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BRNO UNIVERSITY OF TECHNOLOGY

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LINEAR MATRIX DIFFERENTIAL EQUATION WITH DELAY

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DOCTORAL THESIS

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ABSTRAKT

V předložené práci se zabýváme hledáním řešení lineární diferenciální maticové rovnice se zpožděním $\dot{x}(t) = A_0x(t) + A_1x(t-\tau)$, kde A_0, A_1 jsou konstantní matice, $\tau > 0$ je konstantní zpoždění. Dále se zabýváme odvozením podmínek stability řešení systému a řiditelnosti daného systému. Pro řešení tohoto systému byla použita metoda *krok za krokem*. Řešení bylo nalezeno jak v rekurentní formě tak i v obecném tvaru.

Je provedena analýza stability a asymptotické stability řešení systému. Jsou zformulovány podmínky stability. Hlavní roli v analýze stability měla metoda Lyapunovových funkcionálů.

Jsou zformulovány nutné a postačující podmínky řiditelnosti pro případ systémů se stejnými maticemi a je zkonstruována řidící funkce. Jsou odvozeny postačující podmínky pro řiditelnost v případě komutujících matic a v případě obecných matic a je sestrojena řídící funkce.

Všechny výsledky jsou ilustrovány na netriviálních příkladech.

KLÍČOVÁ SLOVA

diferenciální rovnice, systémy diferenciálních rovnic, rovnice se zpožděním, druhá Ljapunovova metoda, stabilita řešení, řiditelnost, zpožděný argument.

ABSTRACT

This work is devoted to computing the solution, stability of the solution and controllability of respective system of linear matrix differential equation with delay $\dot{x}(t) = A_0x(t) + A_1x(t-\tau)$, where A_0, A_1 are constant matrices and $\tau > 0$ is the constant delay. To solve this equation, the *step by step* method was used. The solution was found in recurrent form and in general form.

Stability and the asymptotic stability of the solution of the equation was investigated. Conditions for stability were defined. The Lyapunov's functional theory is basic for the investigation.

Necessary and sufficient condition for controllability in same matrices case was defined and the control was built. Sufficient conditions for controllability in communicative matrices case and general case were defined and controls were built.

All results were illustrated with non-trivial examples.

KEYWORDS

differential equation, systems of differential equations, equations with delay, the second method of Lyapunov, stability of solution, controllability, delayed argument.

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V Brně ....

(podpis autorky)

CONTENTS

1 INTRODUCTION

Individual results for functional-differential equations were obtained more than 250 years ago, and systematic development of the theory of such equations began only in the last 90 years. Before this time there were thousands of articles and several books devoted to the study and application of functional-differential equations. However, all these studies consider separate sections of the theory and its applications (the exception is well-known book Elsgolts L.E., representing the full introduction to the theory, and its second edition published in 1971 in collaboration with Norkin S.B. [34]). There were no studies with single point of view on numerous problems in the theory of functional-differential equations until the book by Hale J. (1977) [45].

Interpretation of solutions of functional-differential equations

$$\dot{x}(t) = f(x(t), t),$$

as integral curve in the space $R \times C$ by Krasovskii N.N. (1968) [64] served as such single point of view. This interpretation is now widespread, proved useful in many parts of the theory, particularly sections of the asymptotic behavior and periodicity of solutions. It clarified the functional structure of the functional-differential equations of delayed and neutral type, provided an opportunity to the deep analogy between the theory of such equations and the theory of ordinary differential equations and showed the reasons for deep differences of these theories.

Classic work on the theory of functional, integral and integro-differential equations is a work by Volterra V. [93]. His book "The Theory of functional, integral and integro-differential equations" first released in Spanish in 1927, then significantly revised version of it released in English in 1929. The last edition was released in U.S. in 1959 and the book released in 1982 is a translation into Russian.

The biggest part of the results obtained during 150 years before works by Volterra V. were related to special properties of very narrow classes of equations. In his studies of "predator-prey" models and studies on viscosity-elasticity Volterra V. got some fairly general differential equation, which include past states of system:

$$\dot{x}(t) = f(x(t), x(t - \tau), t), \quad \tau > 0.$$

In addition, because of the close connection between the equations and specific physical systems Volterra V. tried to introduce the concept of energy function for these models. Then he used the behavior of energy function to study the asymptotic behavior of the system in the distant future.

In late 1930 and early 1940s Minorsky N.F. in his article "Self-excited in dynamical systems possessing retarded actions" [77] very clearly pointed out the importance of considering the delay in feedback mechanism in his works on stabilizing the course of a ship and automatic control its movement.

At the beginning of 1950 Myshkis A.D. introduced general class of equations with delay arguments and laid the foundation for general theory of linear systems. In 1972 he systematized ideas in the paper "Linear differential equations with delay argument" [79]. Bellman R. showed in his monograph [7] a broad applicability of equations that contain information about the past in such fields as economics and biology. He also presented a well-constructed theory of linear equations with constant coefficients and the beginning of stability theory. The most intensive development of these ideas presented in the book of Bellman R. and Cooke K. [8], "Differential-difference equations" (1967). The book describes the theory of linear differential-difference equations with constant and variable coefficients:

$$\dot{x}(t) = f(x(t), \dot{x}(t), ..., x^{(n)}(t), x(t-\tau_1), ...x(t-\tau_m), t), \quad \tau_i > 0, i = 1, ..., m.$$

Considerable attention is paid to asymptotic behavior of the solutions, as well as the stability theory of linear and quasi-linear equations. Most of the results in this area belong to these authors. Large number of problems and examples of the specific problems of the theory probability, economics, nuclear physics, etc. are essential part of the book.

The book "Ordinary differential-difference equation" (1961) by Pinney E. [82] is devoted to differential- difference equations, otherwise known as the equations with deviating argument. The focus of the book is linear equations with constant coefficients, which are most often encountered in the theory of automatic control. The book also presents a new method for studying equations with small nonlinearities found by the author. In particular, this method is applied in control theory of Minorskii equation.

Kurbatov V.G. in 1990 in his book [67] systematized facts about differential and differential-difference equations.

Azbelev N.V., Maksimov V.P., Rakhmatulina L.F. "Introduction to theory of functional differential equations" (1991) [4] and Sabitov K.B. "Functional, differential and integral equations. Textbook for university students majoring in "Applied Mathematics and Informatics" and the direction of "Applied Mathematics and Computer Science" (2005) [87] are relatively new works to the theory. In first the authors try to generalize subclasses of systems differential, integro-differential and difference equations with the operator approach. The second manual presented a purely functional, ordinary differential, integral equations and differential equations in partial derivatives and classical methods of solving them.

1.1 Dynamical systems stability

One of the important characteristic of the dynamic system is stability of this system. The history of stability research is more than one century long and one of the first classical work in this branch of mathematic is book of Lyapunov A.M. "General problem of stability motion" (1892) [74]. This work contains author's results about stability of equilibria and the motion of mechanical systems, the model theory for the stability of uniform turbulent liquid, and the study of particles under the influence of gravity. His work in the field of mathematical physics regarded the boundary value problem of the equation of Laplace. Lyapunov's method acctually produced new branch for researching - Lyapunov stability problem.

In the book "Coarse systems" (1937) [3] Andronov A.A. and Pontryagin L.S. presented their results received from researching motion of dynamic system for which topologically trajectory doesn't change for small preturberation of the system. One of the main results of this work is well-known Andronov-Pontryagin criterion of orbitrally topologically stability of dynamic system.

Krasovskii N.N. in his book on the theory of stability (1956) [61] introduced the theory of Lyapunov functionals, noting the important fact: some problems for such systems become more visual and easier to solve if the motion is considered in a functional space, even when the state variable is a finite-dimensional vector. The paper discusses some problems in the nonlinear systems of ordinary differential equations solutions stability theory. The justification of the Lyapunov functions method is adequately addressed, the existence of functions is clarified. Also the possibility of applying the method to study of the systems described by various ordinary differential equations apparatus is proved. He developed these methods further in his next works [62], [63].

Later Korenevskij D.G. in his book "Stability of Dynamical Systems under Random Perturbations of Parameters. Algebraic criteria." (1989) [60] used the method of Lyapunov-Krasovsii functionals of a special quadratic form and the integral over the interval of delay of a quadratic form.

In 1964 the book was published by Aizerman M.A. and Gantmacher F.R. "Absolute stability of regulator systems" [1] and book by La Salle J.P., Lefshetz S. "Stability By Liapunov's Direct Method, With Applications." [68] it became classical in theory.

Demidovich B.P. worked on the theory of stability systematization in 1970s. In his book "Lectures on the Mathematical Theory of Stability"(1967) [18] a theory of stability framework for ordinary differential equations and some related questions are stated. Also the basics of the almost periodic functions theory and their applications to differential equations were introduces.

In book by Zubov V.I. [96] presented the main problems in the stability theory for the systems defined in functional space and methods for their solutions.

In 1970 the course of lectures of Barbashin E.A. "Lyapunov functions" [6] was published. Emphasis is placed on methods of constructing Lyapunov functions for nonlinear systems. Methods of the region of attraction estimation, solutions estimation, management time, integrated quality control criteria were presented. Sufficient criteria for asymptotic stability in general, absolute stability criteria were recounted. A large number of Lyapunov functions for nonlinear systems of second and third order were presented. The case when the nonlinearity depends on two coordinates of points in phase space was examined. The problem of constructing vector Lyapunov functions for complex systems was also investigated.

In the narrow direction differential equations stability theory was developing in late 1970s by scientists Daletskii J.A., Crane M.G. In 1970 they published monograph "Stability of differential equations solutions in Banach space" [17], which set out a theory of higher Lyapunov exponents and general Bohl indicators for linear nonstationary and close to the nonlinear equations, Lyapunov second method and its interpretation in the spaces with an infinite and definite metric, Floquet's theorem and the localization theorem on the spectrum of the monodromy operator, theory of canonical equations with a periodic Hamiltonian, central stability zone, Lyapunov's stability signs and their various generalizations; Fuchs-Frobenius theory, exponential splitting of the non-stationary linear equations solutions, exponential dichotomy, integral manifolds theory, researches by Bohl P. [11], Bogoliubov N.N. and coauthors [10], generalization of the asymptotic methods of Birkhoff G.D. [9], Tamarkin J.D. [90] et al. All these questions are studied for differential equations in Banach or Hilbert spaces.

Another method of stability research is frequency method. This method is developed in the works of Gelig A.H., Leonov G.A. [38], [39].

In 1980 Rusch H., Abets P., Laloy M. wrote the monograph, "Direct method of Lyapunov in stability theory" [85]. This work devoted to investigation the stability of ordinary differential equations solutions by the direct method of Lyapunov. Much attention is paid to applications for various mechanical systems, nonlinear electrical circuits, problems of mathematical economics. Along with the classical results the monograph presents a series of issues, namely: stability of some variables; theorem about equilibrium and stationary motions stability and their circulation; theorems on the stability of equilibrium and stationary motions and their treatment; the instability theorems, based on the concept of sector and expeller; classification of differential equations solutions properties (stability, attraction, limitations, etc.); classification of properties of solutions of differential equations (stability, attraction, limitations, etc.); attraction for autonomous and nonautonomous differential equations; comparison method; Vector Lyapunov functions; one-parameter family of Lyapunov functions.

One of the classical works in stability theory in this period became book by Chetaev N.G. (1990) [15].

In [55], [58], [59] Kolmanovskii V.B. and Nosov V.R. suggested the way to apply the theory for neutral nonlinear system asymptotic stability investigation, used functionals depending on derivatives. Also, special functions by Lui Mei-Gin [70], Lui Xiu-Xiang and Xu Bugong [71], [72] were used to determine the global asymptotic and exponential stability of nonlinear neutral delayed systems with two time-depend bounded delays.

Delay independent criteria of stability for some classes of neutral systems were developed by Gu K., Kharitonov V.L., Chen J. in their work "Stability of time-delay system" (2003)[44].

The paper of Leonov G.A. "Chaotic dynamics and the classical theory of motion stability" (2006) [69] is relatively new work in the theory of dynamical systems motion stability.

