi & (2.218) \\ i\Big(\beta\frac{\partial\psi}{\partial x^0} + \boldsymbol{\nabla} \cdot \beta\boldsymbol{\alpha}\psi\Big) - \kappa_c\psi &= 0, & (2.219) \end{aligned}$$

where we can denote $\kappa_c = \frac{mc}{\hbar} = \frac{1}{\lambda_C}$ as inverse Compton wavelength of the particles, $[\kappa_c] = \mathrm{m}^{-1}$. Let us define

$$\begin{aligned} \gamma^0 &\equiv \beta & (2.220) \\ \boldsymbol{\gamma} &\equiv \beta\boldsymbol{\alpha} & (2.221) \end{aligned}$$

or

$$\gamma^\mu \equiv (\gamma^0, \boldsymbol{\gamma}) = (\beta, \beta\boldsymbol{\alpha}). \tag{2.222}$$

To have the Hamiltonian $H = H^\dagger$ the gamma matrices must be

$$\begin{aligned} \gamma^0 &= \gamma^{0\dagger} & (2.223) \\ \gamma^{j\dagger} &= -\gamma^j & (2.224) \end{aligned}$$

and may be written in the Weyl (chiral) representation [6] or in the Dirac representation [10] (see Table 2.1).

Table 2.1: Representations of gamma matrices

Matrix	Chiral representation	Dirac representation
γ^0	$\begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$	$\begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$
γ^j	$\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}$
γ^5	$\begin{pmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix}$	$\begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$

The γ^5 is chirality operator (see section 4.2.1) defined as

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{2.225}$$

For gamma matrices applies

$$\begin{aligned}
\{\gamma^\mu, \gamma^\nu\} &= 2\mathbb{I}_{4\times4}g^{\mu\nu} && (2.226)\\
\partial_\mu\partial_\nu\gamma^\mu\gamma^\nu &= \partial_\mu\partial_\nu\Big(\frac{1}{2}[\gamma^\mu, \gamma^\nu] + \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\Big) && (2.227)\\
\partial_\mu\partial_\nu\big[\gamma^\mu, \gamma^\nu\big] &= 0 && (2.228)\\
\partial_\mu\partial_\nu\{\gamma^\mu, \gamma^\nu\} &= 2\mathbb{I}_{4\times4}g^{\mu\nu}\partial_\mu\partial_\nu. && (2.229)
\end{aligned}$$

Dirac equation has then the form

$$i\Big(\gamma^0\partial^0\psi + \boldsymbol{\nabla}\cdot\boldsymbol{\gamma}\psi\mathbb{I}\Big) - \kappa_c\psi = 0 \tag{2.230}$$

Denoting

$$\begin{aligned}
\partial^\mu &\equiv \frac{\partial}{\partial x^\mu} = (\partial_0, -\boldsymbol{\nabla}) && (2.231)\\
\partial_\mu &\equiv \frac{\partial}{\partial x_\mu} = (\partial_0, \boldsymbol{\nabla}) && (2.232)\\
\partial_\mu\gamma^\mu &= \partial_0\gamma^0 + \partial_j\gamma^j = \partial^0\gamma^0 - \partial^j\gamma^j = \partial^0\gamma^0 + \boldsymbol{\nabla}\cdot\boldsymbol{\gamma} && (2.233)
\end{aligned}$$

we get a simplified notation of the Dirac equation

$$(i\gamma^\mu\partial_\mu - \mathbb{I}\kappa_c)\psi = 0 \tag{2.234}$$

Returning back to natural units ($\hbar = c = 1$) we get

$$(i\gamma^\mu\partial_\mu - \mathbb{I}m)\psi = 0 \tag{2.235}$$

and by introducing the Feynman slash symbol

$$\gamma^\mu a_\mu \equiv \not{a} \tag{2.236}$$

we obtain the Dirac equation

$$(i\not{\partial} - \mathbb{I}m)\psi \;=\; 0. \tag{2.237}$$

2.4.3 Solution to the Dirac equation

We take the Dirac equation in the form

$$(i\partial_\mu\gamma^\mu - m)\psi = 0 \tag{2.238}$$

and multiply it by $(i\partial_\nu\gamma^\nu + m)$ from left.

$$\begin{aligned}
(i\partial_\nu\gamma^\nu + m)(i\partial_\mu\gamma^\mu - m)\psi &= 0 && (2.239)\\
\Big[-\partial_\nu\gamma^\nu\partial_\mu\gamma^\mu - im\partial_\nu\gamma^\nu + im\partial_\mu\gamma^\mu - m^2\Big]\psi &= 0 && (2.240)
\end{aligned}$$

One can see that $\partial_\nu\gamma^\nu\partial_\mu\gamma^\mu$ becomes $g^{\mu\nu}\partial_\mu\partial_\nu$, according to (2.226). We can then write

$$\begin{aligned}
(g^{\mu\nu}\partial_\mu\partial_\nu + \mathbb{I}m^2)\psi &= 0 && (2.241)\\
(\partial_\mu\partial^\mu + \mathbb{I}m^2)\psi &= 0, && (2.242)
\end{aligned}$$

which is the Klein-Gordon equation for each component of ψ (2.127). The full solution can be written as $e^{-iP_\mu x^\mu}$, where $P^\mu = (E, \boldsymbol{p})$ and $\boldsymbol{p}^2 = m^2$. [5] One can thus write ψ as

$$\begin{aligned}
\psi &= u(\boldsymbol{p})e^{-iP_\mu x^\mu} && (2.243)\\
u(\boldsymbol{p}) &= \begin{pmatrix}\varphi(\boldsymbol{p})\\ \chi(\boldsymbol{p})\end{pmatrix}, && (2.244)
\end{aligned}$$

where $u(\boldsymbol{p})$ is called bispinor.
We now put solution to final equation and obtain

$$\begin{aligned}
i\partial_\mu\Big(\gamma^\mu u e^{-iP_\mu x^\mu}\Big) - me^{-iP_\mu x^\mu}u\mathbb{I} &= 0 && (2.245)\\
\gamma^\mu uP_\mu e^{-iP_\mu x^\mu} - me^{-iP_\mu x^\mu}u\mathbb{I} &= 0 && (2.246)\\
(\gamma^\mu P_\mu - m\mathbb{I})u &= 0 && (2.247)\\
\Big(\gamma^0 P_0 - \gamma^j P_j - m\mathbb{I}\Big)u &= 0 && (2.248)\\
(\not{P}_\mu - m\mathbb{I})u &= 0, && (2.249)
\end{aligned}$$

[5] Since we hold $c = \hbar = 1$.

From (2.248) it is possible to solve the Dirac equation by rewriting it as

$$\left[\begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} - \begin{pmatrix} 0 & \boldsymbol{p}\cdot\boldsymbol{\sigma} \\ -\boldsymbol{p}\cdot\boldsymbol{\sigma} & 0 \end{pmatrix} - \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\right]\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = 0 \quad (2.250)$$

$$\begin{pmatrix} E-m & -\boldsymbol{p}\cdot\boldsymbol{\sigma} \\ \boldsymbol{p}\cdot\boldsymbol{\sigma} & -E-m \end{pmatrix}\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = 0 \quad (2.251)$$

$$\frac{\boldsymbol{p}\cdot\boldsymbol{\sigma}}{E+m}\varphi - \chi = 0 \quad (2.252)$$

and so for $u(\boldsymbol{p})$ one have

$$u(\boldsymbol{p}) = \begin{pmatrix} \varphi \\ \frac{\boldsymbol{p}\cdot\boldsymbol{\sigma}}{E+m}\varphi \end{pmatrix}. \quad (2.253)$$

Let us show example of functions that satisfy (2.253)

$$\varphi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.254)$$

$$\varphi_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (2.255)$$

Solution was interpreted as prediction of antiparticles. For energy of particles applies $E^2 = (\boldsymbol{p}^2 + m^2)$, which still admits negative energy states, but solves problem with negative probability [6]. Dirac hole theory [11] assume that all possible states with negative energy are already filled by antiparticles. In other words that vacuum is not nothing, but a *sea* [8] of negative energy from which a particle may escape and *annihilate* with another suitable particle while both emit energy-an annihilation of an electron and a positron, for example.

2.4.4 Continuity equation

We are going to deal with the continuity equation in a bit different way that we did with the Schrödinger or the Klein-Gordon equation. The main change will be that we won't use complex conjugation, but hermitian and Dirac conjugation. We also change order of steps, so we first hermitian-conjugate the equation and deal with it and then we will multiply ordinary Dirac equation. In this order it will be easier to understand why we are dealing with this equation as we do.

Continuity equation for Dirac equation without the electromagnetic field

In a first step, we create hermitian-conjugated Dirac equation from (2.234)

$$-i\partial_\mu \psi^\dagger \gamma^{\mu\dagger} - m\psi^\dagger \mathbb{I}^\dagger = 0. \tag{2.256}$$

With advantage we can use

$$\gamma^0 \gamma^{\mu\dagger} \gamma^0 = \gamma^\mu \tag{2.257}$$

by multiplying equation (2.256) by $(\gamma^0)^2$ from left and by γ^0 from right getting

$$-i\partial_\mu (\gamma^0)^2 \psi^\dagger \gamma^{\mu\dagger} \gamma^0 - m{\gamma^0}^2 \psi^\dagger \mathbb{I}^\dagger \gamma^0 = 0 \tag{2.258}$$

$$-i\partial_\mu \gamma^0 \psi^\dagger \gamma^\mu - m{\gamma^0}^2 \psi^\dagger \mathbb{I}^\dagger \gamma^0 = 0. \tag{2.259}$$

It is simple to see that $\mathbb{I}^\dagger = \mathbb{I}$ and also that $\gamma^0 \mathbb{I} \gamma^0 = \mathbb{I}$, because $\gamma^0 = \beta$.
We continue by multiplying this equation by ψ from right.

$$-i\partial_\mu (\gamma^0 \psi^\dagger) \gamma^\mu \psi - m\gamma^0 \psi^\dagger \mathbb{I} \psi = 0 \tag{2.260}$$

In the second step, we take ordinary Dirac equation and multiply it by $\psi^\dagger$ from left. It is important not to multiply it from right, because $\psi\psi^\dagger$ give us a matrix, but $\psi^\dagger\psi$ give us generally a complex number, a Lorentz scalar.

$$i\psi^\dagger \gamma^\mu \partial_\mu \psi - m\psi^\dagger \psi \mathbb{I} = 0 \tag{2.261}$$

From (2.260) we see that we would like to have γ^0 in this equation as well, so we simply multiply this equation by it getting

$$i\psi^\dagger \gamma^0 \gamma^\mu \partial_\mu \psi - m\psi^\dagger \gamma^0 \psi \mathbb{I} = 0. \tag{2.262}$$

There equations are usually written in shorter form where $\psi^\dagger \gamma^0 = \bar{\psi}$ is Dirac conjugate wave function.
As usually while dealing with continuity equation we subtract (2.262) from (2.260) and get

$$-i\partial_\mu \bar{\psi} \gamma^\mu \psi - i\bar{\psi} \gamma^\mu \partial_\mu \psi = 0 \tag{2.263}$$

$$\partial_\mu (\bar{\psi} \gamma^\mu \psi) = 0 \tag{2.264}$$

$$\partial_\mu J^\mu = 0, \tag{2.265}$$

where

$$
\begin{aligned}
J^\mu &= \psi^\dagger\gamma^0\gamma^\mu\psi && (2.266)\\
&= \bar{\psi}\gamma^\mu\psi && (2.267)\\
&= (\rho, \boldsymbol{J}) && (2.268)\\
&= (\rho, \psi^\dagger\boldsymbol{\alpha}\psi) && (2.269)\\
J^0 &= \psi^\dagger{\gamma^0}^2\psi && (2.270)\\
&= \psi^\dagger\psi && (2.271)\\
&= \bar{\psi}\gamma^0\psi && (2.272)\\
&\equiv \rho && (2.273)\\
& && (2.274)
\end{aligned}
$$