Xiaoxin L., Liqiu W., Pei Y. in their work "Stability of Dynamical systems" (2007) [95] investigated the stability of the system with time dependent delays.

Nowadays particular branches of stability are being researched and developed by such scientists as Baštinec J., Khusainov D.Ya., Shatyrko A.V., Diblík J., Dzhalladova I.A., Baštincová A., Piddubna G. [20], [21], [25], [27], [28], [31], [32], [52], [53], [102]-[104].

1.2 Dynamical systems with delay

The future of many processes in the world around us depends not only on the present state, but is also significantly determined by the entire pre-history. Such systems occur in automatic control, economics, medicine, biology and other areas (examples can be found in [16], [43], [50], [56], [57], [76]). Mathematical description of these processes can be done with the help of equations with delay, integral and integro-differential equations. Great contribution to the development of these directions is made by Bellman R., Lunel S.M.V., Mitropolskii U.A., Myshkis A.D., Norkin S.B., Hale J.C. [8], [46], [78], [79], [80].

Classical works in the field of differential equations with retarded argument are work by Myshkis A.D. "Linear differential equations with delay argument "(1972)[79] and Hale J.C. "The Theory of Functional Differential Equations" (1984) [45].

Another branch of differential equations with delay is the systems of differential equations whose parameters change in predefined intervals. The results of Kharitonov V.L., so-called "big and small" Kharitonov's theorem were published in [50], [51].

Boundary value problems for delay differential system are being researched and developed nowadays by such scientists as Boichuk A., Diblík J., Khusainov D.Ya., Růžičková M. [12], [13].

Particular results in representation of solution view are nowadays presented in papers of Baštinec J., Khusainov D.Ya., Piddubna G. [97]-[101].

1.3 Dynamical systems of neutral type

There is also a large number of applications in which retarded argument is included not only as a state variable, but also in its derivative. This is so-called differential-difference equations of neutral type:

$$\dot{x}(t) = f(x(t), \dot{x}(t), \dot{x}(t-\tau)), \tau > 0.$$

Problems that lead to such equations are more difficult to find, although they often appear in studies of two or more oscillatory systems with some links between them. Akhmerov R.R., Kamenskii M.I. and ot. [2], Bellman R., Cooke K. [8] and also Germanovich O.P. [40] raised questions regarding the systems of neutral type in their works.

Work "Linear periodic equation of neutral type and their applications "(1986) by Germanovich O.P. [40] is devoted to linear periodic equations of neutral type with a finite number of concentrated delays which are rationally commensurable with the period of coefficients. The book examines the Floquet Theory for such equations. The method to formulate a sufficient conditions for the existence of Floquet solutions is also proposed. Application of this method allows us to obtain an asymptotic representation for Floquet solutions and their multipliers, to define limit points of multipliers, to establish some properties of the system of Floquet solutions. The approach developed in this paper is illustrated by differential-difference equation that describes wave phenomena in a long line with the parametric conditions on the boundary.

In 1991 Hale J. and Verduyn L.S.M. in their work "Introduction to Functional Differential Equations" [47] have attempted to maintain the spirit of Hale's book [34]. One major change was a completely new presentation of linear systems for retarded and neutral functional differential equations. The theory of dissipative systems and global attractors was thoroughly revamped as well as the invariant manifold theory near equilibrium points and periodic orbits.

The problems of stability of neutral delay-differential system were investigated by Park J.H. and Won S. [81].

Nowadays, new results about estimates of solutions of neutral type equations are being researched and developed by such scientists as Diblík J., Baštinec J., Khusainov D.Ya., Shatyrko A., Baštincová A., Dzhalladova I.A. [21]-[30], [32],[86], [88].

1.4 Optimal dynamic systems control

The challenge of providing restrictions imposed on the movement of a dynamic system remains important task for theory and practice of management for a long time. The best-known approaches to solving this problem are based on the maximum principle and dynamic programming method of Bellman. Moreover, in these approaches, first of all, we seek the optimal control, which in addition to the optimality should also ensure some specified limits. However, the effective management of the system is not necessarily optimal, which allows to speak of a certain narrowness of these approaches. In this case, the procedure of synthesis is quite complex and is ineffective in high-dimensional system. Direct approaches to the synthesis of restrictions control on the system movement are also known. Methods of numerical synthesis (Vasiliev F.P. "Optimization Methods" [91], Gabasov R.F. and Kirillova F.M., "Constructive methods of optimization" [36], Fedorenko R.P. "Approximate solution of optimal control problems" [35], Polak E. "Optimization: Algorithms and Consistent Approximations." [83]), methods based on Lyapunov function (Kuntsevich V.M. and Lychak M.M., [66], Vorotnikov V.I. and Rumyantsev V.V. [94]), methods of inverse dynamics (Krutko P.D. [65]) may be classified as such.

The use of numerical approaches, despite their virtually unlimited applicability to a wide variety of classes of dynamical systems dependens on the construction of efficient approximate models, which is a rather complex problem by itself. In addition, the required solutions search procedure often leads to unusual or extreme problems of mixed algebraic inequalities, which have no effective solutions. Application of methods based on Lyapunov function is related to the problem of forming Lyapunov function and the solution of Lyapunov equations or inequalities. This problem is most easily solved for linear systems, and in more general cases with a fairly arbitrary constraints its solution is linked with considerable difficulties. Use of inverse dynamics methods is connected with serious difficulties due to the problem of choosing the desired motion, which must meet given limitations.

Big research about practical problems of the theory of automatical control was presented by well-known scientist Lurie A.I. in (1951) [73].

One of fundamental works in control theory is the work by one of the primary researchers Kalman R.E. [48]. This work deals with further advances of the author's recent work on optimal design of control systems and Wiener filters. Specifically, the problem of designing a system to control a plant when not all state variables are measurable, or the measured state variables are contaminated with noise, and there are random disturbances is considered. The well-known Kalman filter, also known as linear quadratic estimation (LQE) was at first presented here. The Kalman filter operates recursively on streams of noisy input data to produce a statistically optimal estimate of the underlying system state.

Numerical dynamic programming procedures are based on the Bellman's principle of optimality, which reads: "An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."

In the traditional principle of optimality of Bellman, optimality is understood in the sense of extreme value of the selected scalar criterion. However, currently most important problems can not be reduced to one-criterion formulation, so the problem of Bellman's principle of optimality and numerical schemes of dynamic programming generalization for the case of a broader interpretation of the concept of optimality is on the agenda.

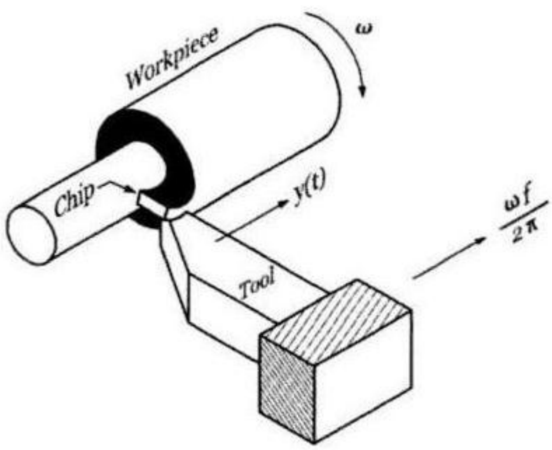
The main drawback of the approach that consists of the direct synthesis of dynamic programming method, for example, for the case of several criteria, is considered by some authors (Velichko D.A. [92], Sysoev V.V. [89]) to be the issue of proportionality and, consequently, a lack of computing resources. For example, in [92] it is shown theoretically that such approach becomes ineffective when the number of criteria is more than three due to avalanche-like increase in the number of conflicting decisions. However, the Pareto set is rarely commensurable with the total number of options, although it is easy to think of a process example in which all possible trajectories will be Pareto optimal, in the real-world conditions, such examples do not occur often. Therefore, a theoretical assessment of the difficulties presented in the case of the exhaustive search is overstated, and drawn conclusions are particular cases.

New particular results in controllability research of linear differential equations with delay are presented in works Baštinec J., Diblík J., Khusainov D.Ya., Lukáčková J., Růžičková M., Dzhalladova I.A., Piddubna G. [19], [20], [31], [54], [105]-[119].

Example
One of examples of such matrix linear differential equation is the regenerative chatter in metal cutting model [50].

Picture 1. Model of regenerative chatter

A cylindrical workpiece rotates with constant angular velocity ω and the cutting tool translates along the axis of the workpiece with constant linear velocity $\omega f/2\pi$, where f is the feed rate in length per revolution corresponding to the normal thickness of the chip removed. The tool generates a surface as the material is removed, shown as shaded, and any vibration of the tool is reflected on this surface. In the regenerative chatter, the surface generated by the previous pass becomes the upper surface of the chip on the subsequent pass.



Picture 2. Geometry of turning

This time-delay system can be described by the equation

$$m\frac{d^2y(t)}{dt^2} + c\frac{dy(t)}{dt} + ky(t) = -F_t(f + y(t) - y(t-\tau))),$$

where m, c, and k reflect the inertia, damping, and stiffness characteristics of the machine tool, the delay time $\tau = 2\pi/\omega$ corresponds to the time for the workpiece to complete one revolution, and $F_t(\cdot)$ is the thrust force depending on the instantaneous chip thickness $f + y(t) - y(t-\tau)$. It is often sufficient to consider $F_t(\cdot)$ to be linear, and techniques for linear time-delay systems are often used

$$F_t(\cdot) = a_1 f + a_2 y(t) + a_3 y(t-\tau),$$

where a_1, a_2, a_3 are constant coefficients. In this case time-delay system can be describes by the equation

$$m\frac{d^2y(t)}{dt^2} + c\frac{dy(t)}{dt} + (k + a_2)y(t) + a_3 y(t-\tau) = -a_1 f,$$

or by the matrix linear differential equation with delay

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} -\frac{c}{m} & -\frac{k+a_2}{m} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} 0 & -\frac{a_3}{m} \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x(t-\tau) \\ y(t-\tau) \end{pmatrix} + \begin{pmatrix} -\frac{a_1}{m} f \\ 0 \end{pmatrix}.$$

$\square$

2 MAIN DEFINITIONS OF THE THEORY

2.1 Definitions of the control theory

Let Z be the state space of a dynamic system, U be the set of control functions, $z = z(z_0, u, t)$ be a vector characterizing the state of the dynamical system at the instant t, starting from the initial state z_0, $z_0 \in Z$, $z_0 = z(t_0)$ and the control function u, $u \in U$. Let X denote a subspace of Z and $x = x(z_0, u, t)$ be the projection of the state vector $z(z_0, u, t)$ onto X.

Definition 2.1.1 *The state z_0 is said to be controllable in the class U (controllable state), if there exist such control u, $u \in U$ and the number T, $t_0 \leq T < \infty$ that $z(z_0, u, T) = 0$.*

Definition 2.1.2 *The state z_0 is said to be controllable in the class U with respect to a given set X (relatively controllable state), if there exist such control $u \in U$ and the number T, $t_0 \leq T < \infty$ that $x(z_0, u, T) = 0$.*

Definition 2.1.3 *If every state z_0, $z_0 \in Z$ of a dynamic system is controllable, then we say that the system is controllable (controllable system).*

Definition 2.1.4 *If every state z_0, $z_0 \in X$ of a dynamic system is relatively controllable, then we say that the system is relatively controllable (relatively controllable system).*

Consider the following Cauchy's problem:

$$\begin{aligned} &\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + Bu(t), \ \ t \in [0, T]\,, \ T < \infty, \\ &x(0) = x_0, \quad x(t) = \varphi(t), \quad -\tau \leq t < 0, \end{aligned} \tag{2.1}$$

where $x = (x_1, ..., x_n)^T$ is the phase coordinates vector, $x \in X$, $u(t) = (u_1, ..., u_r)^T$ is the control function, $u \in U$, U is the set of piecewise-continuous functions; A_0, A_1, B are constant matrices of dimensions $(n \times n)$, $(n \times n)$, $(n \times r)$ respectively, τ is the constant delay.

The state space Z of this system is the set of n-dimensional functions

$$\{x(\theta),\ t - \tau \leq \theta \leq t\} \tag{2.2}$$

The space of the n-dimensional vectors x (phase space X) is a subspace of Z. The initial state z_0 of the system (2.1) is determined by conditions

$$z_0 = \{x_0(\theta),\ x_0(\theta) = \varphi(\theta),\ -\tau \leq \theta < 0,\ x(0) = x_0\}. \tag{2.3}$$

The state $z = z(z_0, u, t)$ of the system (2.1) in the space Z at the instant t is defined by trajectory segment (2.2) of phase space X.

Below we assume that the motions of system (2.1) take place for $t \geq 0$ in the space of continuous function. The initial function $\varphi(\theta)$ is taken to be piecewise-continuous.

In accordance with specified definitions state (2.3) we have defined, the system (2.1) is controllable if there exists such control u, $u \in U$ that $x(t) \equiv 0$, $T - \tau \leq t \leq T$ when $T < \infty$.

The state (2.3) of the system (2.1) is relatively controllable if there exists such control u, $u \in U$ that $x(T) = 0$ for $T < \infty$.

Remark 2.1.5 *The notion of a relatively controllable system follow from the specific nature of differential equations with delay. In the case of the usual differential equations ($A_1 = \Theta$), the sets Z and X coincide and, consequently, the notion of a "relatively controllable state" is equivalent to the well-known* [48] *term "controllable state".*

Let $\mathrm{X}_0(t)$ is a fundamental matrix of solutions of equation (2.1) in case when $B \equiv 0$, normalized in the point t_0, mean $\mathrm{X}_0(t_0) = I$. Let us define following function

$$\omega(t) = \mathrm{X}_0(t)B = \begin{pmatrix} \omega_1(t) \\ ... \\ \omega_n(t) \end{pmatrix}, \tag{2.4}$$

where $\omega_i(t) = (\omega_{i1}(t), ..., \omega_{ir}(t))$, $i = 1, .., n$.

Theorem 2.1.6 [48] *System* (2.1) *will be relatively controllable if and only if vector functions $\omega_i(t)$, $i = 1, .., n$ are linearly independent on all time interval $t_0 \leq t \leq t_1$.*

Proof: sufficiency. Let consider two arbitrary points x_0 and x_1 from phase space X and two arbitrary points t_0 and t_1 of the argument t. Then the solution of the Cauchy problem for equation (2.1) with initial condition $x(t_0) = x_0$ will be

$$x(t) = \int_{t_0}^{t} \omega(\xi)u(\xi)d\xi + \mathrm{X}_0(t)x_0. \tag{2.5}$$

Considering condition $x(t_1) = x_1$, we say that in order for the system (2.1) to be relatively controllable, it is enough to find such control vector function $u(t)$ with which following equality is true

$$x(t_1) = \int_{t_0}^{t_1} \omega(\xi)u(\xi)d\xi + \mathrm{X}_0(t_1)x_0. \tag{2.6}$$

Let's show that this function exists and has the form

$$u_0(t) = u(t) = \omega^*(t) l_0, \tag{2.7}$$

where $\omega^*(t_1)$ is matrix conjugated to (2.4) (obtained from $\omega(t_1)$ by taking the transpose and then taking the complex conjugate of each entry), and l_0 - some n-dimension constant vector. Let us put (2.7) in (2.6) and obtain the system of linear algebraic equations for components of the vector l_0:

$$\int_{t_0}^{t_1} \omega(\xi)\omega^*(\xi) d\xi l_0 = x(t_1) - \mathrm{X}_0(t_1) x_0. \tag{2.8}$$

We now show that the determinant of system (2.8) is nonzero. To do this, let's notice that for arbitrary n-dimensional vector l ($||l|| > 0$) if the linear independence of vectors $\omega_i(t)$, $i = 1, .., n$ the following is true:

$$\int_{t_0}^{t_1} ||\omega^*(\xi)||^2 d\xi = l^* \int_{t_0}^{t_1} \omega(\xi)\omega^*(\xi) d\xi l_0 > 0, \tag{2.9}$$

which mean that the matrix

$$\int_{t_0}^{t_1} \omega(\xi)\omega^*(\xi) d\xi$$

is positive defined, and therefore from the Silvestr's condition we extract inequality

$$det \int_{t_0}^{t_1} \omega(\xi)\omega^*(\xi) d\xi > 0. \tag{2.10}$$

Thus, considering nonsingular matrix in the system of equations (2.8), we write:

$$l_0 = \left(\int_{t_0}^{t_1} \omega(\xi)\omega^*(\xi) d\xi \right)^{-1} \left[x(t_1) - \mathrm{X}_0(t_1) x_0 \right],$$

and this, together with formula (2.7) allows us to find the control function as:

$$u_o(t) = \omega^*(t) \left(\int_{t_0}^{t_1} \omega(\xi)\omega^*(\xi) d\xi \right)^{-1} \left[x(t_1) - \mathrm{X}_0(t_1) x_0 \right]$$

Proof: necessity. The proof is carried out from the opposite. Let us assume that the vector-function $\omega_i(t)$, $i = 1, .., n$ while $t_0 \leq t \leq t_1$ is linearly dependent, but a special selection of control vector-function $u(t)$ can be ensured for an arbitrary boundary conditions $x(t_0) = x_0$, $x(t_1) = x_1$ for a given trajectory of a system. From the linear dependence of vector-functions $\omega_i(t)$, $i = 1, .., n$ on the interval $t_0 \leq t \leq t_1$ followed equality:

$$l^T \omega(\xi) \equiv 0, \quad t_0 \leq \xi \leq t_1, \tag{2.11}$$

where l^T - some special n-dimensional constant vector ($||l^T|| > 0$).

Let us choose arbitrary points x_0 and x_1 using the conditional

$$l^T\left[x_1 - \mathrm{X}_0(t_1)x_0\right] \neq 0.$$

On the basis of identity (2.11) we can confirm that for an arbitrary function $u(\xi)$ next equality is correct:

$$l^T \int_{t_0}^{t_1} \omega(\xi)u(\xi)d\xi \equiv 0,$$

so

$$l^T\left[x_1 - \mathrm{X}_0(t_1)x_0\right] \neq l^T\left[x_1 - \mathrm{X}_0(t_1)x_0\right],$$

or

$$\left[x_1 - \mathrm{X}_0(t_1)x_0\right] \neq x_1 - \mathrm{X}_0(t_1)x_0.$$

Last inequality contradicts the assumption that system (2.1) is relatively controllable, because it asserts that for selected values x_0 and x_1 it is impossible to specify a vector-function $u(t)$ that would meet the condition (2.6). □

Definition 2.1.7 *Let A be a constant matrix of dimension $n \times n$. The matrix exponential is defined by*

$$e^{At} = I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \cdots = \sum_{i=0}^{\infty} A^i\frac{t^i}{i!},$$

where I is the identity matrix.

Lemma 2.1.8 *Let A be a constant matrix of dimension $n \times n$. Then*

$$Ae^{At} = e^{At}A.$$

Proof.

$$Ae^{At} = A\sum_{i=0}^{\infty} A^i\frac{t^i}{i!} = A \cdot \left(I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + A^4\frac{t^4}{4!} + A^5\frac{t^5}{5!} + \ldots\right)$$

$$= A + A^2\frac{t}{1!} + A^3\frac{t^2}{2!} + A^4\frac{t^3}{3!} + A^5\frac{t^4}{4!} + A^6\frac{t^5}{5!} + \ldots$$

$$= \left(I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + A^4\frac{t^4}{4!} + A^5\frac{t^5}{5!} + \ldots\right) \cdot A$$

$$= \sum_{i=0}^{\infty} A^i\frac{t^i}{i!} \cdot A = e^{At}A.$$

□

Lemma 2.1.9 *Let A be a constant matrix of dimension $n \times n$. Then*

$$e^{At}e^{As} = e^{A(t+s)}.$$

Proof.

$$\mathrm{e}^{At}\mathrm{e}^{As} = \sum_{i=0}^{\infty} A^i \frac{t^i}{i!} \sum_{i=0}^{\infty} A^i \frac{s^i}{i!} = \left(I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + A^4\frac{t^4}{4!} + A^5\frac{t^5}{5!} + \dots\right)$$

$$\times \left(I + A\frac{s}{1!} + A^2\frac{s^2}{2!} + A^3\frac{s^3}{3!} + A^4\frac{s^4}{4!} + A^5\frac{s^5}{5!} + \dots\right)$$

$$= I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + A^4\frac{t^4}{4!} + A^5\frac{t^5}{5!} + \dots$$

$$+A\frac{s}{1!} + A^2\frac{ts}{1!1!} + A^3\frac{t^2s}{2!1!} + A^4\frac{t^3s}{3!1!} + A^5\frac{t^4s}{4!1!} + A^6\frac{t^5s}{5!1!} + \dots$$

$$+A^2\frac{s^2}{2!} + A^3\frac{ts^2}{1!2!} + A^4\frac{t^2s^2}{2!2!} + A^5\frac{t^3s^2}{3!2!} + A^6\frac{t^4s^2}{4!2!} + A^7\frac{t^5s^2}{5!2!} + \dots$$

$$+A^3\frac{s^3}{3!} + A^4\frac{ts^3}{1!3!} + A^5\frac{t^2s^3}{2!3!} + A^6\frac{t^3s^3}{3!3!} + A^7\frac{t^4s^3}{4!3!} + \dots$$

$$+A^4\frac{s^4}{4!} + A^5\frac{ts^4}{1!4!} + A^6\frac{t^2s^4}{2!4!} + A^7\frac{t^3s^4}{3!4!} + \dots$$

$$+A^5\frac{s^5}{5!} + A^6\frac{ts^5}{1!5!} + A^7\frac{t^2s^5}{2!5!} + \dots = (*)$$

Now we reorder the sum in accordance with power of matrices.

$$(*) = \sum_{n=0}^{\infty} A^n \left(\frac{t^n}{n!} + \frac{t^{n-1}s}{(n-1)!1!} + \frac{t^{n-2}s^2}{(n-2)!2!} + \dots + \frac{t^2s^{n-2}}{2!(n-2)!} + \frac{ts^{n-1}}{1!(n-1)!} + \frac{s^n}{n!}\right)$$

$$= \sum_{n=0}^{\infty} \frac{A^n}{n!} \left(\frac{n!t^n}{n!} + \frac{n!t^{n-1}s}{(n-1)!1!} + \frac{n!t^{n-2}s^2}{(n-2)!2!} + \dots + \frac{n!t^2s^{n-2}}{2!(n-2)!} + \frac{n!ts^{n-1}}{1!(n-1)!} + \frac{n!s^n}{n!}\right)$$

$$= \sum_{n=0}^{\infty} \frac{A^n}{n!} \left(C_n^0 t^n + C_n^1 t^{n-1}s + C_n^2 t^{n-2}s^2 + \dots + C_n^{n-2} t^2 s^{n-2} + C_n^{n-1} t s^{n-1} + C_n^n s^n\right)$$

$$= \sum_{n=0}^{\infty} \frac{A^n}{n!}(t+s)^n = \mathrm{e}^{A(t+s)},$$

where $C_n^k = \dfrac{n!}{(n-k)!k!}$. □

Lemma 2.1.10 *Let matrices A_0 and A_1 be commutative, (i.e. $A_0A_1 = A_1A_0$). Then*

$$\mathrm{e}^{A_0t}A_1 = A_1e^{A_0t}, \quad t \geq 0.$$

Proof.

$$\mathrm{e}^{A_0t}A_1 = \sum_{i=0}^{\infty} A_0^i \frac{t^i}{i!}A_1 = \left(I + A_0\frac{t}{1!} + A_0^2\frac{t^2}{2!} + A_0^3\frac{t^3}{3!} + A_0^4\frac{t^4}{4!} + A_0^5\frac{t^5}{5!} + \dots\right)A_1$$

$$= A_1 + A_0A_1\frac{t}{1!} + A_0^2A_1\frac{t^2}{2!} + A_0^3A_1\frac{t^3}{3!} + A_0^4A_1\frac{t^4}{4!} + A_0^5A_1\frac{t^5}{5!} + \dots$$

$$= A_1 + A_1A_0\frac{t}{1!} + A_1A_0^2\frac{t^2}{2!} + A_1A_0^3\frac{t^3}{3!} + A_1A_0^4\frac{t^4}{4!} + A_1A_0^5\frac{t^5}{5!} + \dots$$

$$= A_1\left(I + A_0\frac{t}{1!} + A_0^2\frac{t^2}{2!} + A_0^3\frac{t^3}{3!} + A_0^4\frac{t^4}{4!} + A_0^5\frac{t^5}{5!} + \dots\right) = A_1\sum_{i=0}^{\infty} A_0^i\frac{t^i}{i!} = A_1\mathrm{e}^{A_0t}.$$

□

Definition 2.1.11 *Delayed matrix exponential is a matrix function which has the form of a polynomial of degree k in intervals $(k-1)\tau \leq t \leq k\tau$, "glued" in knots $t = k\tau$, $k = 0, 1, 2, \dots$:*

$$e_\tau^{At} = \begin{cases} \Theta, & -\infty < t < -\tau \\ I, & -\tau \leq t < 0 \\ I + A\frac{t}{1!} + A^2\frac{(t-\tau)^2}{2!} + \dots + A^k\frac{(t-(k-1)\tau)^k}{k!}, & (k-1)\tau \leq t < k\tau, \quad k = 1, 2, \dots \end{cases}$$

where Θ is zero matrix.