Continuity equation for the Dirac equation with the electromagnetic field

One would like to involve the electromagnetic field and so substitute covariant derivative $\partial_\mu \to D_\mu$ as minimal introduction. The Dirac equation now has a form

$$
\begin{aligned}
\Big(iD_\mu\gamma^\mu - m\mathbb{I}\Big)\psi &= 0 && (2.275)\\
\Big(i\partial_\mu\gamma^\mu - qA_\mu\gamma^\mu - m\mathbb{I}\Big)\psi &= 0, && (2.276)
\end{aligned}
$$

where

$$
D_\mu = \partial_\mu + iqA_\mu \qquad (2.277)
$$

and q is the charge. Let us remind that A and φ are real function and so $A^\dagger = A$ and $\varphi^\dagger = \varphi$.
As before we first hermitian-conjugate equation (2.275) and multiplying it by $(\gamma^0)^2$ from left and by γ^0 from right

$$
\begin{aligned}
-i\partial_\mu\psi^\dagger(\gamma^0)^2\gamma^{\mu\dagger}\gamma^0 - qA^\dagger_\mu\psi^\dagger(\gamma^0)^2\gamma^{\mu\dagger}\gamma^0 - (\gamma^0)^2\psi^\dagger\gamma^0 m\mathbb{I} &= 0 && (2.278)\\
-i\partial_\mu\bar{\psi}\gamma^\mu - qA^\dagger_\mu\bar{\psi}\gamma^\mu - m\bar{\psi}\mathbb{I} &= 0. && (2.279)
\end{aligned}
$$

We continue by multiplying by ψ from right

$$
-i\partial_\mu\bar{\psi}\gamma^\mu\psi - qA^\dagger_\mu\bar{\psi}\gamma^\mu\psi - m\bar{\psi}\psi\mathbb{I} = 0. \qquad (2.280)
$$

Second we multiply (2.275) by $\psi^\dagger$ from left and by γ^0 from right

$$
i\partial_\mu\bar{\psi}\gamma^\mu\psi - qA_\mu\bar{\psi}\gamma^\mu\psi - m\bar{\psi}\psi\mathbb{I} = 0. \qquad (2.281)
$$

Third we subtract (2.281) from (2.280), while still having on mind that (2.65)

$$\partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0 \tag{2.282}$$

$$\partial_\mu J^\mu = 0, \tag{2.283}$$

where

$$J^\mu = \bar{\psi}\gamma^\mu\psi. \tag{2.284}$$

We got the same result as for Dirac equation without the electromagnetic field.

Continuity equation for the Dirac equation with the electromagnetic field with a gauge-calibrated potential

Dirac equation is now in form

$$(iD'_\mu\gamma^\mu - m\mathbb{I})\psi' = 0, \tag{2.285}$$

where one have local phase change

$$D'_\mu = \partial_\mu + iqA'_\mu \tag{2.286}$$

$$\psi' = \psi e^{iq\chi(x)}. \tag{2.287}$$

To have an easier "job" we transfer the equation into more detail since $A'_\mu = A_\mu + \partial_\mu\chi_\mu(x)$

$$i\partial_\mu\gamma^\mu\psi e^{iq\chi} - qA'_\mu\gamma^\mu\psi e^{iq\chi} - m\psi e^{iq\chi}\mathbb{I} = 0 \tag{2.288}$$

$$i(\partial_\mu\gamma^\mu\psi) - q\gamma^\mu\psi(A_\mu - 2\partial_\mu\chi_\mu) - m\psi\mathbb{I} = 0 \tag{2.289}$$

First we hermitian-conjugate (2.289)

$$-i(\partial_\mu\psi^\dagger\gamma^{\mu\dagger}) - q\psi^\dagger\gamma^{\mu\dagger}(A^\star_\mu - 2\partial_\mu\chi^\star_\mu) - m\psi^\dagger\mathbb{I} = 0 \tag{2.290}$$

and continue by multiplying by $\gamma^{0 2}$ from left and γ^2 from right

$$-i(\partial_\mu\bar{\psi}\gamma^\mu) - q\bar{\psi}\gamma^\mu(A^\star_\mu - 2\partial_\mu\chi^\star_\mu) - m\bar{\psi}\mathbb{I} = 0. \tag{2.291}$$

Then we simply multiply this equation by ψ from right

$$-i(\partial_\mu\bar{\psi}\gamma^\mu)\psi - q\bar{\psi}\gamma^\mu\psi(A^\star_\mu - 2\partial_\mu\chi^\star_\mu) - m\bar{\psi}\psi\mathbb{I} = 0. \tag{2.292}$$

Second we take (2.289) and multiply it by $\psi^\dagger$ from left and by γ^0 from right

$$i\bar{\psi}(\partial_\mu\gamma^\mu\psi) - q\bar{\psi}\gamma^\mu\psi(A_\mu - 2\partial_\mu\chi_\mu) - m\bar{\psi}\psi\mathbb{I} = 0. \tag{2.293}$$

As usual we subtract (2.293) from (2.292)

$$-i(\partial_\mu\bar{\psi}\gamma^\mu)\psi - i\bar{\psi}(\partial_\mu\gamma^\mu\psi) = 0 \quad (2.294)$$

$$-i\partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0. \quad (2.295)$$

It is interesting that we got the same form of continuity equation for the Dirac equation both with or without electromagnetic field and also with gauge-calibrated potential. This is because in (2.284) there is no derivative.

2.4.5 Lagrangian, Hamiltonian and Euler-Lagrange equation

Lagrangian density for the Dirac equation has the form [12]

$$\mathcal{L}_0 = \frac{1}{2}\Big[i\bar{\psi}\gamma^\mu\partial_\mu\psi - i(\partial_\mu\bar{\psi})\gamma^\mu\psi\Big] - m\bar{\psi}\psi.^6 \quad (2.296)$$

In other literature ([10], [6] or [13], for example) the Lagrangian is written only as

$$\mathcal{L}_0 = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi. \quad (2.298)$$

The form (2.298) will be discussed in chapter 6 and on. In this chapter, we will use (2.297). By computing

$$\frac{\partial\mathcal{L}_0}{\partial\psi} = -m\bar{\psi} - \frac{i}{2}(\partial_\mu\bar{\psi})\gamma^\mu \quad (2.299)$$

$$\frac{\partial\mathcal{L}_0}{\partial\bar{\psi}} = -m\psi + \frac{i}{2}\gamma^\mu\partial_\mu\psi \quad (2.300)$$

$$\frac{\partial\mathcal{L}_0}{\partial(\partial_\mu\psi)} \equiv \pi^\mu \quad (2.301)$$

$$= \frac{i}{2}\bar{\psi}\gamma^\mu \quad (2.302)$$

$$\frac{\partial\mathcal{L}_0}{\partial(\partial_\mu\bar{\psi})} \equiv \bar{\pi}^\mu \quad (2.303)$$

$$= -\frac{i}{2}\gamma^\mu\psi \quad (2.304)$$

[6] In may be also written in a compact form as

$$L = \frac{i}{2}\bar{\psi}\gamma^\mu\overleftrightarrow{\partial^\mu}\psi - m\bar{\psi}\psi. \quad (2.297)$$

one gets Euler-Lagrange equations

$$
\begin{aligned}
0 &= \frac{\partial \mathcal{L}_0}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}_0}{\partial(\partial_\mu \psi)} && (2.305)\\
0 &= -m\bar{\psi} - \frac{i}{2}(\partial_\mu \bar{\psi})\gamma^\mu - \frac{i}{2}\partial_\mu(\bar{\psi}\gamma^\mu) && (2.306)\\
0 &= -m\psi + \frac{i}{2}\gamma^\mu \partial_\mu \psi + \frac{i}{2}\partial_\mu(\gamma^\mu \psi). && (2.307)
\end{aligned}
$$

The last (or the last but one) equation is Dirac equation

$$
\begin{aligned}
0 &= -m\psi + \frac{i}{2}\gamma^\mu \partial_\mu \psi + \frac{i}{2}\partial_\mu(\gamma^\mu \psi) && (2.308)\\
0 &= (i\gamma^\mu \partial_\mu - m)\psi. && (2.309)
\end{aligned}
$$

The Hamiltonian density is then

$$
\begin{aligned}
\mathcal{H}_0 &\equiv \frac{\partial \mathcal{L}_0}{\partial(\partial_0 \psi)}\partial_0 \psi + \partial_0 \bar{\psi} \frac{\partial \mathcal{L}_0}{\partial(\partial_0 \bar{\psi})} - \mathcal{L}_0 && (2.310)\\
&= \frac{i}{2}\bar{\psi}\gamma^0 \partial_0 \psi - \frac{i}{2}(\partial_0 \bar{\psi})\gamma^0 \psi - \frac{i}{2}\bar{\psi}\gamma^\mu \partial_\mu \psi + \\
&\quad \frac{i}{2}(\partial_\mu \bar{\psi})\gamma^\mu \psi + m\bar{\psi}\psi && (2.311)\\
&= \frac{i}{2}(\partial_j \bar{\psi})\gamma^j \psi - \frac{i}{2}\bar{\psi}\gamma^j \partial_j \psi + m\bar{\psi}\psi. && (2.312)
\end{aligned}
$$

Chapter 3

Noether's theorem

3.1 Conservation laws, symmetries and group theory

There are important laws in nature that cannot been broken: conservation laws, such as energy conservation, momentum conservation. Such laws are connected with symmetry.
Symmetries are best described by a group theory (Lie groups, for example). The important group of symmetry are the unitary group $U(n)$ ($n \times n$ unitary matrices) or the $O(n)$ ($n \times n$ orthogonal matrices) and its subgroup $SO(n)$. The letter S stands for special, meaning that their determinant is ± 1, and U means unitary. We would like to emphasize $SU(2)$ that appears in the electroweak theory ($U(1) \times SU(2)$) and $SU(3)$ that describes quantum chromodynamics.
Another important aspect of symmetry is when a Hamiltonian of system commute with the operator of the symmetry: then there is a preserved quantity, an integral of motion.
As an example one can name a few transformations and preserved quantities in Table 3.1. For a long time it had been thought that there is another example-space inverse and conserved parity-P symmetry. But it was shown that does not hold true for the weak interaction [14].

Table 3.1: Symmetries

Transformation	Preserved quantity
translation	momentum
rotation	angular momentum
time translation	energy

3.2 Mathematical description

Let us consider an infinitesimal transformation of a field in the Lagrangian

$$\phi(x) \to \phi'(x') = \phi(x) + \epsilon\delta\phi(x) \tag{3.1}$$

$$\phi'(x) - \phi(x) = \delta\psi(x) \tag{3.2}$$

under which the action

$$S = \int \mathcal{L}(\phi(x), \partial_\mu \phi(x))\, d^4x \tag{3.3}$$

is invariant

$$S' = \int \mathcal{L}(\phi'(x'), \partial'_\mu \phi'(x'))\, d'^4x \tag{3.4}$$

$$= S \tag{3.5}$$

and so the dynamics and Euler-Lagrange equation are still the same.
Let's have a look at the Lagrangians' difference

$$\delta\mathcal{L} = \mathcal{L}(\phi'(x), \partial_\mu \phi'(x)) - \mathcal{L}(\phi(x), \partial_\mu \phi(x)) \tag{3.6}$$

$$= \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\partial_\mu \delta\phi \tag{3.7}$$

$$= \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_0 \phi}\partial_0 \delta\phi + \frac{\partial \mathcal{L}}{\partial \boldsymbol{\nabla} \phi}\boldsymbol{\nabla}\delta\phi. \tag{3.8}$$

The last two terms express the continuity equation (in case that $\delta\mathcal{L}$=0)

$$\partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\delta\phi(x)\Big) = 0 \tag{3.9}$$

$$\partial_0 \frac{\partial \mathcal{L}}{\partial \partial_0 \phi}\delta\phi + \boldsymbol{\nabla}\frac{\partial \mathcal{L}}{\partial \boldsymbol{\nabla} \phi}\delta\phi = 0 \tag{3.10}$$

and one may thus rename

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\delta\phi \equiv J^\mu \tag{3.11}$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\delta\phi \equiv \partial_\mu J^\mu, \tag{3.12}$$

where J^μ is Noether current.