Delayed matrix exponential was at first defined in [54].

Theorem 2.1.12 *Delayed matrix exponential is the fundamental matrix of solutions of the matrix differential equation with pure delay*

$$\dot{x}(t) = Ax(t-\tau), \quad 0 < \tau < t,$$

where A is a constant matrix of dimension $n \times n$.

Proof. To prove the theorem statement let us differentiate the delayed matrix exponential

$$\left(e_\tau^{At}\right)' = \left(I + A\frac{t}{1!} + A^2\frac{(t-\tau)^2}{2!} + \dots + A^k\frac{(t-(k-1)\tau)^k}{k!}\right)'$$

$$= \Theta + A\frac{1}{1!} + A^2\frac{(t-\tau)^1}{1!} + \dots + A^k\frac{(t-(k-1)\tau)^{k-1}}{(k-1)!}$$

$$= A\left(I + A\frac{(t-\tau)^1}{1!} + \dots + A^{k-1}\frac{(t-(k-1)\tau)^{k-1}}{(k-1)!}\right) = Ae_\tau^{A(t-\tau)}.$$

This means that the delayed matrix exponential is the fundamental matrix of solutions for differential equation with pure delay. □

2.2 Definitions of the stability theory

Consider an autonomous nonlinear dynamical system

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0 \tag{2.12}$$

where $x(t) \in D \in R^n$ denotes the system state vector, D is an open set containing the origin, and $f : D \to R^n$ is continuous on D. Suppose (2.12) has a solution $\varphi(t)$.

Definition 2.2.1 *The solution $\varphi(t)$ of the system* (2.12) *is said to be Lyapunov's stable, if, for each $\varepsilon > 0$, there exists $\delta = \delta(e) > 0$ such that for every other solution $x(t)$ if*

$$||x(t_0) - \varphi(t_0)|| < \delta,$$

then for each $t \geq 0$

$$||x(t) - \varphi(t)|| < \varepsilon,$$

where $||\cdot||$ is a norm.

Definition 2.2.2 *The solution $\varphi(t)$ of the system* (2.12) *is said to be asymptotically stable if it is Lyapunov's stable and if there exists $\delta > 0$ such that for every other solution $x(t)$ if*

$$||x(t_0) - \varphi(t_0)|| < \delta,$$

then for each $t \geq 0$

$$\lim_{x \to \infty} ||x(t) - \varphi(t)|| = 0.$$

Definition 2.2.3 *The solution $\varphi(t)$ of the system* (2.12) *is said to be exponentially stable if it is asymptotically stable and if there exist positive constants α, β, δ such that for every other solution $x(t)$ if*

$$||x(t_0) - \varphi(t_0)|| < \delta,$$

then for each $t \geq 0$

$$||x(t) - \varphi(t)|| \leq \alpha ||x(t_0) - \varphi(t_0)|| e^{-\beta t}.$$

Remark 2.2.4 *The stability investigation of an arbitrary solution $\varphi(t)$ can be easy reduced to the stability investigation of a zero solution $\dot{y}(t) \equiv 0$ using a simple substitute $x(t) = y(t) + \varphi(t)$, where $y(t)$ is a new unknown function.*

Conceptually, the meanings of the above terms are the following:

1. The Lyapunov's stability of an equilibrium means that solutions starting "close enough" to the equilibrium (within a distance δ from it) remain "close enough" forever (within a distance ε from it). Note that this must be true for any ε that one may want to choose.

2. The asymptotic stability means that solutions that start close enough not only remain close enough but also eventually converge to the equilibrium.

3. The exponential stability means that solutions not only converge, but in fact converge faster than or at least as fast as a particular known rate

$$\alpha||x(t_0) - \varphi(t_0)||e^{-\beta t}.$$

Lyapunov's second method for stability

Lyapunov A.M. (6. 6. 1857 - 3. 11. 1918), in his original work "General problem of stability of motion" (1892) [74] proposed two methods to demonstrate the stability. The first method developed the solution in a series which was then proved convergent within limits. The second method, which is almost universally used nowadays, makes use of a Lyapunov's functional $V(x)$ which has an analogy to the potential function of classical dynamics. It is introduced as follows.

Definition 2.2.5 *Consider a functional $V(x) : R^n \to R$ such that:*

1. $V(x) \geq 0$ *with equality if and only if* $x = 0$ *(positive definite)*
2. $\dot{V}(x) = \dfrac{dV(x)}{dt} \leq 0$ *with equality if and only if* $x = 0$ *(negative definite).*

Then $V(x)$ is called a Lyapunov's functional.

Theorem (First Lyapunov's theorem) *If there exists a positive definite Lyapunov's functional V(x) with a negative definite first derivative along the trajectories of a system of differential equations, then the solution of the system of differential equations is stable.*

Theorem (Second Lyapunov's theorem) *If there exists a positive definite Lyapunov's functional V(x) such that for the derivation along the trajectories of a system of differential equations*

$$\dot{V}(x) = \frac{dV(x)}{dt} \leq W(x) < 0,$$

where $W(x)$ is some bounded function, then the solution of the system of differential equations is asymptotically stable.

It is easier to visualize that method of analysis by thinking of a physical system (e.g. the vibrating string and the mass) and considering the energy of such a system. If the system loses energy over time and the energy is never restored, then eventually the system must grind to a stop and reach some final resting state. The final state is called the attractor. However, finding a function that gives the precise energy of a physical system can be difficult, and for abstract mathematical systems, economic systems or biological systems, the concept of energy may not be applicable.

Lyapunov's discovery was that stability can be proven without requiring knowledge of the true physical energy if Lyapunov functional can be found to satisfy the above constraints.

3 REPRESENTATION OF THE SOLUTION

3.1 Systems with same matrices

Let us consider the following Cauchy problem

$$\dot{x}(x) = Ax(t) + Ax(t-\tau) + f(t), \quad t \geq 0, \tag{3.1}$$

$$x(t) = \varphi(t), \quad -\tau \leq t \leq 0, \tag{3.2}$$

where $x(t) = (x_1(t), x_2(t), .., x_n)^T$ is vector of states of the system,
$f(t) = (f_1(t), f_2(t), .., f_n(t))^T$ is known function of disturbance, A is constant matrix of dimension $(n \times n)$, $\tau > 0, \tau \in R$ is a constant delay.

To solve Cauchy problem (3.1) - (3.2) let us find the fundamental matrix of solution of this equation. Fundamental matrix would be a solution of the following matrix equation

$$\dot{X}(t) = AX(t) + AX(t-\tau), \quad t \geq 0, \tag{3.3}$$

with initial condition

$$X(t) = I, \qquad -\tau \leq t \leq 0, \tag{3.4}$$

where $X(t)$ is a matrix of type $n \times n$ and I is the identity matrix.

Theorem 3.1.1 [99] *The solution of equation* (3.3) *with identity initial condition* (3.4) *has the recurrent form:*

$$X_{k+1}(t) = \mathrm{e}^{A(t-k\tau)} X_k(k\tau) + \int_{k\tau}^{t} \mathrm{e}^{A(t-s)} A X_k(s-\tau) ds,$$

where $X_k(t)$ is defined on the interval $(k-1)\tau \leq t \leq k\tau$, $k = 0, 1, ...$

Proof. Let us have the solution $X_k(t)$ of the equation (3.3) on the time interval $(k-1)\tau \leq t \leq k\tau$. Then, equation (3.3) on the next time interval is

$$\dot{X}_{k+1}(t) = AX_{k+1}(t) + AX_{k+1}(t-\tau)$$

and, because on time interval $(k-1)\tau \leq t \leq k\tau$ we have $X_{k+1}(t) = X_k(t)$, the last equation can be rewritten as

$$\dot{X}_{k+1}(t) = AX_{k+1}(t) + AX_k(t-\tau).$$

So now we get the non-homogeneous equation with unknown function $X_{k+1}(t)$ and function $AX_k(t-\tau)$ is a know function.

According to the theory of ordinary differential equations, the solution of non-homogeneous equation $\dot{x}(t) = Ax(t) + f(t)$ have the solution in the form

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-s)}f(s)ds.$$

As far as we have $f(t) = Ax(t-\tau)$, on every time interval $(k-1)\tau \le t \le k\tau$ we have following recurrent form for solution

$$X_{k+1}(t) = \mathrm{e}^{A(t-k\tau)}X_k(k\tau) + \int_{k\tau}^{t} \mathrm{e}^{A(t-s)}AX_k(s-\tau)ds.$$

□

Theorem 3.1.2 [99] *The fundamental matrix of solutions of equation* (3.3) *has the form:*

$$\mathrm{X}_0 = \begin{cases} \Theta, & -\infty \le t < -\tau \\ I, & -\tau \le t < 0 \\ 2e^{At} - I, & 0 \le t \le \tau \\ 2e^{At} + 2e^{A(t-\tau)}\left(A(t-\tau) - I\right) + I, & \tau \le t \le 2\tau \\ \dots & \\ \sum\limits_{m=0}^{k-1} 2\mathrm{e}^{A(t-m\tau)} \sum\limits_{p=0}^{m} (-1)^{p+m} A^p \frac{(t-m\tau)^p}{p!} + (-I)^k, & (k-1)\tau \le t < k\tau, \\ & k = 3, 4, \dots \end{cases} \tag{3.5}$$

Proof. We proved Theorem 3.1.2 using mathematical induction method.
1. Let $0 \le t \le \tau$. Then the equation (3.3) has the form

$$\dot{X}_1(t) = AX_1(t) + AX_1(t-\tau) = AX_1(t) + AX_0(t-\tau).$$

Because $X_0(t-\tau) = I$ for $0 \le t \le \tau$, we have

$$\dot{X}_1(t) = AX_1(t) + A.$$

Then from the statement of the Theorem 3.1.1 follows that for the solution of equation (3.3) on this interval holds for $n = 0$

$$X_1(t) = e^{At}X_0(0) + \int_0^t e^{A(t-s)}AX_0(s-\tau)ds = e^{At}I + \int_0^t e^{A(t-s)}AIds$$

$$= e^{At} + A\int_0^t e^{A(t-s)}ds = e^{At} - A\sum_{i=0}^{\infty} A^i \frac{(t-s)^{i+1}}{(i+1)!}\Bigg|_0^t = e^{At} + e^{At} - I.$$

Finally, we have

$$X_1(t) = 2e^{At} - I.$$

2. Let $\tau \le t \le 2\tau$. Then the equation (3.3) has the form

$$\dot{X}_2(t) = AX_2(t) + AX_2(t-\tau) = AX_2(t) + AX_1(t-\tau),$$

because $X_2(t-\tau) = X_1(t-\tau)$ for $\tau \le t \le 2\tau$. Then from the statement of the Theorem 3.1.1 there follows that for the solution of equation (3.3) on this interval there holds for $n = 1$

$$X_2(t) = e^{A(t-\tau)}X_1(\tau) + \int_\tau^t e^{A(t-s)}AX_1(s-\tau)ds.$$

After substitution $X_1(t)$ we have

$$X_2(t) = e^{A(t-\tau)}\left[2e^{A\tau} - I\right] + \int_\tau^t e^{A(t-s)}A\left[2e^{A(s-\tau)} - I\right]ds$$

$$= 2e^{A(t-\tau)}e^{A\tau} - e^{A(t-\tau)} + 2A\int_\tau^t e^{A(t-s)}e^{A(s-\tau)}ds - A\int_\tau^t e^{A(t-s)}ds = (*).$$

Using results of Lemma 2.1.8 we can write

$$(*) = 2e^{At} - e^{A(t-\tau)} + 2A\int_\tau^t e^{A(t-\tau)}ds - A\int_\tau^t e^{A(t-s)}ds$$

$$= 2e^{At} - e^{A(t-\tau)} + 2Ae^{A(t-\tau)}(t-\tau) + I - e^{A(t-\tau)}$$

$$= 2e^{At} - 2e^{A(t-\tau)} + 2Ae^{A(t-\tau)}(t-\tau) + I = 2e^{At} + 2e^{A(t-\tau)}\left(A(t-\tau) - I\right) + I.$$

So we have

$$X_2(t) = 2e^{At} + 2e^{A(t-\tau)}\left(A(t-\tau) - I\right) + I.$$

k. Let $(k-1)\tau \le t \le k\tau$.
Assumption: Let for k holds

$$X_k(t) = \sum_{m=0}^{k-1} 2e^{A(t-m\tau)} \sum_{p=0}^{m} (-1)^{p+m} A^p \frac{(t-m\tau)^p}{p!} + (-I)^k.$$

$\mathbf{k+1}$. Then for $k+1$ we get $k\tau \le t \le (k+1)\tau$ and the equation (3.3) has the form

$$\dot{X}_{k+1}(t) = AX_{k+1}(t) + AX_{k+1}(t-\tau) = AX_{k+1}(t) + AX_k(t-\tau).$$

Then from the Theorem 3.1.1 there follows that for the solution of equation (3.3) on this interval holds for $n = k$

$$X_{k+1}(t) = e^{A(t-k\tau)}X_k(k\tau) + \int_{k\tau}^t e^{A(t-s)}AX_k(s-\tau)ds.$$

After substitution $X_k(t)$ from (3.5) we have

$$X_{k+1}(t) = e^{A(t-k\tau)}\left[\sum_{m=0}^{k-1} 2e^{A(k\tau-m\tau)}\sum_{p=0}^{m}(-1)^{p+m}A^p\frac{(k\tau-m\tau)^p}{p!}+(-I)^k\right]$$

$$+\int_{k\tau}^{t} e^{A(t-s)}A\left[\sum_{m=0}^{k-1} 2e^{A(s-\tau-m\tau)}\sum_{p=0}^{m}(-1)^{p+m}A^p\frac{(s-\tau-m\tau)^p}{p!}+(-I)^k\right]ds$$

$$=\sum_{m=0}^{k-1} e^{A(t-k\tau)}2e^{A(k\tau-m\tau)}\sum_{p=0}^{m}(-1)^{p+m}A^p\frac{(k\tau-m\tau)^p}{p!}+(-1)^k e^{A(t-k\tau)}$$

$$+\sum_{m=0}^{k-1}\int_{k\tau}^{t} e^{A(t-s)}A2e^{A(s-\tau-m\tau)}\sum_{p=0}^{m}(-1)^{p+m}A^p\frac{(s-\tau-m\tau)^p}{p!}ds$$

$$+(-1)^k\int_{k\tau}^{t} e^{A(t-s)}Ads = (*)$$

Using the result of Lemma 2.1.8 we write

$$(*) = \sum_{m=0}^{k-1} 2e^{A(t-m\tau)}\sum_{p=0}^{m}(-1)^{p+m}A^p\frac{(k\tau-m\tau)^p}{p!}+(-1)^k e^{A(t-k\tau)}$$

$$+\sum_{m=0}^{k-1}\int_{k\tau}^{t} 2e^{A(t-(m+1)\tau)}\sum_{p=0}^{m}(-1)^{p+m}A^{p+1}\frac{(s-(m+1)\tau)^p}{p!}ds+(-1)^{k+1}\left[e^{A(t-s)}-I\right]\Big|_{k\tau}^{t}$$

$$=\sum_{m=0}^{k-1} 2e^{A(t-m\tau)}\sum_{p=0}^{m}(-1)^{p+m}A^p\frac{(k\tau-m\tau)^p}{p!}+(-1)^k e^{A(t-k\tau)}$$

$$+\sum_{m=0}^{k-1} 2e^{A(t-(m+1)\tau)}\sum_{p=0}^{m}(-1)^{p+m}A^{p+1}\int_{k\tau}^{t}\frac{(s-(m+1)\tau)^p}{p!}ds$$

$$-(-1)^{k+1}e^{A(t-k\tau)}+(-1)^{k+1}I$$

$$=\sum_{m=0}^{k-1} 2e^{A(t-m\tau)}\sum_{p=0}^{m}(-1)^{p+m}A^p\frac{(k\tau-m\tau)^p}{p!}+(-1)^k 2e^{A(t-k\tau)}+(-I)^{k+1}$$

$$+\sum_{m=0}^{k-1} 2e^{A(t-(m+1)\tau)}\sum_{p=0}^{m}(-1)^{p+m}A^{p+1}\frac{(t-(m+1)\tau)^{p+1}}{(p+1)!}$$

$$-\sum_{m=0}^{k-1} 2e^{A(t-(m+1)\tau)}\sum_{p=0}^{m}(-1)^{p+m}A^{p+1}\frac{(k\tau-(m+1)\tau)^{p+1}}{(p+1)!}=(**)$$

Now we change the bound of sums

$$(**) = \sum_{m=0}^{k-1} 2e^{A(t-m\tau)} \sum_{p=0}^{m} (-1)^{p+m} A^p \frac{(k\tau - m\tau)^p}{p!} + (-1)^k 2e^{A(t-k\tau)} + (-I)^{k+1}$$

$$+ \sum_{m=1}^{k} 2e^{A(t-m\tau)} \sum_{p=1}^{m} (-1)^{p+m} A^p \frac{(t - m\tau)^p}{p!}$$

$$- \sum_{m=1}^{k-1} 2e^{A(t-m\tau)} \sum_{p=1}^{m} (-1)^{p+m} A^p \frac{(k\tau - m\tau)^p}{p!} = (***)$$

Then we group elements as follows

$$(***) = 2e^{At} + \sum_{m=1}^{k-1} 2e^{A(t-m\tau)} (-1)^m + \sum_{m=1}^{k} 2e^{A(t-m\tau)} \sum_{p=1}^{m} (-1)^{p+m} A^p \frac{(t - m\tau)^p}{p!}$$

$$+ (-1)^k 2e^{A(t-k\tau)} + (-I)^{k+1}$$

$$= \left(2e^{At} + \sum_{m=1}^{k-1} 2e^{A(t-m\tau)} \sum_{p=0}^{m} (-1)^{p+m} A^p \frac{(t - m\tau)^p}{p!} \right)$$

$$+ \left(2e^{A(t-k\tau)} \sum_{p=0}^{m} (-1)^{p+k} A^p \frac{(t - k\tau)^p}{p!} \right) + (-I)^{k+1}$$

$$= \sum_{m=0}^{k-1} 2e^{A(t-m\tau)} \sum_{p=0}^{m} (-1)^{p+m} A^p \frac{(t - m\tau)^p}{p!}$$

$$+ 2e^{A(t-k\tau)} \sum_{p=0}^{m} (-1)^{p+k} A^p \frac{(t - k\tau)^p}{p!} + (-I)^{k+1}.$$

Finally, we get

$$X_{k+1}(t) = \sum_{m=0}^{k} 2e^{A(t-m\tau)} \sum_{p=0}^{m} (-1)^{p+m} A^p \frac{(t - m\tau)^p}{p!} + (-I)^{k+1}.$$

And we got the expression (3.5) for fundamental matrix of solutions for time interval $k\tau \leq t \leq (k+1)\tau$. $\square$

Example 3.1.1
Let us have the system of differential equations with a constant delay:

$$\dot{x}(t) = Ax(t) + Ax(t-1),$$

where $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. So we have $n = 3$, $\tau = 1$.
Using definition of the matrix exponential we can write

$$e^{A(t-m\tau)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{(t-m)^1}{1!} + \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 \frac{(t-m)^2}{2!} + \ldots$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{(t-m)^1}{1!} + \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{(t-m)^2}{2!} + \ldots$$

$$= \begin{pmatrix} \sum\limits_{i=0}^{\infty} \frac{(t-m)^i}{i!} & 0 & (t-m)\sum\limits_{i=0}^{\infty} \frac{(t-m)^i}{i!} \\ 0 & \sum\limits_{i=0}^{\infty} \frac{(t-m)^i}{i!} & 0 \\ 0 & 0 & \sum\limits_{i=0}^{\infty} \frac{(t-m)^i}{i!} \end{pmatrix}$$

$$= \begin{pmatrix} e^{t-m} & 0 & (t-m)e^{t-m} \\ 0 & e^{t-m} & 0 \\ 0 & 0 & e^{t-m} \end{pmatrix}.$$

Now we could write the solution of the system. According to (3.5) for $0 \le t \le 1$ there follows:

$$X_1(t) = 2\begin{pmatrix} e^t & 0 & te^t \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2e^t - 1 & 0 & 2te^t \\ 0 & 2e^t - 1 & 0 \\ 0 & 0 & 2e^t - 1 \end{pmatrix}.$$

Again according the (3.5) for $1 \le t \le 2$ follows:

$$X_2(t) = \begin{pmatrix} 2e^t + 2(t-2)e^{t-1} + 1 & 0 & 2te^t + 2(t-1)^2 e^{t-1} \\ 0 & 2e^t + 2(t-2)e^{t-1} + 1 & 0 \\ 0 & 0 & 2e^t + 2(t-2)e^{t-1} + 1 \end{pmatrix}.$$

And finally for $(k-1)\tau \le t \le k\tau$ according to (3.5) there follows that

$$X_k(t) = \sum_{m=0}^{k-1}\sum_{p=o}^{m} 2e^{t-m} \begin{pmatrix} (-1)^{p+m}\frac{(t-m)^p}{p!} & 0 & t(-1)^{p+m}\frac{(t-m)^p}{(p-1)!} \\ 0 & (-1)^{p+m}\frac{(t-m)^p}{p!} & 0 \\ 0 & 0 & (-1)^{p+m}\frac{(t-m)^p}{p!} \end{pmatrix}$$

$$+\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}^k .$$

□

Remark 3.1.3 *Maple software was used in this and following examples.*

Theorem 3.1.4 [99] *The solution of homogeneous equation for equation* (3.1) *(mean $f(t) \equiv 0$) with initial condition* (3.2) *have the form:*

$$x(t) = \mathrm{X}_0(t)\varphi(-\tau) + \int_{-\tau}^{0} \mathrm{X}_0(t-\tau-s)\varphi'(s)ds,$$

where $\mathrm{X}_0(t)$ *is the fundamental solutions matrix* (3.5)*.*

Proof. Solution of the system (3.1), when $f(t) \equiv 0$, which satisfies the initial conditions $x(t) \equiv \varphi(t)$, $-\tau \le t \le 0$, can be described in the form

$$x(t) = \mathrm{X}_0(t)c + \int_{-\tau}^{0} \mathrm{X}_0(t-\tau-s)y'(s)ds, \tag{3.6}$$

where c is the vector of unknown constants, $y(t)$ is an unknown continuously differentiable vector-function and $\mathrm{X}_0(t)$ is the matrix defined in (3.5). Since the matrix $\mathrm{X}_0(t)$ is the fundamental matrix of solutions of system (3.3), then, for any c and $y(t)$, expression (3.6) is a solution of system (3.1) with condition $f(t) \equiv 0$. We choose c and $y(t)$ such that the initial conditions is in the next form

$$x(t) = \mathrm{X}_0(t)c + \int_{-\tau}^{0} \mathrm{X}_0(t-\tau-s)y'(s)ds \equiv \varphi(t).$$