We must note there that Lagrangians does not need to be invariant, but can be modified by four-divergence $\partial_\mu K^\mu$ and still give the same dynamics. The change will appear in equation (3.9)

$$\partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi(x)\Big) = \partial_\mu K^\mu \tag{3.13}$$

$$\partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi(x) - K^\mu\Big) = 0 \tag{3.14}$$

and we obtain

$$\delta\phi(x)\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi(x)} - K^\mu = J^\mu \tag{3.15}$$

$$\partial_\mu\Big(\delta\phi(x)\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi(x)} - K^\mu\Big) \equiv \partial_\mu J^\mu. \tag{3.16}$$

From such current it is possible to define a conserved quantity called charge as

$$Q \equiv \int d^4x\, J^0 \tag{3.17}$$

$$= \int d^4x\, \rho \tag{3.18}$$

and because we know the charge conservation law, we know that

$$\frac{dQ}{dt} = 0 \tag{3.19}$$

$$= \int d^4x\, \partial_0 J^0 \tag{3.20}$$

$$= -\int d^4x\, \partial_j J^j. \tag{3.21}$$

which is seen from continuity equation [6] [11] [13] [15].
As an example (executed according to [6]) one may take the Lagrangian of the Dirac field (2.298)

$$\mathcal{L}_0 = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi \tag{3.22}$$

and verify that it is invariant under the global ($\partial_\mu\chi = 0$) infinitesimal transformation

$$
\begin{aligned}
\psi &\rightarrow \psi e^{i\chi} && (3.23)\\
\psi e^{i\chi} &\sim \psi + i\epsilon\psi && (3.24)\\
\bar{\psi} &\rightarrow \bar{\psi} e^{-i\chi} && (3.25)\\
\bar{\psi} e^{-i\chi} &\sim \bar{\psi} - i\epsilon\bar{\psi} && (3.26)\\
\mathcal{L}_0' &\sim i\bar{\psi}(1-i\epsilon)\gamma^\mu(1+i\epsilon)\partial_\mu\psi - m\bar{\psi}(1-i\epsilon)(1+i\epsilon)\psi && (3.27)\\
&= (i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi)(1+\epsilon^2) && (3.28)\\
&\sim \mathcal{L}_0 && (3.29)
\end{aligned}
$$

We used $\epsilon^2 = 0$ in the last step, because the transformation is infinitesimal. In this case we see that

$$K^\mu = 0. \tag{3.30}$$

From this Lagrangian it is possible to derive the Noether current according to (3.11)

$$
\begin{aligned}
\frac{\partial\mathcal{L}'}{\partial\partial_\mu\psi} &\equiv \pi_\mu && (3.31)\\
&= i\bar{\psi}\gamma^\mu && (3.32)\\
\frac{\partial\mathcal{L}'}{\partial\partial_\mu\bar{\psi}} &\equiv \bar{\pi}_\mu && (3.33)\\
&= 0 && (3.34)\\
J^0 &= \bar{\psi}\gamma^0\psi && (3.35)\\
&= \psi^\dagger\psi && (3.36)\\
J^j &= \bar{\psi}\boldsymbol{\gamma}\psi && (3.37)\\
J^\mu &= \bar{\psi}\gamma^\mu\psi && (3.38)
\end{aligned}
$$

and the charge is

$$Q = \int d^3x\, \psi^\dagger\psi. \tag{3.39}$$

The Klein-Gordon Lagrangian may be treated similarly. Such example can be found in chapter 6 in [11].

3.3 CPT symmetry

According to the P symmetry, object (or a physical law) and its mirror image should be the same and there shouldn't be any way how to tell the left and the right (mirror image and reality). But T. D. Lee and C. D. Yang realized that such requirement was not in any theory, so they proposed that there need not be a conservation of the P symmetry in weak interaction and C. S. Wu executed an experiment in which it was proven that weak interaction violate the P symmetry [14]. Another symmetry is the change of reality and its mirror (left and right) but also particle for its antiparticle (C symmetry), and the weak interaction should conserve the CP symmetry. Another experiment was executed and J. Cronin and V. Fitch proved that such combined symmetry is also violated [16].

For all that we know so far (2020) the CPT symmetry holds. One must realize that since CP symmetry is violated there must by another violation in time (T symmetry) to cancel it out and be violated, too. The CPT symmetry means that if an experiment is executed and another one in which everything has the opposite parity, antiparticles instead of particles and is executed inversely in time, there is no way how to distinguish these two experiments.

If the nature is symmetric, all four fundamental forces should act as only one interaction on high-level energies.

Chapter 4

Short introduction to particles and their properties

In this thesis we would like to bound all particles into Standard particle model by electroweak unification, with ambition to briefly mention quantum electrodynamics (QED) and quantum chromodynamics (QCD). To construct such model one needs a fundamental knowledge about properties of elementary particles, such as charge, mass and spin.

4.1 Fermions and bosons

The first thing is that we distinguish two different types of particles-fermions and bosons.

4.1.1 Fermions

Fermions, particles with half-integer spin are described by Fermi-Dirac statistical distribution and obey the Pauli exclusion principle. That means that there can by only one fermion in particular quantum state and the others must be in a different state. The wave function describing fermions is antisymmetric-meaning that it changes sign when two particles are switched

$$\psi_1 = \psi_1(x_1, x_2, x_3, ..., x_n) \tag{4.1}$$

$$\psi_2 = \psi_2(x_2, x_1, x_3, ..., x_n) \tag{4.2}$$

$$\psi_1 = -\psi_2. \tag{4.3}$$

Quarks, leptons and baryons are fermions.

4.1.2 Bosons

Bosons are particles with spin n, $n \in \mathbb{N}$ and are described by Bose-Einstein statistical distribution. They can be in the same quantum state and their wave function is symmetric when switching two particles

$$\psi_1 = \psi_1(x_1, x_2, x_3, ..., x_n) \tag{4.4}$$
$$\psi_2 = \psi_2(x_2, x_1, x_3, ..., x_n) \tag{4.5}$$
$$\psi_1 = \psi_2. \tag{4.6}$$

Mesons and intermediate particles like W or Z are bosons.

4.2 Fundamental particles

4.2.1 Leptons

Leptons may be described by SU(2) doublet (representing their left-handed components)

$$L = \frac{1}{2}(1 - \gamma_5) \begin{pmatrix} \nu^\ell \\ \ell \end{pmatrix} \tag{4.7}$$

and by two SU(2) singlets (representing their right-handed component)

$$\ell_R = \frac{1}{2}(1 + \gamma_5)\ell \tag{4.8}$$
$$\nu^\ell_R = \frac{1}{2}(1 + \gamma_5)\nu^\ell, \tag{4.9}$$

for $\ell = e, \mu, \tau$ and γ_5 is the chirality operator. Chirality describes handedness of particles and it is closely connected with helicity. Helicity is projection of spin to the momentum. For massless particles is the same as chirality.
All leptons in the Standard Model have spin $\frac{1}{2}$.
Leptons appear as electron e, muon μ and tau τ in ascending order by their mass (which is non-zero and its precise form will be shown in 6.3.4).
The right-handed part for neutrinos is without physical meaning since all neutrinos have only left-handed chirality.
Electron, muon and tau have negative electric charge $q = -e$, where e is positron charge. According to this they "feel" the electromagnetic and the weak interaction (which, as will be shown in 6.1.3, becomes a single electroweak interaction at high-energy levels) besides gravity.[1]

[1] Every massive particle feels gravity, so with this being said we won't mention it again in this thesis.

There are antiparticles to all leptons: positron $\bar{e}^- = e^+$, anti-muon $\bar{\mu}^- = \mu^+$, anti-tau $\bar{\tau}^- = \tau^+$ and anti-neutrinos $\bar{\nu}$. The difference is that positron, anti-muon and anti-tau have positive charge $q = e$.
Neutrinos have zero electric charge, so they feel only the weak interaction. If one introduces non-zero masses for neutrinos by Yukawa theory (see 6.3.4), there is phenomenon called *neutrino oscillations*. Briefly described, it allows neutrinos to transform $\nu_e \leftrightarrow \nu_\mu \leftrightarrow \nu_\tau \leftrightarrow \nu_e$.
There is special conserved quantum number for leptons $L = 1$, anti-leptons $L = -1$ and for other particles is $L = 0$. In every reaction this lepton number must be conserved.[2] One could ask what is this good for-well, due to this number, one can tell difference between neutrino ν and antineutrino $\bar{\nu}$.

4.2.2 Quarks

In ascending order by mass, there are six quarks[3]: up u, down d, strange s, charm c, bottom (or beauty) b and top (truth) t. [4] With s quark we introduce *strangeness* $S_s = -1$, $S_{\bar{s}} = 1$ and other particles S=0. Analogically there are *truth* for t quark, *beauty* for the b quark and *charm* for the c quark. The top quark is the heaviest elementary particle, its mass is approximately mass of 176 protons.
All quarks have spin $\frac{1}{2}$. Their electric charge is either $q = \frac{2}{3}e$ or $q = -\frac{1}{3}e$ and due to this they are unable to exist alone. They also carry a quantum number called *colour charge* and they feel strong interaction! [5] According to this they may be combined into hadrons. There are two ways how hadrons can by made-baryons are composed of three quarks (or antiquarks) meanwhile mesons are composed of quark and anti-quark. Example of baryons may be proton (uud) or neutron (udd), and examples of meson is π ($\pi^+ = u\bar{d}$, $\pi^- = \bar{u}d$) [20].
Analogically to a lepton number, there is a baryon number B which is $B = 1$ for any baryon , $B = -1$ for any anti-baryon and $B = 0$ otherwise.
Interesting fact is that mass of u and d quarks is approximately $m_u = 2.2$ MeV and $m_d = 4.7$ MeV [20], the mass of proton is 1 GeV. This is because of gluons that tie quarks together have a huge energy that contributes to such

[2] Lepton number is conserved in perturbative renormalized theories, but there is idea that $B - L$, where B is baryon number, is conserved. In other theories lepton number is violated. Proton decay, for example, could violate lepton number conservation up to three units $\delta L = \pm 3$. [17] [18]

[3] The name *quark* is from book by James Joyce *Finnegans Wake*: "...Three quarks for Muster Mark". [19]

[4] In GWS model, there are only u, d and s quarks.

[5] Since quarks have mass, they feel all four interactions.

big proton mass.
The most common quarks are u, d and s. Those were predicted and discovered and were thought to be only ones, but then theory predicted another three to satisfy symmetry with leptons. The t quark, according to its huge mass, decay very fast, so is impossible to be tamed into hadrons.

4.2.3 Intermediate particles

There are four fundamentals forces in universe: gravity, weak, electromagnetic and strong. Each of these interactions is believed to be manifested by relevant intermediate bosons. [6] There is heavy scalar (spin 0) particle called the Higgs boson (6.3). It have been discovered in 2013 by ATLAS and CMS experiments at CERN and its mass is 126 GeV.
Another particles are gauge vector (spin 1) and belongs to weak, electromagnetic and strong interaction.
Photon, quantum of the electromagnetic interaction, is a massless particle and in this thesis is described by field A_μ. It has no electric charge so it is its own antiparticle.
Three massive vector bosons $W^\pm$, Z belong to weak interaction. The electric charge of $W^\pm$ is $\pm e$, Z boson is neutral.
Those three are responsible for weak decays.
Gluon, a massless particle, is taming quarks together in hadrons. It has no electric charge, but it carries *colour charge* similarly to quarks. There three different types: q_r, q_g, q_b. The rule is that resulting particle containing quarks and gluons must have a zero *colour charge* q_w. That can be achieved by combinations

$$q_r + q_{\bar{r}} = q_w \tag{4.10}$$
$$q_g + q_{\bar{g}} = q_w \tag{4.11}$$
$$q_b + q_{\bar{b}} = q_w \tag{4.12}$$
$$q_r + q_g + q_b = q_w. \tag{4.13}$$

According to QCD, there are 8 possible states (octet) in which free gluon can be found.
The very last should be a tensor (spin 2) particle called *graviton*. It should have zero mass and carry no charge. This particle has only been predicted but never been neither observed in experiment nor confirmed by theory.

[6] This is according to Standard Model of particles (SMP). There are theories beyond SMP giving another prediction, but this thesis will hold its form within SMP derived by Glasgow, Salam and Weinberg.