Let put $t = -\tau$. From (3.5) there follows that

$$\mathrm{X}_0(-\tau) = I, \quad \mathrm{X}_0(-2\tau - s) = \begin{cases} \Theta, & -\tau < s \le 0, \\ I, & s = -\tau. \end{cases}$$

So we have $\varphi(-\tau) = c$, and formula (3.6) takes the form

$$x(t) = \mathrm{X}_0(t)\varphi(-\tau) + \int_{-\tau}^{0} \mathrm{X}_0(t-\tau-s)y'(s)ds.$$

Since $-\tau \le t \le 0$, let us divide the interval in two parts. Getting

$$\varphi(t) = \varphi(-\tau) + \int_{-\tau}^{t} \mathrm{X}_0(t-\tau-s)y'(s)ds + \int_{t}^{0} \mathrm{X}_0(t-\tau-s)y'(s)ds.$$

In the first integral $-\tau \le s \le t$, so $-\tau \le t-\tau-s \le t$ and late matrix exponential equals

$$\mathrm{X}_0(t-\tau-s) \equiv I, \quad -\tau \le s \le t.$$

In the second integral $t \le s \le 0$, so $t-\tau \le t-\tau-s \le -\tau$ and late matrix exponential is equal

$$\mathrm{X}_0(t-\tau-s) = \begin{cases} \Theta, & 0 \le s < t, \\ I, & s = t. \end{cases}$$

Hence in the interval $-\tau \le t \le 0$ we get

$$\varphi(-\tau) + \int\limits_{-\tau}^{t} y'(s)ds = \varphi(t). \tag{3.7}$$

We get

$$\varphi(-\tau) + y(t) - y(-\tau) = \varphi(t). \tag{3.8}$$

Solving the system of equations (3.7), (3.8), we obtain that $y(t) = \varphi(t)$. Substituting this in (3.6), we obtain the statement of the theorem. □

Theorem 3.1.5 [99] *The solution of the heterogeneous equation* (3.1) *with zero initial condition* $x(t) \equiv 0, -\tau < t < 0$, *has the form*

$$x(t) = \int_0^t \mathrm{X}_0(t-\tau-s)f(s)ds, \quad t \ge 0,$$

where $\mathrm{X}_0(t)$ *is the fundamental solutions matrix* (3.5).

Proof. Since $\mathrm{X}_0(t)$ is the solution of the homogeneous system (3.3), using the method of variation of arbitrary constant, the solution $x(t)$ of the heterogeneous system will have the form

$$x(t) = \int\limits_0^t \mathrm{X}_0(t-\tau-s)c(s)ds,$$

where $c(s)$, $0 \le s \le t$ is an unknown vector-function. According to Leibniz integral rule differential of the expression will be

$$\dot{x}(t) = \mathrm{X}_0(t-\tau-s)c(s)\big|_{s=t} + \int\limits_0^t \frac{\partial \mathrm{X}_0'(t-\tau-s)}{\partial t} c(s)ds$$

$$= \mathrm{X}_0(-\tau)c(t) + \int_0^t \left[A \sum_{m=0}^{k-1} 2\mathrm{e}^{A(t-\tau-s-m\tau)} \sum_{p=0}^{m} (-1)^{p+m} A^p \frac{(t-\tau-s-m\tau)^p}{p!} \right.$$

$$+\sum_{m=1}^{k-1} 2\mathrm{e}^{A(t-\tau-s-m\tau)}\sum_{p=0}^{m-1}(-1)^{p+m+1}A^{p+1}\frac{(t-\tau-s-m\tau)^p}{p!}\Bigg] c(s)ds$$

$$= \mathrm{X}_0(-\tau)c(t) + \int_0^t A\mathrm{X}_0(t-\tau-s)c(s)ds$$

$$+\int_0^t A\left[\sum_{m=1}^{k-2} 2\mathrm{e}^{A(t-2\tau-s-m\tau)}\sum_{p=0}^{m}(-1)^{p+m}A^p\frac{(t-2\tau-s-m\tau)^p}{p!} + (-I)^{k+1}\right] c(s)ds$$

$$= \mathrm{X}_0(-\tau)c(t) + \int_0^t A\mathrm{X}_0(t-\tau-s)c(s)ds + \int_0^t A\mathrm{X}_0(t-2\tau-s)c(s)ds.$$

After substitution in (3.1), we get

$$\mathrm{X}_0(-\tau)c(t) + \int_0^t A\mathrm{X}_0(t-\tau-s)c(s)ds + \int_0^t A\mathrm{X}_0(t-2\tau-s)c(s)dsc(s)ds$$

$$= A\left[\int_0^t \mathrm{X}_0(t-\tau-s)c(s)ds\right] + A\left[\int_0^{t-\tau} \mathrm{X}_0(t-2\tau-s)c(s)ds\right] + f(t).$$

Since $\mathrm{X}_0(-\tau) = I$, we get

$$c(t) + A_1\int_{t-\tau}^t \mathrm{X}_0(t-2\tau-s)c(s)ds = f(t),$$

$$\mathrm{X}_0(t-2\tau-s) = \begin{cases} \Theta, & t-\tau < s \le t, \\ I, & s = t-\tau, \end{cases}$$

then we get $c(t) = f(t)$. Hence the statement of the theorem follows. □

Theorem 3.1.6 [99] *The solution of the heterogeneous equation* (3.1) *with the initial condition* (3.2) *has the form*

$$x(t) = \mathrm{X}_0(t)\varphi(-\tau) + \int_{-\tau}^0 \mathrm{X}_0(t-\tau-s)\varphi'(s)ds + \int_0^t \mathrm{X}_0(t-\tau-s)f(s)ds, \quad (3.9)$$

where $\mathrm{X}_0(t)$ *is the fundamental solutions matrix* (3.5).

Proof. The proof of the theorem follows from theorems 3.1.4 and 3.1.5. □

Example 3.1.2
Let us have the system of differential equations with a constant delay:

$$\dot{x}(t) = Ax(t) + Ax(t-1) + f(t),$$

where $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\quad f(t) = \begin{pmatrix} 0 \\ e^{t+1} \\ 2e^{t+1} \end{pmatrix}$,
with initial conditions $\varphi(t) = (1,2,0)^T$, $-1 \le t \le 0$. So we have $n = 3$, $\tau = 1$. Using the fundamental solutions matrix from the Example 3.1.1 we can use the formula (3.9). Then for $(k-1)\tau \le t \le k\tau$ follows

$$x(t) = \mathrm{X}_0(t)\varphi(-\tau) + \int\limits_{-\tau}^{0} \mathrm{X}_0(t-\tau-s)\varphi'(s)ds + \int\limits_{0}^{t} \mathrm{X}_0(t-\tau-s)f(s)ds$$

$$= \mathrm{X}_0(t)\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \int\limits_{-1}^{0} \mathrm{X}_0(t-1-s)\cdot 0ds + \int\limits_{0}^{t} \mathrm{X}_0(t-1-s)\begin{pmatrix} 0 \\ e^{s+1} \\ 2e^{s+1} \end{pmatrix} ds$$

$$= \sum_{m=0}^{k-1}\sum_{p=o}^{m} 2e^{t-m}(-1)^{p+m}\begin{pmatrix} \dfrac{(t-m)^p}{p!} \\ 2\dfrac{(t-m)^p}{p!} \\ 0 \end{pmatrix} + \begin{pmatrix} (-1)^k \\ 2(-1)^k \\ 0 \end{pmatrix}$$

$$+ \sum_{m=0}^{k-1}\sum_{p=o}^{m} 2e^{t-m}(-1)^{p+m}\int\limits_{0}^{t}\begin{pmatrix} (t-1-s)\dfrac{(t-1-s-m)^p}{p!} \\ \dfrac{(t-1-s-m)^p}{p!} \\ 2\dfrac{(t-1-s-m)^p}{p!} \end{pmatrix} ds$$

$$+ \int\limits_{0}^{t}\begin{pmatrix} 0 \\ (-1)^k e^{s+1} \\ 2(-1)^k e^{s+1} \end{pmatrix} ds$$

$$= \sum_{m=0}^{k-1}\sum_{p=o}^{m} 2e^{t-m}(-1)^{p+m}\begin{pmatrix} \dfrac{(t-m)^p}{p!} \\ 2\dfrac{(t-m)^p}{p!} \\ 0 \end{pmatrix}$$

$$+ \sum_{m=0}^{k-1}\sum_{p=o}^{m} 2e^{t-m}(-1)^{p+m}\begin{pmatrix} \frac{(t-2)((t-1-m)^{p+1}-(-1-m)^{p+1})}{(p+1)!} + \frac{(t-1-m)^{p+2}-(-1-m)^{p+2})}{(p+2)!} \\ \frac{(t-1-m)^{p+1}}{(p+1)!} - \frac{(-1-m)^{p+1}}{(p+1)!} \\ 2\frac{(t-1-m)^{p+1}}{(p+1)!} - 2\frac{(-1-m)^{p+1}}{(p+1)!} \end{pmatrix}$$

$$+ \begin{pmatrix} (-1)^k \\ (-1)^k(e^{t+1}-e+2) \\ 2(-1)^k(e^{t+1}-e) \end{pmatrix}.$$

□

3.2 Systems with commutative matrices

Let us consider the following Cauchy problem

$$\dot{x}(x) = A_0 x(t) + A_1 x(t-\tau) + f(t), \quad t \geq 0, \tag{3.10}$$

$$x(t) = \varphi(t), \quad -\tau \leq t \leq 0, \tag{3.11}$$

where $x(t) = (x_1(t), x_2(t), .., x_n)^T$ is a vector of states of the sysem,
$f(t) = (f_1(t), f_2(t), .., f_n(t))^T$ is known function of disturbance, A_0, A_1 are commutative constant matrices of dimensions $(n \times n)$, $\tau > 0$, $\tau \in R$ is a constant delay.

To solve Cauchy problem (3.10) - (3.11) let us find the fundamental matrix of solution of this equation. Fundamental matrix would be a solution of matrix equation

$$\dot{X}(t) = A_0 X(t) + A_1 X(t-\tau), \quad t \geq 0, \tag{3.12}$$

with initial condition

$$X(t) = I, \qquad -\tau \leq t \leq 0, \tag{3.13}$$

where $X(t) \in R^{n \times n}$, I is identity matrix.

Now let us obtain the explicit form of the fundamental matrix of the system (3.12) for commutative matrices A_0, A_1.

Theorem 3.2.1 [99] *The solution of equation* (3.12) *with identity initial condition* (3.13) *has the recurrent form:*

$$X_{k+1}(t) = \mathrm{e}^{A_0(t-k\tau)} X_k(k\tau) + \int_{k\tau}^{t} \mathrm{e}^{A_0(t-s)} A_1 X_k(s-\tau) ds,$$

where $X_k(t)$ is defined on the interval $(k-1)\tau \leq t \leq k\tau$, $k = 0, 1, 2...$

Proof. Let us have the solution $X_k(t)$ of the equation (3.12) on the time interval $(k-1)\tau \leq t \leq k\tau$, $k = 0, 1, 2...$ Then, equation (3.12) on the next time interval is

$$\dot{X}_{k+1}(t) = A_0 X_{k+1}(t) + A_1 X_{k+1}(t-\tau)$$

and, because on time interval $(k-1)\tau \leq t \leq k\tau$ we have $X_{k+1}(t) = X_k(t)$, the last equation can be rewritten as

$$\dot{X}_{k+1}(t) = A_0 X_{k+1}(t) + A_1 X_k(t-\tau).$$

So now we get the non-homoheneous equation with unknown function $X_{k+1}(t)$ and function $A_1 X_k(t-\tau)$ is a know function.

According to the theory of ordinary differential equations, the solution of non-homogeneous equation $\dot{x}(t) = A_0 x(t) + f(t)$ have the solution in the form

$$x(t) = e^{A_0(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{A_0(t-s)} f(s) ds.$$

As far as we have $f(t) = A_1 x(t-\tau)$, on every time interval $(k-1)\tau \leq t \leq k\tau$ we have following recurent form for solution

$$X_{k+1}(t) = \mathrm{e}^{A_0(t-k\tau)} X_k(k\tau) + \int_{k\tau}^{t} \mathrm{e}^{A_0(t-s)} A_1 X_k(s-\tau) ds.$$

□

Theorem 3.2.2 [105] *Let matrices* A_0, A_1 *of system* (3.12) *be commutative. Then the matrix*

$$\mathrm{X}_0 = \begin{cases} \Theta, & -\infty \leq t < -\tau \\ I, & -\tau \leq t < 0 \\ e^{A_0 t}\left[I + Dt\right], & 0 \leq t < \tau \\ \dots & \\ e^{A_0 t} e_{\tau}^{Dt}, & (k-1)\tau \leq t < k\tau, \quad k = 1, 2, ..., \end{cases} \tag{3.14}$$

where $D = e^{-A_0\tau} A_1$, $t \geq 0$ *is the solution of the system* (3.12), *satisfying the initial conditions* (3.13).

Proof. The view of matrix $\mathrm{X}_0(t)$ follows from the definitions of exponents $e^{A_0 t}$ and e_{τ}^{Dt}. We will show that when $t \geq 0$ matrix $\mathrm{X}_0(t)$ is a solution of the system (3.12). After differentiation of (3.14), we get

$$\begin{aligned} \left(e^{A_0 t} e_{\tau}^{Dt}\right)_t' &= A_0 \cdot e^{A_0 t} e_{\tau}^{Dt} + e^{A_0 t} D e_{\tau}^{D(t-\tau)} \\ &= A_0 \cdot e^{A_0 t} e_{\tau}^{Dt} + e^{A_0 t} e^{-A_0 \tau} A_1 e_{\tau}^{D(t-\tau)} \\ &= A_0 \cdot e^{A_0 t} e_{\tau}^{Dt} + A_1 \cdot e^{A_0(t-\tau)} e_{\tau}^{D(t-\tau)}. \end{aligned}$$

Using the notation (3.14), we get $\dot{\mathrm{X}}_0(t) = A_0 \mathrm{X}_0(t) + A_1 \mathrm{X}_0(t-\tau)$, so we obtain the statement of Theorem 3.2.2. □

Example 3.2.1
Let us have the system of differential equations with a constant delay:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-1),$$

where $A_0 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$
So we have $n = 3$, $A_0 A_1 = A_1 A_0$, $\tau = 1$. Using definition of the matrix exponential we can write

$$e^{A_0(t-m\tau)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{(t-m)^1}{1!} + \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 \frac{(t-m)^2}{2!} + \dots$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{(t-m)^1}{1!} + \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{(t-m)^2}{2!} + \dots$$

$$= \begin{pmatrix} \sum_{i=0}^{\infty} \frac{(t-m)^i}{i!} & 0 & (t-m)\sum_{i=0}^{\infty} \frac{(t-m)^i}{i!} \\ 0 & \sum_{i=0}^{\infty} \frac{(t-m)^i}{i!} & 0 \\ 0 & 0 & \sum_{i=0}^{\infty} \frac{(t-m)^i}{i!} \end{pmatrix}$$

$$= \begin{pmatrix} e^{t-m} & 0 & (t-m)e^{t-m} \\ 0 & e^{t-m} & 0 \\ 0 & 0 & e^{t-m} \end{pmatrix}.$$

Now we could write the solution of the system. According to (3.14) for $0 \le t \le 1$ there follows:

$$\mathrm{X}_0(t) = \begin{pmatrix} e^t & 0 & te^t \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{pmatrix} \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} e^{-1} & e^{-1} & 0 \\ 0 & e^{-1} & 0 \\ 0 & 0 & e^{-1} \end{pmatrix} (t-\tau) \right]$$

$$= \begin{pmatrix} e^t + (t-1)e^{t-1} & (t-1)e^{t-1} & te^t + (t^2+t)e^{t-1} \\ 0 & e^t + (t-1)e^{t-1} & 0 \\ 0 & 0 & e^t + (t-1)e^{t-1} \end{pmatrix}.$$

And for $(k-1)\tau \le t \le k\tau$ according to (3.14) there follows that

$$\mathrm{X}_0(t) = \begin{pmatrix} e^t & 0 & te^t \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{pmatrix} \sum_{i=0}^{k} \begin{pmatrix} e^{-i} & ie^{-i} & 0 \\ 0 & e^{-i} & 0 \\ 0 & 0 & e^{-i} \end{pmatrix} \frac{(t-(i-1))^i}{i!}.$$

$\square$

Theorem 3.2.3 *Let matrices A_0, A_1 of system (3.10) be commutative. Then the solution of the Cauchy problem for system (3.10) in case $f(t) \equiv 0$ with initial conditions (3.11) has the form*

$$x(t) = \mathrm{X}_0(t)\varphi(-\tau) + \int_{-\tau}^{0} \mathrm{X}_0(t-\tau-s)\varphi'(s)ds,$$

where $\mathrm{X}_0(t)$ is the fundamental solutions matrix (3.14).

Proof. Solution of system (3.10), which satisfies the initial conditions $x(t) \equiv \varphi(t)$, $-\tau \le t \le 0$, can be described in the form

$$x(t) = \mathrm{X}_0(t)c + \int_{-\tau}^{0} \mathrm{X}_0(t-\tau-s)y'(s)ds, \tag{3.15}$$

where c is a vector of unknown constants, $y(t)$ is an unknown continuously differentiable vector-function and $\mathrm{X}_0(t)$ is the matrix defined in (3.14). Since the matrix $\mathrm{X}_0(t)$ is a solution of system (3.12), then, for any c and $y(t)$ expression (3.15) is also a solution of system (3.12). We choose c and $y(t)$ such that the initial conditions is in the form

$$x(t) = \mathrm{X}_0(t)c + \int_{-\tau}^{0} \mathrm{X}_0(t-\tau-s)y'(s)ds \equiv \varphi(t).$$

Put $t = -\tau$. From definition of the matrix delayed exponential (2.1.11) and from (3.14) there follows that

$$\mathrm{X}_0(-\tau) = I, \quad \mathrm{X}_0(-2\tau - s) = \begin{cases} \Theta, & -\tau < s \le 0, \\ I, & s = -\tau. \end{cases}$$

So we have $\varphi(-\tau) = c$, and formula (3.15) takes the form

$$x(t) = \mathrm{X}_0(t)\varphi(-\tau) + \int_{-\tau}^{0} \mathrm{X}_0(t-\tau-s)y'(s)ds.$$

Since $-\tau \le t \le 0$, let us break the interval in two parts. We get

$$\varphi(t) = \varphi(-\tau) + \int_{-\tau}^{t} \mathrm{X}_0(t-\tau-s)y'(s)ds + \int_{t}^{0} \mathrm{X}_0(t-\tau-s)y'(s)ds.$$

In the first integral $-\tau \le s \le t$, so $-\tau \le t-\tau-s \le t$ and the late matrix exponential equals

$$\mathrm{X}_0(t-\tau-s) \equiv I, \quad -\tau \le s \le t.$$

In the second integral $t \leq s \leq 0$, so $t-\tau \leq t-\tau-s \leq -\tau$ and the late matrix exponential equals

$$\mathrm{X}_0(t-\tau-s) = \begin{cases} \Theta, & 0 \leq s < t, \\ I, & s = t. \end{cases}$$

Hence in the interval $-\tau \leq t \leq 0$ we get

$$\varphi(-\tau) + \int\limits_{-\tau}^{t} y'(s)ds = \varphi(t). \tag{3.16}$$

We get

$$\varphi(-\tau) + y(t) - y(-\tau) = \varphi(t). \tag{3.17}$$

Solving the system of equations (3.16), (3.17), we obtain that $y(t) = \varphi(t)$. Substituting this in (3.15), we obtain the statement of the theorem. □

Theorem 3.2.4 *Let matrices A_0, A_1 of system* (3.10) *be commutative. Then the solution of the heterogeneous system* (3.10) *that satisfies zero initial conditions has the form*

$$x(t) = \int\limits_0^t e^{A_0(t-\tau-s)} e_\tau^{D(t-\tau-s)} f(s)ds, \quad t \geq 0, \tag{3.18}$$

where D is defined in Theorem 3.2.2.

Proof. Since $\mathrm{X}_0(t)$ is the solution of the homogeneous system (3.12), using the method of variation of arbitrary constant, the solution of the heterogeneous system has the form

$$x(t) = \int\limits_0^t e^{A_0(t-\tau-s)} e_\tau^{D(t-\tau-s)} c(s)ds,$$

where $c(s)$, $0 \leq s \leq t$ is an unknown vector-function. According to Leibniz integral rule differential of the expression will be

$$\dot{x}(t) = \mathrm{X}_0(t-\tau-s)c(s)\big|_{s=t} + \int\limits_0^t \frac{\partial \mathrm{X}_0'(t-\tau-s)}{\partial t} c(s)ds$$

$$= e^{A_0(t-\tau-s)} e_\tau^{D(t-\tau-s)} c(s) \;\big|_{s=t}$$

$$+ \int\limits_0^t \left[A_0 e^{A_0(t-\tau-s)} e_\tau^{D(t-\tau-s)} + e^{A_0(t-\tau-s)} D e_\tau^{D(t-2\tau-s)}\right] c(s)ds.$$

After substitution in (3.10), we get

$$e^{A_0(-\tau)} e_\tau^{D(-\tau)} c(t) + \int\limits_0^t \left[A_0 e^{A_0(t-\tau-s)} e_\tau^{D(t-\tau-s)} + e^{A_0(t-\tau-s)} D e_\tau^{D(t-2\tau-s)}\right] c(s)ds$$

$$= A_0 \int_0^t e^{A_0(t-\tau-s)} e_\tau^{D(t-\tau-s)} c(s)ds + A_1 \int_0^{t-\tau} e^{A_0(t-2\tau-s)} e_\tau^{D(t-2\tau-s)} c(s)ds + f(t).$$

Hence $e^{A_0(-\tau)} e_\tau^{D(-\tau)} = I$ and $e^{A_0(t-2\tau-s)} e_\tau^{D(t-2\tau-s)} c(s) = \mathrm{X}_0(t-2\tau-s)$, we get

$$c(t) + A_1 \int_{t-\tau}^t \mathrm{X}_0(t-2\tau-s)c(s)ds = f(t),$$

and as far as

$$\mathrm{X}_0(t-2\tau-s) = \begin{cases} \Theta, & t-\tau < s \le t, \\ I, & s = t-\tau, \end{cases}$$

then we get $c(t) = f(t)$. Hence follows the dependence (3.18). □

Theorem 3.2.5 *Let matrices A_0, A_1 of system* (3.10) *be commutative. The solution of heterogeneous system* (3.10) *which satisfies the initial solutions* (3.11) *has the form*

$$x(t) = \mathrm{X}_0(t)\varphi(-\tau) + \int_{-\tau}^0 \mathrm{X}_0(t-\tau-s)\varphi'(s)ds + \int_0^t \mathrm{X}_0(t-\tau-s)f(s)ds, \qquad (3.19)$$

where $\mathrm{X}_0(t)$ is the fundamental solutions matrix (3.14).