Chapter 5

Weak interaction

As mentioned in chapter 4, all fermions (quarks and leptons) feel the weak interaction that is responsible for decays. A well-known example of neutron β-decay will be discussed in this section. Let us start with the Fermi theory improved by Gell-Mann and Feynman [10]

$$\mathcal{L}_{\text{int}} = -\frac{G}{\sqrt{2}} J^{\rho\dagger} J_\rho, \tag{5.1}$$

where J^ρ is charged current of hadrons and leptons and G is constant corresponding to interaction. One can take the

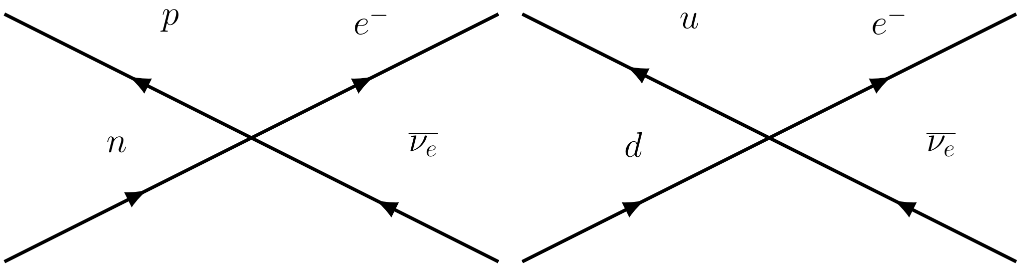
$$n \to p^+ + e^- + \bar{\nu_e} \tag{5.2}$$

process, for example. The (5.1) will have a form

$$\begin{aligned}
\mathcal{L}_{\text{int}} &= -\frac{G_\beta}{\sqrt{2}} J^{\rho\dagger} J_\rho & (5.3)\\
&= \frac{G_\beta}{\sqrt{2}}[\bar{\psi_p}\gamma^\rho(1-\gamma_5)\psi_n][\bar{\psi_e}\gamma_\rho(1-\gamma_5)\psi_\nu] + \\
&\quad \frac{G_\beta}{\sqrt{2}}[\bar{\psi_n}\gamma^\rho(1-\gamma_5)\psi_p][\bar{\psi_\nu}\gamma_\rho(1-\gamma_5)\psi_e]. & (5.4)
\end{aligned}$$

From 4.2.2 we know that n consist of udd quarks so the interaction (5.2) is in fact

$$udd \to uud^+ + e^- + \bar{\nu_e}. \tag{5.5}$$



(a) Neutron β-decay (b) Quark d β-decay

Figure 5.1: Diagrams of β decays, drawn in [7]

The constant for $u \leftrightarrow d$ interaction G_β is connected with Fermi constant G_F

$$G_\beta \quad = \quad G_F \cos\Theta_C, \tag{5.6}$$

where $\Theta_C \doteq 13°$ is Cabbibo angle. [10] The Fermi constant has a meaning of "strength" (coupling constant) of the weak force and its value is measured $G_F = 1.166 \cdot 10^{-5}\ \mathrm{GeV}^{-2}$.[10]
In the Standard Model such interaction is possible due to three massive particles $W^\pm$ and Z. Its interaction term in Lagrangian may we written as [10]

$$\mathcal{L}_{\mathrm{int}}{}^W \quad = \quad \frac{g}{2\sqrt{2}}\Big(J^\mu W_\mu^+ + J^{\mu\dagger} W_\mu^-\Big) \tag{5.7}$$

with new coupling constant g.
We introduce another interaction described by Fermi theory

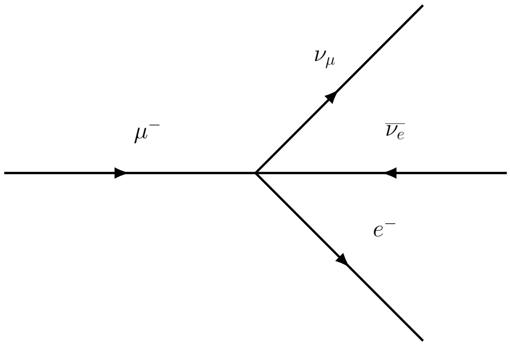
$$\mu^- \to \nu_\mu + e^- + \bar{\nu}_e$$

with Lagrangian

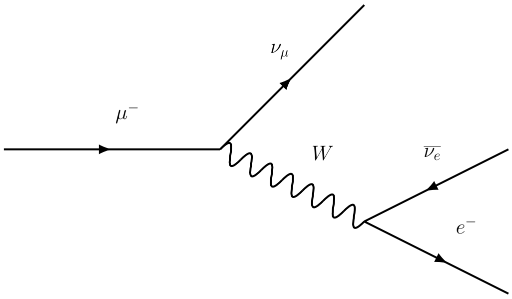
$$\mathcal{L}_{\mathrm{int}} \quad = \quad -\frac{G_F}{\sqrt{2}} J_e^\rho J_\rho^\mu \tag{5.8}$$

$$\begin{aligned} &= -\frac{G_F}{\sqrt{2}}[\bar{\psi}_e \gamma^\rho (1-\gamma_5)\psi_{\nu_e}][\bar{\psi}_{\nu_\mu}\gamma_\rho(1-\gamma_5)\psi_\mu] + \\ &\quad -\frac{G_F}{\sqrt{2}}[\bar{\psi}_{\nu_e}\gamma^\rho(1-\gamma_5)\psi_e][\bar{\psi}_\mu \gamma_\rho (1-\gamma_5)\psi_{\nu_\mu}]. \end{aligned} \tag{5.9}$$

The Feynman diagrams of this scattering are shown in 5.2. Interaction in Fig. 5.2a is according to Fermi theory with amplitude $\mathcal{M} \propto \frac{G_F}{\sqrt{2}}$ and in 5.2b, there is such interaction mediated by W boson. Its amplitude is $\mathcal{M} \propto \frac{g^2}{8m_W^2}$.



(a) Interaction according to the Fermi theory



(b) Interaction according to the WS theory

Figure 5.2: Diagrams describing the μ decay, drawn in [7]

After comparing these two one can see that there is connection between the Fermi and the GWS coupling constant

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2}, \tag{5.10}$$

where m_W is mass of W boson and $\alpha = \frac{e^2}{4\pi} \sim \frac{1}{137}$ [1] is a fine structure constant.

[1]Emphasizing that we work in natural units. In SI it is $\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}$.

Chapter 6

Electroweak unification and mass term

6.1 Gauge invariance, gauge symmetry, electroweak unification

In this section, we will be led by chapter 4 and 5 from [10].

6.1.1 Abelian gauge invariance

Let us take Lagrangian of Dirac field (2.298) (meaning with no interaction)

$$\mathcal{L}_0 = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi \tag{6.1}$$

where ψ is bispinor-solution of corresponding Dirac equation. We introduce global phase transformation

$$\psi' = e^{i\chi}\psi \tag{6.2}$$

$$\bar{\psi}' = \bar{\psi}e^{-i\chi}, \tag{6.3}$$

where $\psi = \psi(x)$, $\bar{\psi} = \bar{\psi}(x)$ and $\chi \in \mathbb{R}$. Then $\partial_\mu\chi = 0$, the Lagrangian will be still the same

$$\begin{aligned}
\mathcal{L}'_0 &= i\bar{\psi}'\gamma^\mu\partial_\mu\psi' - m\bar{\psi}'\psi' && (6.4)\\
&= i\bar{\psi}e^{-i\chi}\gamma^\mu\partial_\mu\left(e^{i\chi}\psi\right) - m\bar{\psi}e^{-i\chi}e^{i\chi}\psi && (6.5)\\
&= i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi && (6.6)\\
&= \mathcal{L}_0. && (6.7)
\end{aligned}$$

It was shown (2.295) that in this case we obtain flux density

$$J_\mu = \bar{\psi}\gamma_\mu\psi. \tag{6.8}$$

Let us now examine what happens once there is local transformation $\chi = \chi(x)$ and so $\partial_\mu \chi \neq 0$

$$\begin{aligned}
\mathcal{L}'_0 &= i\bar{\psi}e^{-i\chi(x)}\gamma^\mu\partial_\mu(e^{i\chi(x)}\psi) - me\bar{\psi}^{-i\chi(x)}e^{i\chi(x)}\psi & (6.9)\\
&= i\bar{\psi}e^{-i\chi(x)}\gamma^\mu e^{i\chi}\Big[\partial_\mu\psi + i\psi\partial_\mu\chi(x)\Big] - m\bar{\psi}e^{-i\chi}e^{i\chi}\psi & (6.10)\\
&= i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi - \bar{\psi}\gamma^\mu\psi\partial_\mu\chi & (6.11)\\
&= \mathcal{L}_0 - \bar{\psi}\gamma^\mu\psi\partial_\mu\chi. & (6.12)
\end{aligned}$$

To have an invariant Lagrangian one must add the interaction term

$$\mathcal{L}_{\mathtt{int}} = g\bar{\psi}\gamma^\mu\psi A_\mu \tag{6.13}$$

where A_μ transforms as

$$A'_\mu = A_\mu + \frac{1}{g}\partial_\mu\chi. \tag{6.14}$$

The Lagrangian is then

$$\begin{aligned}
\mathcal{L} &= i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi + g\bar{\psi}\gamma^\mu\psi A_\mu & (6.15)\\
\mathcal{L}' &= i\bar{\psi}'\gamma^\mu\partial_\mu\psi' - m\bar{\psi}'\psi + g\bar{\psi}'\gamma^\mu\partial_\mu\psi' A'_\mu & (6.16)\\
&= i\bar{\psi}e^{-i\chi(x)}\gamma^\mu e^{i\chi}\Big(\partial_\mu\psi + i\psi\partial_\mu\chi(x)\Big) - me^{-i\chi}\bar{\psi}e^{i\chi}\psi + \\
&\quad g\bar{\psi}e^{-i\chi(x)}\gamma^\mu e^{i\chi(x)}\psi\Big(A_\mu + \frac{1}{g}\partial_\mu\chi\Big) & (6.17)\\
&= i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi + g\bar{\psi}\gamma^\mu\psi A_\mu & (6.18)\\
&= \mathcal{L}. & (6.19)
\end{aligned}$$

From there a covariant derivative can be seen

$$\begin{aligned}
\mathcal{L} &= i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi + g\bar{\psi}\gamma^\mu\psi A_\mu & (6.20)\\
&= \bar{\psi}\gamma^\mu\Big(i\partial_\mu + gA_\mu\Big)\psi - m\bar{\psi}\psi & (6.21)\\
&= i\bar{\psi}\gamma^\mu\Big(\partial_\mu + \frac{g}{i}A_\mu\Big)\psi - m\bar{\psi}\psi & (6.22)\\
&= i\bar{\psi}\gamma^\mu\Big(\partial_\mu - igA_\mu\Big)\psi - m\bar{\psi}\psi & (6.23)\\
&= i\bar{\psi}\gamma^\mu D_\mu\psi - m\bar{\psi}\psi & (6.24)
\end{aligned}$$

and so the covariant derivative is

$$D_\mu = \partial_\mu - igA_\mu. \tag{6.25}$$

The Lagrangian may by now rewritten as

$$\begin{aligned}
\mathcal{L} &= i\bar{\psi}\gamma^\mu D_\mu \psi - m\bar{\psi}\psi && (6.26)\\
&= i\bar{\psi}\not{D}\psi - m\bar{\psi}\psi && (6.27)\\
&= \bar{\psi}\Big(i\not{D} - m\Big)\psi. && (6.28)
\end{aligned}$$

Now it is easy to see that such Lagrangian will be invariant under calibration

$$\begin{aligned}
A'_\mu &= A_\mu + \frac{1}{g}\partial_\mu \chi(x) && (6.29)\\
\psi' &= e^{i\chi(x)}\psi && (6.30)\\
\bar{\psi}' &= \bar{\psi}e^{-i\chi(x)}. && (6.31)
\end{aligned}$$

We were allowed to calibrate A with four-divergence of an arbitrary function $\chi(x)$ so this is our calibration choice and we are free to do so.
Note that it is possible to rewrite this Lagrangian density with covariant derivative once again

$$\begin{aligned}
\mathcal{L}' &= ie^{-i\chi(x)}\bar{\psi}\gamma^\mu D'_\mu e^{i\chi(x)}\psi - me^{-i\chi(x)}\bar{\psi}e^{i\chi(x)}\psi && (6.32)\\
&= \mathcal{L} && (6.33)\\
&= i\bar{\psi}\gamma^\mu D_\mu\psi - m\bar{\psi}\psi && (6.34)
\end{aligned}$$

and so we see that

$$e^{-i\chi(x)}D'_\mu e^{i\chi(x)} = D_\mu. \tag{6.35}$$

To fully describe this topic we must derive Euler-Lagrange equation

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\psi} &= g\bar{\psi}\gamma^\mu A_\mu - m\bar{\psi} && (6.36)\\
\frac{\partial\mathcal{L}}{\partial\bar{\psi}} &= g\gamma^\mu A_\mu\psi - m\psi && (6.37)\\
\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi} &= i\bar{\psi}\gamma^\mu && (6.38)\\
\frac{\partial\mathcal{L}}{\partial\partial_\mu\bar{\psi}} &= 0 && (6.39)
\end{aligned}$$

and so the Euler-Lagrange equation is

$$\frac{\partial\mathcal{L}}{\partial\psi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi} = 0 \tag{6.40}$$

$$g\bar{\psi}\gamma^\mu A_\mu - m\bar{\psi} - i\partial_\mu\bar{\psi}\gamma^\mu = 0. \tag{6.41}$$

If the last equation is multiplied by γ^0 from left and hermitian-conjugated, one obtain

$$gA_\mu\gamma^\mu\psi + i\partial_\mu\gamma^\mu\psi - m\psi = 0 \tag{6.42}$$

$$(i\partial_\mu + gA_\mu)\gamma^\mu\psi - m\psi = 0 \tag{6.43}$$

$$(iD_\mu\gamma^\mu - m)\psi = 0 \tag{6.44}$$

which is the Dirac equation with the electromagnetic field (2.275).
The Hamiltonian density is

$$\mathcal{H} = \frac{\partial\mathcal{L}}{\partial\partial_0\psi}\partial_0\psi - \mathcal{L} \tag{6.45}$$

$$= -\bar{\psi}(i\gamma^j\partial_j - m) - g\bar{\psi}\gamma^\mu\psi A_\mu. \tag{6.46}$$

One would like to know more about A^μ and so we must introduce kinetic term (antisymmetric tensor) that include first derivative of A_μ

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -cB_z & cB_y \\ -E_y & cB_z & 0 & -cB_x \\ -E_z & -cB_y & cB_x & 0 \end{bmatrix} \tag{6.47}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = -F_{\nu\mu}. \tag{6.48}$$

Its inner product is

$$F_{\mu\nu}F^{\mu\nu} = 2(\boldsymbol{B}^2 - \boldsymbol{E}^2). \tag{6.49}$$

From this tensor is possible to derive the Maxwell equations (1.1) - (1.4) from the Euler-Lagrange equation as

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu, [1] \tag{6.50}$$

and from the identity as

$$\partial_\rho F^{\mu\nu} + \partial_\mu F^{\rho\nu} + \partial_\nu F^{\mu\rho} = 0. \tag{6.51}$$

Such equations describe a massless, spin 1 particles.

[1] In vacuum it would be $\partial_\mu F^{\mu\nu} = 0$.

Spin-1 particles with non-zero mass are described by Proca equation [8]

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu \;=\; 0,^{2} \tag{6.52}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. The Proca Lagrangian is [11]

$$\mathcal{L} \;=\; -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu. \tag{6.53}$$

Gauge field tensor $F^{\mu\nu}$ may be also expressed as commutator of covariant derivatives [10]

$$F^{\mu\nu} \;=\; \frac{i}{g}[D^\mu, D^\nu], \tag{6.54}$$

emphasizing that in Abelian theory

$$[A_\mu, A_\nu] = 0. \tag{6.55}$$

We can now complete our Lagrangian for field with Abelian gauge symmetry by including field tensor

$$\mathcal{L} \;=\; -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi} \not{D} \psi - m\bar{\psi}\psi. \tag{6.56}$$

6.1.2 Non-Abelian gauge invariance

We discussed case where (6.55) holds true. But what happens when these fields don't commute? Let's start once again with the Lagrangian (2.296), but now ψ is not an ordinary spinor field but a doublet of bispinors (Dirac spinors) that describes two particles ψ_1 and ψ_2

$$\Psi \;=\; \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{6.57}$$

and so the free Lagrangian has two parts

$$\begin{aligned}
\mathcal{L} \;&\equiv\; \mathcal{L}_1 + \mathcal{L}_2 && (6.58)\\
&=\; i\bar{\psi}_1 \gamma^\mu \partial_\mu \psi_1 - m\bar{\psi}_1\psi_1 + i\bar{\psi}_2 \gamma^\mu \partial_\mu \psi_2 - m\bar{\psi}_2\psi_2 && (6.59)\\
&=\; i\bar{\Psi}\mathbb{I}\gamma^\mu \partial_\mu \Psi - m\bar{\Psi}\mathbb{I}\Psi, && (6.60)
\end{aligned}$$

where

$$\bar{\Psi} \;\equiv\; (\bar{\psi}_1, \bar{\psi}_2). \tag{6.61}$$

[2] In literature may be found as $(\Box + m^2)A^\mu = 0$, $\partial_\mu A^\mu = 0$.

Such a Lagrangian is invariant under a global matrix transformation

$$\Psi' = S\Psi \tag{6.62}$$
$$\bar{\Psi}' = \bar{\Psi}S^{-1}, \tag{6.63}$$

where $S \in SU(2)$ is arbitrary constant matrix multiplied by Abelian phase factor $U(1) \in \mathbb{C}$. We can thus write

$$S = e^{i\omega^a T^a} \tag{6.64}$$
$$= e^{i\boldsymbol{\omega}\cdot\boldsymbol{T}} \in U(2), \tag{6.65}$$

where $T^a = \dfrac{\sigma^a}{2}$ and σ^a are Pauli matrices and $\omega^a \in \mathbb{R}$.
Applying (2.168) we get

$$[T_a, T_b] = i f_{abc} T_c, \tag{6.66}$$

where in this case f_{abc} is a generalized Levi-Civita symbol ϵ_{abc}.
Analogically to the prior Abelian theory case we look at local transformation $\omega = \omega(x)$. We see that such Lagrangian will not be invariant and to fix this, we must introduce vector field $A_\mu(x)$ that in fact consist of triplet of vector fields A_μ^a.

$$A_\mu(x) \equiv A_\mu^a(x) T^a. \tag{6.67}$$

The covariant derivative is now

$$D_\mu = \partial_\mu - igA_\mu. \tag{6.68}$$

Let's see the calibrated Lagrangian

$$\mathcal{L}' = i\bar{\Psi}'\gamma^\mu D_\mu \Psi' - m\bar{\Psi}'\Psi' \tag{6.69}$$
$$= i\bar{\Psi}S^{-1}\gamma^\mu D_\mu S\Psi - m\bar{\Psi}S^{-1}S\Psi \tag{6.70}$$

and to keep the Lagrangian invariant, the covariant derivative must be transformed as

$$D'_\mu = S D_\mu S^{-1} \tag{6.71}$$
$$= \partial_\mu - igA'_\mu. \tag{6.72}$$

From there it is possible to derive [10]

$$A'_\mu = S A_\mu S^{-1} + \frac{i}{g} S \partial_\mu S^{-1}. \tag{6.73}$$

For infinitesimal transformation

$$A'^a_\mu = A^a_\mu - f^{abc}\epsilon^b A^c_\mu + \frac{1}{g}\partial_\mu \epsilon^a, \tag{6.74}$$

where we used the linearisation

$$S(x) = \mathbb{I} + i\epsilon^a(x)T^a \tag{6.75}$$
$$S^{-1}(x) = \mathbb{I} - i\epsilon^a(x)T^a. \tag{6.76}$$

As before, we would like to introduce kinetic term for A_μ of Lagrangian $\mathcal{L}_{\text{kin}}$ including first derivatives of A^a_μ, but in non-Abelian case, the commutator

$$[A_\mu, A^\nu] \neq 0 \tag{6.77}$$

and we must derive our kinetic term from formula mentioned before (6.54). The antisymmetric tensor $F_{\mu\nu}$ is now

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \tag{6.78}$$
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu, \tag{6.79}$$

where $F_{\mu\nu} = F^a_{\mu\nu}T^a$. The last term leads to self-interaction of the field A_μ. To construct the gauge invariant kinetic interaction term, we must take $F^a_{\mu\nu}$ instead of just $F_{\mu\nu}$ and so the Lagrangian will be

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + i\bar{\Psi}\not{D}\Psi - m\bar{\Psi}\Psi. \tag{6.80}$$

The lower index YM references to C. N. Yang and R. Mills, who described this theory [10]. In this Lagrangian we obtained, apart from kinetic term, also a new quadratic and cubic term that describe self-interaction of Yang-Mills field. This theory is based on the gauge group $U(2) = SU(2) \times U(1)$ in which $SU(2)$ is generated by Pauli matrices and $U(1)$ is corresponding phase factor ω^a. Coupling constant in this theory must be a real parameter g.
We haven't include any mass term for A_μ in Lagrangian, yet. One cannot add it by hand, that would break symmetry, so a new theory is needed-Higgs mechanism.

6.1.3 Electroweak unification

In this chapter we would like to unify parity-conserving electromagnetic current and the weak current that according to [14] violates parity. The solution is to separate left-and right-handed parts of the fermion field so that the left-handed fermion field is $SU(2)$ doublet (leading to three mass fields $A_{\mu^a T^a}$)

and the right-handed is $SU(2)$ singlet (leading to massless particle B_μ). [10] One can consider fermion field for two corresponding particles, electron e and its neutrino ν, for example

$$e_L = \frac{1}{2}(1-\gamma_5)e \tag{6.81}$$

$$\nu_L = \frac{1}{2}(1-\gamma_5)\nu \tag{6.82}$$

$$e_R = \frac{1}{2}(1+\gamma_5)e \tag{6.83}$$

$$\nu_R = \frac{1}{2}(1+\gamma_5)\nu, \tag{6.84}$$

and since we treat e_L and ν_L as an $SU(2)$ doublet we arrange them as

$$L = \begin{pmatrix} \nu_L \\ e_l \end{pmatrix}. \tag{6.85}$$

The covariant derivate is then

$$D^L_\mu = \partial_\mu - igA^a_\mu T^a - ig'Y_L B_\mu \tag{6.86}$$

$$D^R_\mu = \partial_\mu - igA^a_\mu T^a - ig'Y_R B_\mu \tag{6.87}$$

$$\tag{6.88}$$

where $Y_L, Y_R \in \mathbb{R}$. By condition $Y_L \neq Y_R$ we mean that the left-handed field may transform with different phase factor than the right-handed one. The Lagrangian is

$$\mathcal{L} = i\bar{L}\gamma^\mu D^L_\mu L + i\bar{e_R}\gamma^\mu D^R_\mu e_R + i\bar{\nu_R}\gamma^\mu D^R_\mu \nu_R. \tag{6.89}$$

As in previous sections we would like to include kinetic term that may be written as

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}. \tag{6.90}$$

Now the Lagrangian consists of (6.89) and (6.90)

$$\begin{aligned}\mathcal{L} = {} & i\bar{L}\gamma^\mu D^L_\mu L + i\bar{e_R}\gamma^\mu D^R_\mu e_R + i\bar{\nu_R}\gamma^\mu D^R_\mu \nu_R - \\ & -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}.\end{aligned} \tag{6.91}$$

Let us take only $A_\mu - \psi$ interaction part of (6.89) and introduce

$$\frac{T^{\pm}}{2} \equiv \sigma^{\pm} \quad (6.92)$$

$$\equiv \frac{1}{\sqrt{2}}(\sigma_x \pm i\sigma_y) \quad (6.93)$$

$$\sigma^{+} = \sqrt{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (6.94)$$

$$\sigma^{-} = \sqrt{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (6.95)$$

and also

$$A_\mu^{\pm} \equiv \frac{1}{\sqrt{2}}(A_\mu^1 \pm iA_\mu^2) \quad (6.96)$$

$$\equiv W_\mu^{\mp} \quad (6.97)$$

According to these we find

$$\mathcal{L}_{\text{int}} = \frac{g}{\sqrt{2}}(\bar{\nu_L}\gamma^\mu e_L W_\mu^+ + \bar{e_L}\gamma^\mu \nu_L W_\mu^-) + \mathcal{L}_{\text{diag}} \quad (6.98)$$

$$\begin{aligned} \mathcal{L}_{\text{diag}} &= \frac{1}{2}g\bar{L}\gamma^\mu\sigma_z L A_\mu^3 + g'Y_L\bar{L}\gamma^\mu L B_\mu + g'Y_R^e\bar{e_R}\gamma^\mu e_R B_\mu \\ &+ g'Y_R^\nu\bar{\nu_R}\gamma^\mu\nu_R B_\mu. \end{aligned} \quad (6.99)$$

Once this is set, the kinetic term of Lagrangian (6.90) will now become

$$\mathcal{L}_{\text{kin}} = -\frac{1}{2}W_{\mu\nu}^- W^{+\mu\nu} - \frac{1}{4}A_{\mu\nu}^3 A^{3\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}. \quad (6.100)$$

It is important to emphasize that neither A_μ nor B_μ has no direct physical meaning, but their linear combination shall, as it will be shown. We introduce a new vector field Z_μ

$$A_\mu^3 = \cos\Theta_W Z_\mu + \sin\Theta_W A_\mu \quad (6.101)$$

$$B_\mu = -\sin\Theta_W Z_\mu + \cos\Theta_W A_\mu, \quad (6.102)$$

where Θ_W is the Weinberg mixing angle. This can be also rewritten as

$$\begin{pmatrix} A_\mu^3 \\ B_\mu \end{pmatrix} = \begin{pmatrix} \cos\Theta_W & \sin\Theta_W \\ -\sin\Theta_W & \cos\Theta_W \end{pmatrix}\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix}. \quad (6.103)$$

One thus get Lagrangian in form

$$\mathcal{L}_{\text{kin}} = -\frac{1}{2}W_{\mu\nu}^- W^{+\mu\nu} - \frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu}. \quad (6.104)$$

It is good to also realize that since A^1_μ and A^2_μ couple with anti-diagonal Pauli matrices, they allow us to exchange ν for e and vice versa.
According to the orthogonality of (6.103) we obtained no mixing term in (6.104). If we now substitute (6.101) into the diagonal part of the Lagrangian (6.99), we obtain two independent components

$$
\begin{aligned}
\mathcal{L}_{\text{diag}} &= \mathcal{L}^{\text{A}}_{\text{diag}} + \mathcal{L}^{\text{Z}}_{\text{diag}} && (6.105)\\
\mathcal{L}^{\text{A}}_{\text{diag}} &= \Big(\frac{1}{2} g \sin\Theta_W \bar{\nu_L}\gamma^\mu \nu_L - \frac{1}{2} g \sin\Theta_W \bar{e_L}\gamma^\mu e_L + \\
&\quad g'\cos\Theta_W Y_L \bar{\nu_L}\gamma^\mu \nu_L + g'\cos\Theta_W Y_L \bar{e_L}\gamma^\mu e_L + \\
&\quad g'\cos\Theta_W Y^e_R \bar{e_R}\gamma^\mu e_R + g'\cos\Theta_W Y^\nu_R \bar{\nu_R}\gamma^\mu \nu_R\Big) A_\mu && (6.106)\\
\mathcal{L}^{\text{Z}}_{\text{diag}} &= \Big(\frac{1}{2} g \cos\Theta_W \bar{\nu_L}\gamma^\mu \nu_L - \frac{1}{2} g \cos\Theta_W \bar{e_L}\gamma^\mu e_L - \\
&\quad -g'\sin\Theta_W Y_L \bar{\nu_L}\gamma^\mu \nu_L - g'\sin\Theta_W Y_L \bar{e_L}\gamma^\mu e_L - \\
&\quad -g'\sin\Theta_W Y^e_R \bar{e_R}\gamma^\mu e_R - g'\sin\Theta_W Y^\nu_R \bar{\nu_R}\gamma^\mu \nu_R\Big) Z_\mu. && (6.107)
\end{aligned}
$$

First, we will look at only $\mathcal{L}^{\text{A}}_{\text{diag}}$ We require that there are no right-handed neutrinos

$$Y^\nu_R = 0, \qquad (6.108)$$

that there is no interaction between left-handed neutrino and a vector field A_μ (photon-neutrino interaction, for example).

$$\frac{1}{2} g \sin\Theta_W + g'\cos\Theta_W Y_L = 0 \qquad (6.109)$$

and that left-and right-handed electron has the same coupling to A_μ (since it describes one particle)

$$-\frac{1}{2} g \sin\Theta_W + g'\cos\Theta_W Y_L = g'\cos\Theta_W Y^e_R. \qquad (6.110)$$

According to these requirement we see that

$$
\begin{aligned}
Y^e_R &= 2Y_L && (6.111)\\
\tan\Theta_W &= -2\frac{g'}{g} Y_L && (6.112)\\
&= -\frac{g'}{g} Y^e_R. && (6.113)
\end{aligned}
$$

We now compare such Lagrangian with the QED Lagrangian

$$\mathcal{L}_{\text{QED}} = e\bar{\psi}\gamma^\mu A_\mu \psi, \qquad (6.114)$$

where ψ describes fermion field. One can see that for coupling constant must apply

$$\mathrm{e} = g \sin\Theta_W \tag{6.115}$$
$$\mathrm{e} < g, \tag{6.116}$$

which are so called unification conditions. The e is positron charge. If we substitute (6.115) into (5.10), we obtain

$$m_w = \Big(\frac{\sqrt{2}e^2}{8G_F \sin^2\Theta_W}\Big)^{\frac{1}{2}} \tag{6.117}$$
$$= \Big(\frac{\pi\alpha}{G_F\sqrt{2}\sin^2\Theta_W}\Big)^{\frac{1}{2}}. \tag{6.118}$$

6.2 Weak hypercharge

Let us introduce weak hypercharge Y_W [6] that consist of third component of weak isospin I_3 and electric charge Q

$$Q = I_3 + Y_W.^3 \tag{6.119}$$

Their values for $SU(2)$ can be found in Table 6.1.

	e_L	ν_L	e_R	ν_R
I_3	$-\frac{1}{2}$	$\frac{1}{2}$	0	0
Y_W	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	0

Table 6.1: Values Y_W and I_3 for $SU(2)$

For exact choice $Y_L = -\frac{1}{2}$ (see Table 6.1 and corresponding formula (6.119)) one get expression for g, g' and Θ_W

$$\tan\Theta_W = \frac{g'}{g} \tag{6.120}$$
$$\cos\Theta_W = \frac{g}{\sqrt{g^2+g'^2}} \tag{6.121}$$
$$\sin\Theta_W = \frac{g'}{\sqrt{g^2+g'^2}}. \tag{6.122}$$

[3] It is also possible to find it in form of $Q = I_3 + \frac{1}{2}Y_W$.

If we now take a look at $\mathcal{L}^{\mathrm{Z}}_{\mathrm{diag}}$ (6.107) with respect to (6.108), (6.111) and (6.112), we obtain

$$\begin{aligned}\mathcal{L}^{\mathrm{Z}}_{\mathrm{diag}} &= \Big[\frac{g}{\cos\Theta_W}\frac{1}{2}\bar{\nu_L}\gamma^\mu\nu_L - \frac{g}{\cos\Theta_W}\frac{\cos 2\Theta_W}{2}\bar{e_L}\gamma^\mu e_L + \\ &\quad \frac{g}{\cos\Theta_W}\sin^2\Theta_W\bar{e_R}\gamma^\mu e_R\Big]Z_\mu. \end{aligned} \tag{6.123}$$

From this very important relations for $SU(2)\times U(1)$ can be seen

$$\frac{1}{2} = I^\nu_{3L} - Q^\nu\sin^2\Theta_W \tag{6.124}$$

$$0 = I^\nu_{3R} - Q^\nu\sin^2\Theta_W \tag{6.125}$$

$$\sin^2\Theta_W - \frac{1}{2} = I^e_{3L} - Q^e\sin^2\Theta_W \tag{6.126}$$

$$\sin^2\Theta_W = I^e_{3r} - Q^e\sin^2\Theta_W. \tag{6.127}$$

Since $Q^e = -1$ and $Q^\nu = 0$, this can by easily verify according to Table 6.1. If we now introduce new coupling constants g^ν_L, g^e_L and g^e_R respecting (6.115)

$$g^\nu_L = \frac{g}{2\cos\Theta_W} \tag{6.128}$$

$$= \frac{g^2}{2\sqrt{g^2-e^2}} \tag{6.129}$$

$$g^e_L = \frac{g}{\cos\Theta_W}(-\cos^2\Theta_W) \tag{6.130}$$

$$= \frac{2e^2-g^2}{2\sqrt{g^2-e^2}} \tag{6.131}$$

$$g^e_R = \frac{g}{2\cos\Theta_W}\sin^2\Theta_W \tag{6.132}$$

$$= \frac{e^2}{\sqrt{g^2-e^2}}, \tag{6.133}$$

it is possible to rewrite our Lagrangian into a shorted form

$$\mathcal{L}^{\mathrm{Z}}_{\mathrm{diag}} = \Big(g^\nu_L\bar{\nu_L}\gamma^\mu\nu_L + g^e_L\bar{e_L}\gamma^\mu e_L + g^e_R\bar{e_R}\gamma^\mu e_R\Big)Z_\mu. \tag{6.134}$$

The main parameter of electroweak unification is then the mixing angle $\sin\Theta_W = \frac{e}{g}$. For zero angle one obtain only electromagnetism and for $\sin\Theta_W = 1$ is interaction strictly weak with no electromagnetic impact.

6.3 Higgs mechanism

This section is inspired by chapter 6 in [10].
Having suitable Lagrangian for electroweak theory, one would like to add a mass term. First we show how to do it by Goldstone theorem, then execute Abelian theory and finally came up with mass for vector bosons ($W^{\pm}$, Z) and leptons (e^{-}, ν). Photon, that is also vector particle included in A_{μ}, has a zero (invariant) mass. A new scalar particles will be examined-massless Goldstone boson and Higgs boson with non-zero mass.

6.3.1 Goldstone model

Let's start with Lagrangian density for complex scalar field

$$\mathcal{L} = \partial^{\mu}\varphi^{\star}\partial_{\mu}\varphi - V(\varphi\varphi^{\star}) \tag{6.135}$$
$$V(\varphi\varphi^{\star}) = \lambda(\varphi\varphi^{\star})^2 - \mu^2\varphi\varphi^{\star}, \tag{6.136}$$

where $V(\varphi\varphi^{\dagger})$ is potential term constisting of dimensionless coupling constant $\lambda > 0$ and real parameter μ, $[\mu] = \text{GeV}$. We can thus rewrite Lagrangian in a form of

$$\mathcal{L} = \partial_{\mu}\varphi\partial^{\mu}\varphi^{\star} - \lambda(\varphi\varphi^{\star})^2 + \mu^2\varphi\varphi^{\star}. \tag{6.137}$$

The first and third terms looks similar to Klein-Gordon Lagrangian, but the last term has opposite sign (2.95). Lets have a look at potential term written with substitution $\sqrt{\rho} = \varphi\varphi^{\star}$, $[\rho] = \text{GeV}$, c. f. 6.1

$$V(\rho) = \lambda\rho^4 - \mu^2\rho^2. \tag{6.138}$$

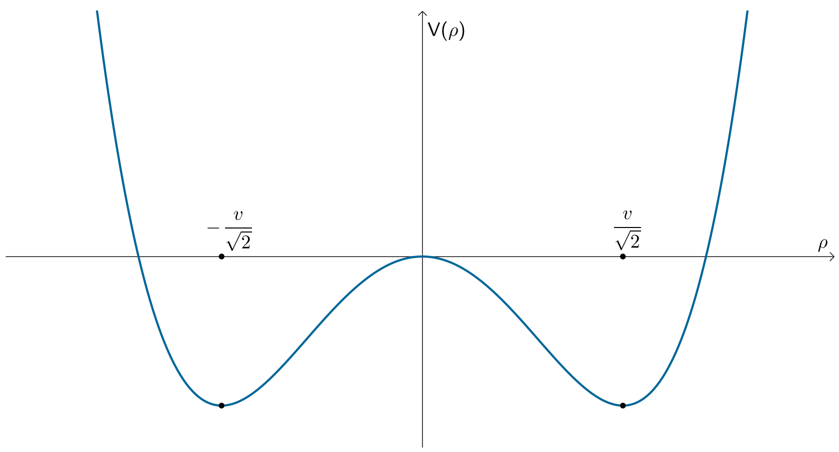


Figure 6.1: Potential $V(\rho)$

We would like to find minimum

$$\frac{\partial V(\rho)}{\partial\rho} = 0 \tag{6.139}$$

$$4\lambda\rho^3 - 2\mu^2\rho = 0 \tag{6.140}$$

$$\rho = 0 \quad \vee \quad \rho^2 = \frac{\mu^2}{2\lambda}. \tag{6.141}$$

The first give us maximum and the second is global minimum. We may thus substitute

$$v^2 = \frac{\mu^2}{\lambda}, \tag{6.142}$$

where v stand for *vacuum*, to obtain

$$\frac{v^2}{\sqrt{2}} = \rho^2 \tag{6.143}$$

$$= \varphi\varphi^\star \tag{6.144}$$

and so

$$\varphi_0 = \frac{v}{\sqrt{2}}e^{ia} \tag{6.145}$$

$$\varphi_0^\star = \frac{v}{\sqrt{2}}e^{-ia}, \tag{6.146}$$

where $a \in \mathbb{R}$. This v will be very important in following sections as well.
In the light of this observation we would like to have the *vacuum* state as a starting point, not the unstable $\rho = 0$. To do so, we recalibrate $\rho \to \rho'$ where

$$\rho' = \frac{\sigma + v}{\sqrt{2}} \tag{6.147}$$

and so we obtain

$$\varphi = \rho' e^{\frac{i\pi(x)}{v}} \text{ [4]} \tag{6.148}$$

We now put (6.148) into (6.137) and use (6.142)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\pi\partial^\mu\pi - \lambda v^2\sigma^2 + \mathcal{L}_{\text{int}} \tag{6.149}$$

$$\mathcal{L}_{\text{int}} = -\frac{1}{4}\lambda\sigma^4 - \lambda v\sigma^3 + \frac{\sigma^2}{2v^2}\partial_\mu\pi\partial^\mu\pi + \frac{\sigma}{v}\partial_\mu\pi\partial^\mu\pi \tag{6.150}$$

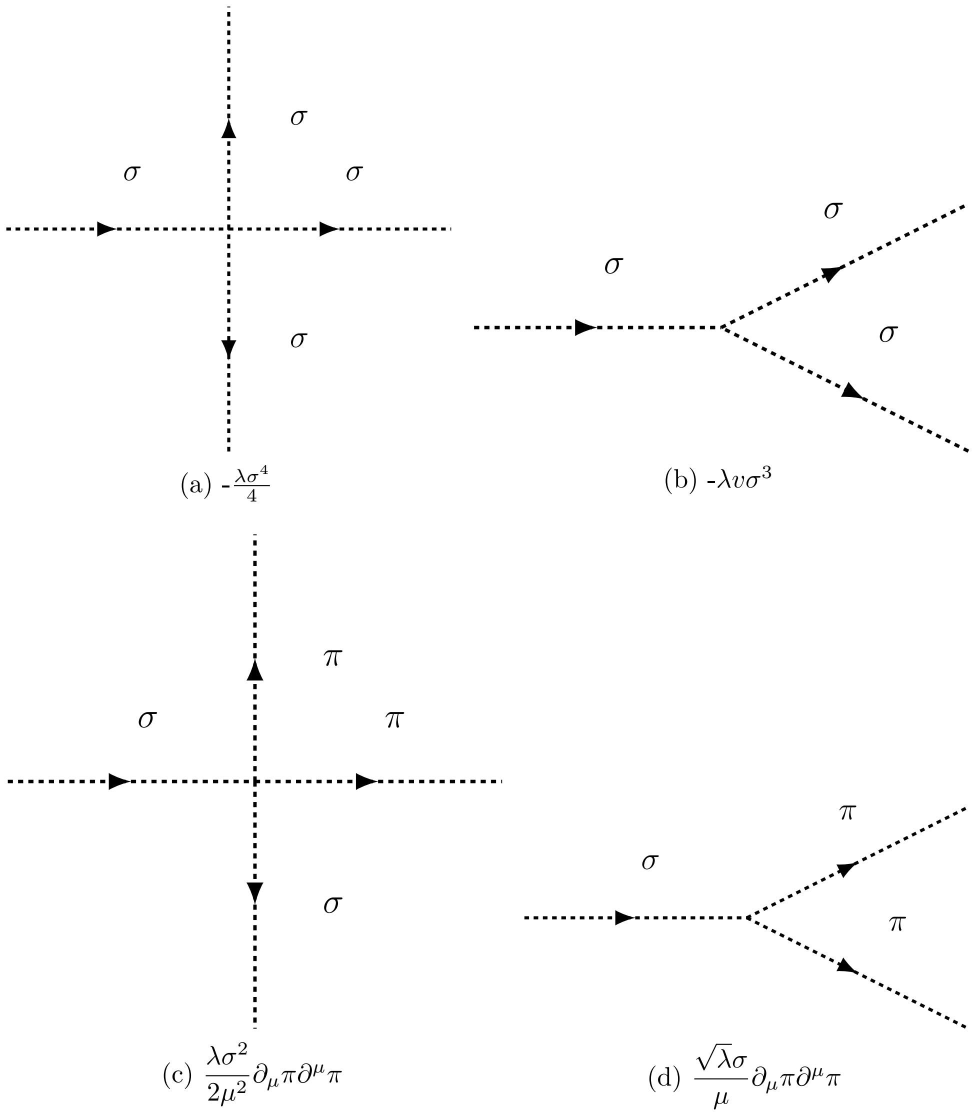
$$= -\frac{1}{4}\lambda\sigma^4 - \sqrt{\lambda}\mu\sigma^3 + \frac{\lambda\sigma^2}{2\mu^2}\partial_\mu\pi\partial^\mu\pi + \frac{\sqrt{\lambda}\sigma}{\mu}\partial_\mu\pi\partial^\mu\pi \tag{6.151}$$

[4] Term $\frac{1}{v}$ is introduced to get right dimension of angular field $\pi(x)$. [10]

and introduce $\frac{m_\sigma^2}{2} = \lambda v^2 = \mu^2$. Now we have the same sign of the mass term as in (2.95) and we see that there are two scalar fields σ and π with masses

$$
\begin{aligned}
m_\sigma &= \mu\sqrt{2} && (6.152)\\
m_\pi &= 0. && (6.153)
\end{aligned}
$$

In the Fig. 6.2 there are Feynman diagrams for all interactions in (6.151).



(a) -$\frac{\lambda\sigma^4}{4}$ (b) -$\lambda v\sigma^3$

(c) $\frac{\lambda\sigma^2}{2\mu^2}\partial_\mu\pi\partial^\mu\pi$ (d) $\frac{\sqrt{\lambda}\sigma}{\mu}\partial_\mu\pi\partial^\mu\pi$

Figure 6.2: Interactions in (6.151), drawn in [7]

We would like to see, whether such Lagrangian is invariant under transformation

$$\begin{aligned} \varphi' &= e^{i\omega}\varphi && (6.154) \\ \varphi'^{\star} &= e^{-i\omega}\varphi^{\star} && (6.155) \\ \sigma' &= \sigma && (6.156) \\ \pi' &= \pi + v\omega, && (6.157) \end{aligned}$$

where $\omega \in \mathbb{R}$. We see that our ground state *vacuum* isn't invariant, while the Lagrangian is. Such a case is called *spontaneous symmetry breaking.* Such theory, according to Goldstone theorem, implies that there must be scalar massless particle called Goldstone boson. In forthcoming section we shall see that such particle doesn't exist and is replaces by Higgs boson.

6.3.2 Abelian model

In this theory we work with covariant derivatives instead of ordinary ones and we must add a kinetic term, so our starting Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\varphi D^\mu\varphi^\star - \lambda(\varphi\varphi^\star - \frac{v^2}{2})^2. \tag{6.158}$$

Such Lagrangian is invariant under local transformation

$$\begin{aligned} \varphi' &= e^{i\chi}\varphi && (6.159) \\ \varphi'^{\star} &= e^{-i\chi}\varphi^{\star} && (6.160) \\ A'_\mu &= A_\mu + \frac{1}{g}\partial_\mu\chi, && (6.161) \end{aligned}$$

where $\chi = \chi(x)$.
We process analogically to previous case: introduce reparametrization for field

$$\varphi = \rho e^{i\frac{\pi}{v}}, \tag{6.162}$$

still having (6.142) in mind. To remove phase factor, we choose

$$\chi(x) = -\frac{\pi(x)}{v} \tag{6.163}$$

to obtain

$$\begin{aligned} \rho' &= \rho && (6.164) \\ A'_\mu &= A_\mu - \frac{1}{gv}\partial_\mu\pi && (6.165) \\ &= B_\mu. && (6.166) \end{aligned}$$

As in previous case we use shift (6.147) and by renaming

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{6.167}$$

we obtain final form of Lagrangian

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \partial_\mu\sigma\partial^\mu\sigma + \frac{g^2\sigma^2}{2}B_\mu B^\mu + g^2\sigma v B_\mu B^\mu + \\ &\quad \frac{g^2v^2}{2}B_\mu B^\mu - \frac{\lambda\sigma^4}{4} - \lambda\sigma^3 v - \lambda\sigma^2 v^2.\end{aligned} \tag{6.168}$$

In the Fig. 6.3 there are Feynman diagrams for all interactions in (6.168).

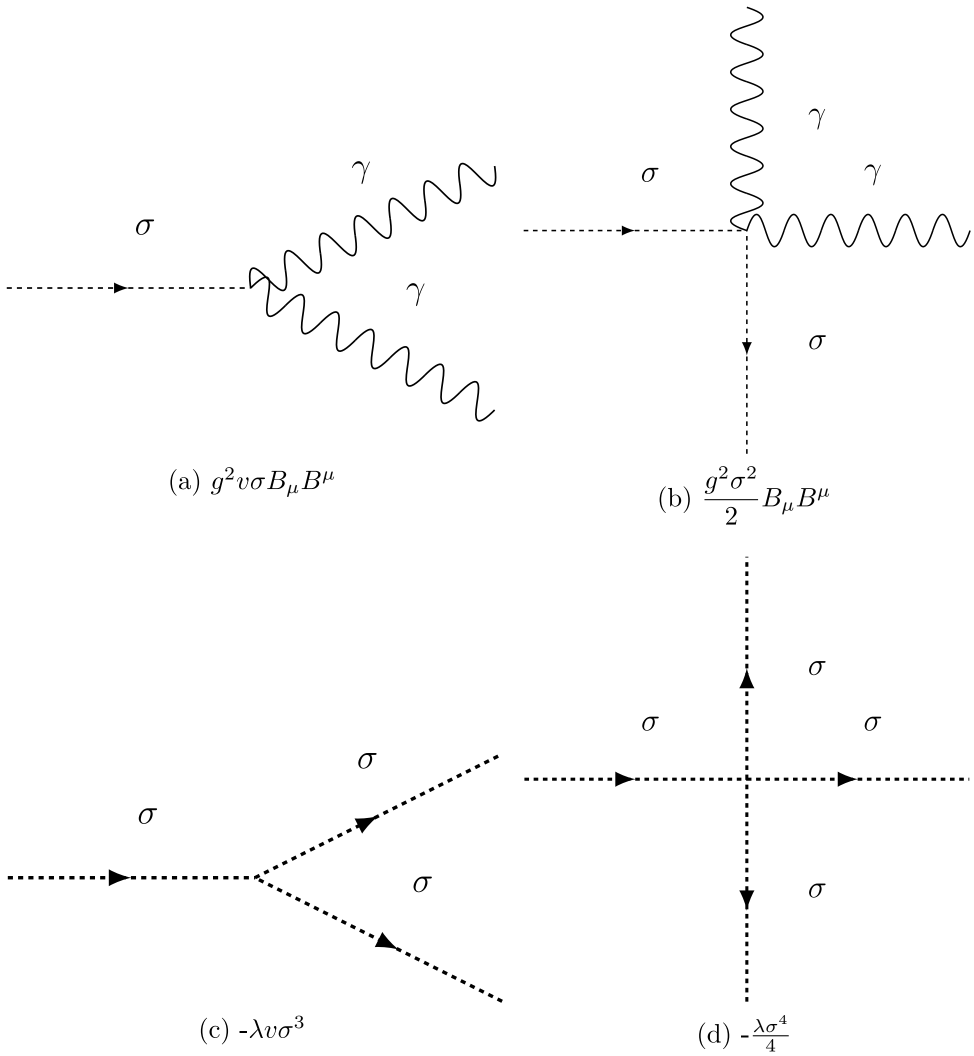


Figure 6.3: Interactions in (6.168), drawn in [7]

As in prior case one obtain mass term for scalar field σ, but in this Lagrangian also contains mass term for vector field B_μ with respective coupling constant g. The sign is here accurate, because such particle B_μ is described by Proca equation Lagrangian (6.53)

$$m_\sigma = \mu\sqrt{2} \tag{6.169}$$
$$m_B = gv. \tag{6.170}$$

The main difference is need for massive scalar particle σ instead of massless.

6.3.3 Electroweak $SU(2) \times U(1)$ model

In this chapter, we would like to work with a doublet of two complex fields

$$\Phi = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix} \tag{6.171}$$

consisting of complex field carrying +1 and 0 charge respectively. In fact, it can easily be

$$L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \tag{6.172}$$

as well. Each component may be rewritten in form of two real fields

$$\varphi^+ = \varphi_1 + i\varphi_2 \tag{6.173}$$
$$\varphi^0 = \varphi_3 + i\varphi_4. \tag{6.174}$$

We start with Goldstone Lagrangian

$$\mathcal{L} = \Big(D_\mu\Phi\Big)^\dagger D^\mu\Phi - \lambda\Big(\Phi^\dagger\Phi - \frac{v^2}{2}\Big)^2 \tag{6.175}$$
$$\Phi^\dagger\Phi = \varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_4^2 \tag{6.176}$$

with respect to (6.142), but we included the covariant derivatives instead of ordinary ones

$$D_\mu = \partial_\mu + igA_\mu^a\frac{\sigma^a}{2} + ig'YB_\mu. \tag{6.177}$$

A_μ is vector field corresponding to $SU(2)$, B_μ is scalar field corresponding to $U(1)$ and g, g' and coupling constant.

Once again we shift the ground state $\Phi_0^\dagger\Phi_0$ and we get rid of expanding around minimum od $V(\Phi)$ one gets

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ (v+H) \end{pmatrix} e^{\frac{i\boldsymbol{\pi}\cdot\boldsymbol{T}}{v}} \tag{6.178}$$

where $H = H(x)$ is scalar Higgs field. Since we have a freedom in choosing $\boldsymbol{\chi}$, we can easily get rid of this whole exponential term including non-physical π to have only

$$\Phi = \frac{1}{\sqrt{2}}(v+H)\xi, \tag{6.179}$$

where $\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The mass of Higgs particle is found to be

$$m_H = \sqrt{2\lambda}v \tag{6.180}$$
$$= \sqrt{2}\mu. \tag{6.181}$$

If we now rewrite our Lagrangian with respect to all of this, anticommutativity and unitarity of Pauli matrices and according to

$$\xi^\dagger\xi = 1 \tag{6.182}$$
$$\xi^\dagger\sigma^x\xi = 0 \tag{6.183}$$
$$\xi^\dagger\sigma^y\xi = 0 \tag{6.184}$$
$$\xi^\dagger\sigma^z\xi = -1 \tag{6.185}$$

we obtain Lagrangian in form of

$$\mathcal{L} = \frac{1}{2}\partial_\mu H\partial^\mu H - \lambda v^2H^2 - \lambda vH^3 - \frac{1}{4}\lambda H^4 + \mathcal{L}_{\mathrm{g}} \tag{6.186}$$
$$\mathcal{L}_{\mathrm{g}} = \frac{(v+H)^2}{8}\Big(g^2A_\mu^aA^{a\mu} - 4Ygg'A_\mu^3B^\mu + 4Y^2g'^2B_\mu B^\mu\Big), \tag{6.187}$$

where the second term in $\mathcal{L}$ is a mass term for Higgs boson. If we take into account (6.103) and (6.112) and realize that $Y = -Y_L = \frac{1}{2}$ (since we want to recall electroweak theory from 6.1.3), it is possible to rewrite

$$\mathcal{L}_{\mathrm{g}} = \frac{v^2}{8}(g^2+g'^2)Z_\mu Z^\mu + \frac{v^2g^2}{4}W_\mu^-W^{+\mu} + \mathcal{L}_{\mathrm{qH}} \tag{6.188}$$
$$Z_\mu = \frac{1}{\sqrt{g^2+g'^2}}(gA_\mu^3 - g'B_\mu) \tag{6.189}$$
$$W_\mu^\pm = \frac{1}{\sqrt{2}}(A_\mu^1 \mp iA_\mu^2) \tag{6.190}$$

from which we directly see that

$$m_W = \frac{gv}{2} \tag{6.191}$$

$$m_Z = \frac{\sqrt{g^2+g'^2}v}{2} \tag{6.192}$$

$$\frac{m_W}{m_Z} = \cos\Theta_W. \tag{6.193}$$

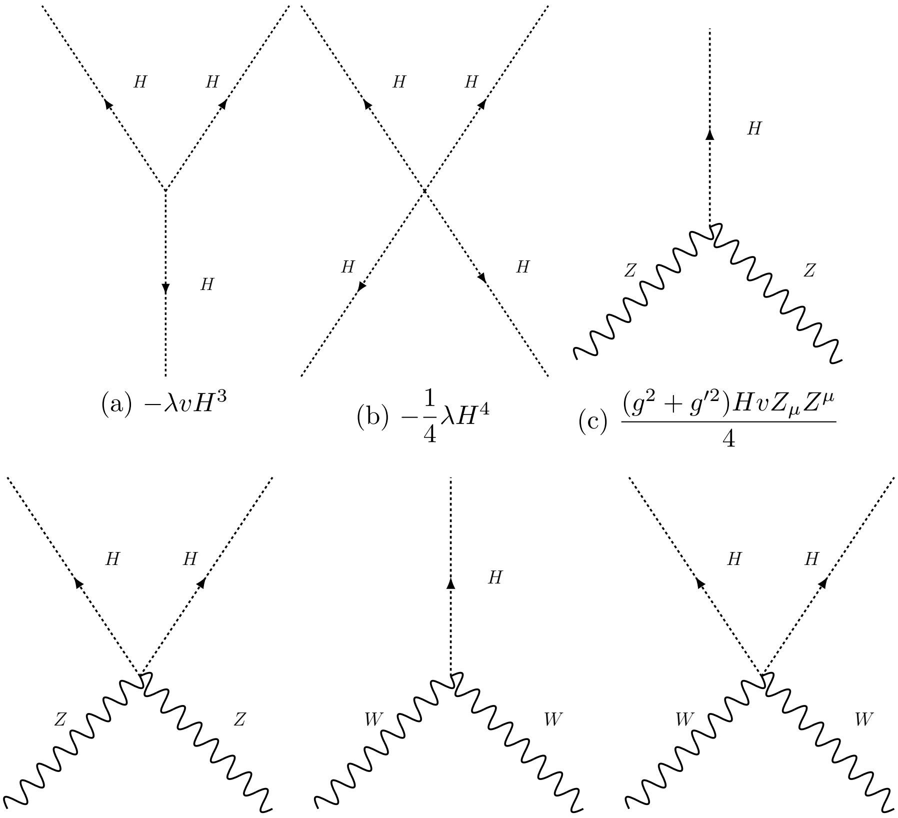
The term with Higgs interaction $\mathcal{L}_{\mathrm{qH}}$ is

$$\mathcal{L}_{\mathrm{qH}} = \frac{(2vH+H^2)}{8}\Big((g^2+g'^2)Z_\mu Z^\mu + 2g^2 W_\mu^- W^{+\mu}\Big) \tag{6.194}$$

and so the full Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu H \partial^\mu H - \lambda v^2 H^2 - \lambda v H^3 - \frac{1}{4}\lambda H^4 + \frac{(v+H)^2}{8}\Big[(g^2+g'^2)Z_\mu Z^\mu + 2g^2 W_\mu^- W^{+\mu}\Big]. \tag{6.195}$$

In the Fig. 6.4 there are Feynman diagrams for all interactions in (6.195).



(a) $-\lambda v H^3$ (b) $-\frac{1}{4}\lambda H^4$ (c) $\frac{(g^2+g'^2)HvZ_\mu Z^\mu}{4}$

(d) $\frac{(g^2+g'^2)H^2 Z_\mu Z^\mu}{8}$ (e) $\frac{g^2 H v W^-_\mu W^{+\mu}}{2}$ (f) $\frac{g^2 H^2 W^-_\mu W^{+\mu}}{4}$

Figure 6.4: Interactions in (6.195), drawn in [7]

If we put m_W into (5.10), we get exact value for v

$$v = \sqrt{\frac{1}{G_F\sqrt{2}}} \quad (6.196)$$

$$\doteq 246 \text{ GeV} \quad (6.197)$$

and one can rename it as Higgs field VEV (*vacuum expected value*). It is scale for electroweak unification. Knowing this, one can easily solve m_Z remembering (6.117)

$$m_Z = \Big(\frac{\sqrt{2}e^2}{2G_F \sin^2 2\Theta_W}\Big)^{\frac{1}{2}}. \quad (6.198)$$

6.3.4 Yukawa coupling

As mentioned above, also lepton mass term couldn't have been added to Lagrangian by hand, because it would spoil a symmetry. In this section we will use Yukawa-coupling to achieve non-zero lepton masses. Yukawa interaction term in Lagrangian is

$$\begin{aligned}
\mathcal{L}^e_{\text{Yukawa}} &= -h_e \bar{L}\Phi e_R - h_e \Phi^\dagger L \bar{e_R} && (6.199)\\
&= -\frac{h_e}{\sqrt{2}}(v+H)(e_m \bar{m} L e_R + \bar{e_R} e_L) && (6.200)\\
&= -m_e \bar{e} e + g_{eeH} \bar{e} e H, && (6.201)
\end{aligned}$$

where $L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$, Φ is defined as (6.179), h_e is dimensionless coupling constant, e_R is right-handed electron singlet, H is Higgs boson, m_e is electron mass

$$m_e = = -\frac{1}{\sqrt{2}} h_e v \qquad (6.202)$$

and g_{eeH} is Yukawa coupling for 2 electron and Higgs boson. Note that every term in (6.199) is invariant under $SU(2) \times U(1)$ group symmetry.
According to (6.191) this may be recast as

$$\begin{aligned}
g_{eeH} &= -\frac{m_e}{v} && (6.203)\\
&= -\frac{m_e g}{2m_W}. && (6.204)
\end{aligned}$$

It is possible to rewrite for muon and tau respectively as

$$\begin{aligned}
g_{\mu\mu H} &= -\frac{m_\mu g}{2m_W} && (6.205)\\
g_{\tau\tau H} &= -\frac{m_\tau g}{2m_W}. && (6.206)
\end{aligned}$$

From definition for Φ is clear that there cannot be any mass for a neutrino, but it can by obtained by redefining

$$\begin{aligned}
\tilde{\Phi} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Phi^\dagger && (6.207)\\
&= \frac{v+H}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. && (6.208)
\end{aligned}$$

That is anti-doublet to $L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$. The Lagrangian is

$$\begin{aligned}
\mathcal{L}^{\nu}_{\text{Yukawa}} &= -h_\nu \bar{L}\tilde{\Phi}\nu_R - h_e \tilde{\Phi}^\dagger L \bar{\nu_R} && (6.209)\\
&= -\frac{h_\nu}{\sqrt{2}}(v+H)(\bar{\nu_L}\nu_R + \bar{\nu_R}\nu_L) && (6.210)\\
&= -m_\nu \bar{\nu}\nu + g_{\nu\nu H}\bar{\nu}\nu H. && (6.211)
\end{aligned}$$

Analogically to prior case

$$\begin{aligned}
m_\nu &= = -\frac{1}{\sqrt{2}} h_\nu v && (6.212)\\
g_{\nu\nu H} &= -\frac{g m_\nu}{2 m_W}. && (6.213)
\end{aligned}$$

Chapter 7

Symmetry in unification theories

This thesis is based only on relativistic quantum mechanics, the GWS theory of Standard particle model and the electroweak unification. But there are further study to accomplish such theories based on a quantum field theory (QFT): the Super-symmetry (SUSY), the M-theory (String theory), the Grand unification theory (GUT) and the Quantum gravity. The author would like to briefly introduce such theories.

7.1 GUT

Since we have unified the electromagnetic and the weak interaction, one would like to continue and add the electroweak and the strong force as well. Such theory is predicted to happen at scale around 10^{15} GeV and is called Great unification theory. The nature is thought to be in such state at 10^{-36} s after the Big bang. There are different theories of GUT: theory based on $SU(5)$, [16] or the Spin(4)×Spin(6) theory and both can be extended to the Spin(10) theory [21].

7.2 SUSY

There is proposed unification theory based on symmetry in nature called a *Supersymmetry* (*SUSY* for short). It suggests that fermions and bosons do not differ, but are the same particle in different state. Therefore there must be a symmetrical partner for every particle - fermion to boson and boson to fermion. Such a symmetrical fermion should have a letter $s-$ before its name (*squark*, for example), and a symmetrical boson is to have "-ion" at the end (*gluino*, for example.) [16]

7.3 Quantum gravity

Once there is theory where all non-gravitational forces, described by quantum field theory, are tamed, one would like to add the gravity, described by general relativity, as well. But there comes problem with infinities that start to appear in equations describing the ToE-Theory of everything. Many research are focused on bringing together quantum mechanics and general relativity, but none of them succeeded, yet.

Summary

In this thesis we first dealt with relativistic quantum mechanics equations and we examined their continuity equations with and without the electromagnetic field and a gauge calibration. We mentioned elementarily the Noether theorem and briefly introduced particles in the Standard Model. Then we built up the Lagrangian for the GWS model, add the kinetic and the interaction term. The mass term was included by using Higgs mechanism.
At the end we emphasized the impact of the electroweak interaction by mentioning other unification theories and their possible influence in further understanding of the nature.

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