Proof. The proof of the theorem follows from theorems 3.2.3 and 3.2.4. □

Example 3.2.2

Let us have the system of differential equations with a constant delay:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-1) + f(t),$$

where $A_0 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $f(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$,

with initial conditions $\varphi(t) = (0, 1, 2)^T$, $-1 \le t \le 0$. So we have $n = 3$, $A_0 A_1 = A_1 A_0$, $\tau = 1$. Using the fundamental solutions matrix from the Example 3.2.1 we can use the formule (3.19). Then for $(k-1)\tau \le t \le k\tau$ follows

$$x(t) = \mathrm{X}_0(t)\varphi(-\tau) + \int_{-\tau}^0 \mathrm{X}_0(t-\tau-s)\varphi'(s)ds + \int_0^t \mathrm{X}_0(t-\tau-s)f(s)ds$$

$$= \mathrm{X}_0(t)\varphi(-1) + \int_{-1}^0 \mathrm{X}_0(t-1-s)\varphi'(s)ds + \int_0^t \mathrm{X}_0(t-1-s)f(s)ds$$

$$= \begin{pmatrix} e^t & 0 & te^t \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{pmatrix} \sum_{i=0}^{k} \begin{pmatrix} e^{-i} & ie^{-i} & 0 \\ 0 & e^{-i} & 0 \\ 0 & 0 & e^{-i} \end{pmatrix} \frac{(t-(i-1))^i}{i!} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$$+ \int_{-1}^{0} \begin{pmatrix} e^{t-1-s} & 0 & (t-1-s)e^{t-1-s} \\ 0 & e^{t-1-s} & 0 \\ 0 & 0 & e^{t-1-s} \end{pmatrix} \sum_{i=0}^{k} \begin{pmatrix} e^{-i} & ie^{-i} & 0 \\ 0 & e^{-i} & 0 \\ 0 & 0 & e^{-i} \end{pmatrix} \frac{(t-s-i)^i}{i!} \cdot 0 ds$$

$$+ \int_{0}^{t} \begin{pmatrix} e^{t-1-s} & 0 & (t-1-s)e^{t-1-s} \\ 0 & e^{t-1-s} & 0 \\ 0 & 0 & e^{t-1-s} \end{pmatrix} \sum_{i=0}^{k} \begin{pmatrix} e^{-i} & ie^{-i} & 0 \\ 0 & e^{-i} & 0 \\ 0 & 0 & e^{-i} \end{pmatrix} \frac{(t-s-i)^i}{i!} \begin{pmatrix} s \\ s^2 \\ s^3 \end{pmatrix} ds$$

$$= \sum_{i=0}^{k} \begin{pmatrix} i+2t \\ 1 \\ 2 \end{pmatrix} e^{t-i} \frac{(t-(i-1))^i}{i!}$$

$$+ \int_{0}^{t} \sum_{i=0}^{k} e^{t-1-s-i} \frac{(t-s-i)^i}{i!} \begin{pmatrix} s + is^2 + (t-1-s)s^3 \\ s^2 \\ s^3 \end{pmatrix} ds.$$

And finally for $(k-1)\tau \le t \le k\tau$ we have

$$x(t) = \sum_{i=0}^{k} e^{t-(i-1)} \frac{(t-(i-1))^i}{i!} \begin{pmatrix} i+2t \\ 1 \\ 2 \end{pmatrix}$$

$$+ \sum_{i=0}^{k} e^{-1-i} \frac{1}{i!} \begin{pmatrix} -142 - 2i + i^2 - (89 + 19i - i^2)t - (24 + 8i - i^2)t^2 - (3+2i)t^3 \\ 6 + 2i + (4+2i)t + (1+i)t^2 \\ 24 + 6i + (18+6i)t + (6+3i)t^2 + (1+i)t^3 \end{pmatrix}$$

$$- \sum_{i=0}^{k} e^{t-1-i} \frac{1}{i!} \begin{pmatrix} -142 - 23i + (53+4i)t - 4t^2 \\ 6 + 2i - 2t \\ 24 + 6i - 6t \end{pmatrix}.$$

$\square$

3.3 Systems with general matrices

Let us consider the following Cauchy problem

$$\dot{x}(x) = A_0x(t) + A_1x(t-\tau) + f(t), \quad t \geq 0, \tag{3.20}$$

$$x(t) = \varphi(t), \quad -\tau \leq t \leq 0, \tag{3.21}$$

where $x(t) = (x_1(t), x_2(t), .., x_n)^T$ is a vector of states of the sysem,
$f(t) = (f_1(t), f_2(t), .., f_n(t))^T$ is a known function of disturbance, A_0, A_1 are constant matrices of dimensions $(n \times n)$, $\tau > 0, \tau \in R$ is a constant delay.

To solve Cauchy problem (3.20) - (3.21) let us find the fundamental matrix of solution of this equation. Fundamental matrix would be a solution of the following matrix equation

$$\dot{X}(t) = A_0X(t) + A_1X(t-\tau), \quad t \geq 0, \tag{3.22}$$

with initial condition

$$X(t) = I, \qquad -\tau \leq t \leq 0, \tag{3.23}$$

where $X(t) \in R^{n \times n}$, I is identity matrix.

Theorem 3.3.1 [98] *The solution of equation* (3.22) *with initial condition* (3.23) *has the recurrent form:*

$$X_{k+1}(t) = \mathrm{e}^{A_0(t-k\tau)}X_k(k\tau) + \int_{k\tau}^{t} \mathrm{e}^{A_0(t-s)}A_1X_k(s-\tau)ds,$$

where $X_k(t)$ is defined on the interval $(k-1)\tau \leq t \leq k\tau$, $k = 0, 1, ...$

Proof. Let us have the solution $X_k(t)$ of the equation (3.22) on the time interval $(k-1)\tau \leq t \leq k\tau$. Then, equation (3.22) on the next time interval is

$$\dot{X}_{k+1}(t) = A_0X_{k+1}(t) + A_1X_{k+1}(t-\tau)$$

and, because on time interval $(k-1)\tau \leq t \leq k\tau$ we have $X_{k+1}(t) = X_k(t)$, last equation can be rewritten as

$$\dot{X}_{k+1}(t) = A_0X_{k+1}(t) + A_1X_k(t-\tau).$$

So now we get the non-homoheneous equation with unknown function $X_{k+1}(t)$ and the function $A_1X_k(t-\tau)$ is a know function.
According to the theory of ordinary differential equations, the solution of non-homogeneous equation $\dot{x}(t) = A_0x(t) + f(t)$ have the solution in the form

$$x(t) = e^{A_0(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A_0(t-s)}f(s)ds.$$

As far as we have $f(t) = A_1 x(t-\tau)$, on every time interval $(k-1)\tau \le t \le k\tau$ we have following recurent form for solution

$$X_{k+1}(t) = \mathrm{e}^{A_0(t-k\tau)} X_k(k\tau) + \int_{k\tau}^{t} \mathrm{e}^{A_0(t-s)} A_1 X_k(s-\tau) ds.$$

□

Theorem 3.3.2 *Fundamental matrix of solutions of equation* (3.22) *with identity initial conditions* (3.23) *has the following form:*

$$\mathrm{X}_0 = \begin{cases} \Theta, & -\infty \le t < -\tau \\ I, & -\tau \le t < 0 \\ e^{A_0 t} + f_1(t), & 0 \le t \le \tau \\ e^{A_0 t} + e^{A_0(t-\tau)} f_1(\tau) + f_2(t), & \tau \le t \le 2\tau \\ \dots & \\ \sum\limits_{m=0}^{k-1} e^{A_0(t-m\tau)} f_m(m\tau) + f_k(t), & (k-1)\tau \le t < k\tau, \\ & k = 3, 4, \dots, \end{cases} \tag{3.24}$$

where

$$f_p(t) = \sum_{\sum i_j = 1}^{p} \prod_{j=p}^{1} \left(\sum_{k_j=0}^{\infty} A_0^{k_j} A_1^{i_j} \right) \frac{(t-(p-1)\tau)^{K(p)}}{K(p)!} \prod_{s=p-1}^{1} \frac{\tau^{(1-i_{s+1})K(s)}}{(1-i_{s+1})K(s)!},$$

$$K(v) = k_v + i_v(1 + k_{v-1} + i_{v-1}(1 + \dots + i_2(1 + k_1 + i_1)\dots)), \quad i_p = 1, i_j \in \{0, 1\}.$$

Proof. We proved Theorem 3.3.2 using mathematical induction method.
1. Let $0 \le t \le \tau$. Then there, according to Theorem 3.3.1, for the solution of equation (3.22) on this interval holds for $n = 0$

$$X_1(t) = e^{A_0 t} X_0(0) + \int_0^t e^{A_0(t-s)} A_1 X_0(s-\tau) ds$$

$$= e^{A_0 t} + \int_0^t \sum_{k_1=0}^{\infty} A_0^{k_1} \frac{(t-s)^{k_1}}{k_1!} \cdot A_1 ds = e^{A_0 t} + \sum_{k_1=0}^{\infty} A_0^{k_1} A_1 \frac{t^{k_1+1}}{(k_1+1)!}$$

Or

$$X_1(t) = e^{A_0 t} + f_1(t), \qquad f_1(t) = \sum_{k_1=0}^{\infty} A_0^{k_1} A_1 \frac{t^{k_1+1}}{(k_1+1)!}.$$

2. Let $\tau \le t \le 2\tau$. Again there, according the Theorem 3.3.1, for the solution of equation (3.22) on this interval holds for $n = 1$

$$X_2(t) = e^{A_0(t-\tau)} X_1(\tau) + \int_{\tau}^{t} e^{A_0(t-s)} A_1 X_1(s-\tau) ds.$$

After substitution $X_1(t)$ we have

$$X_2(t) = e^{A_0(t-\tau)}\left[e^{A_0\tau} + f_1(\tau)\right] + \int_\tau^t e^{A_0(t-s)}A_1\left[e^{A_0(s-\tau)} + f_1(s-\tau)\right]ds$$

$$= e^{A_0(t-\tau)}\left[e^{A_0\tau} + f_1(\tau)\right] + \sum_{k_2=0}^{\infty} A_0^{k_2}A_1 \sum_{k_1=0}^{\infty} A_0^{k_1} \int_\tau^t \frac{(t-s)^{k_2}}{k_2!}\frac{(s-\tau)^{k_1}}{k_1!}ds$$

$$+ \sum_{k_2=0}^{\infty} A_0^{k_2}A_1 \sum_{k_1=0}^{\infty} A_0^{k_1}A_1 \int_\tau^t \frac{(t-s)^{k_2}}{k_2!}\frac{(s-\tau)^{k_1+1}}{(k_1+1)!}ds$$

$$= e^{A_0(t-\tau)}\left[e^{A_0\tau} + f_1(\tau)\right] + \sum_{k_2=0}^{\infty} A_0^{k_2}A_1 \sum_{k_1=0}^{\infty} A_0^{k_1}\frac{(t-\tau)^{k_1+k_2+1}}{(k_1+k_2+1)!}$$

$$+ \sum_{k_2=0}^{\infty} A_0^{k_2}A_1 \sum_{k_1=0}^{\infty} A_0^{k_1}A_1\frac{(t-\tau)^{k_1+k_2+2}}{(k_1+k_2+2)!}$$

Or

$$X_2(t) = e^{A_0 t} + e^{A_0(t-\tau)}f_1(\tau) + f_2(t),$$

$$f_2(t) = \sum_{k_2=0}^{\infty} A_0^{k_2}A_1 \sum_{k_1=0}^{\infty} A_0^{k_1}\frac{(t-\tau)^{k_1+k_2+1}}{(k_1+k_2+1)!} + \sum_{k_2=0}^{\infty} A_0^{k_2}A_1 \sum_{k_1=0}^{\infty} A_0^{k_1}A_1\frac{(t-\tau)^{k_1+k_2+2}}{(k_1+k_2+2)!}.$$

n. Let $(n-1)\tau \le t \le n\tau$.
Assumption: Let for n holds

$$X_n(t) = \sum_{m=0}^{n-1} e^{A_0(t-m\tau)}f_m(m\tau) + f_n(t)$$

and

$$f_p(t) = \sum_{\sum i_j=1}^{p} \prod_{j=p}^{1}\left(\sum_{k_j=0}^{\infty} A_0^{k_j}A_1^{i_j}\right)\frac{(t-(p-1)\tau)^{K(p)}}{K(p)!}\prod_{s=p-1}^{1}\frac{\tau^{(1-i_{s+1})K(s)}}{(1-i_{s+1})K(s)!},$$

$$K(v) = k_v + i_v(1 + k_{v-1} + i_{v-1}(1 + ... + i_2(1 + k_1 + i_1)...)), \quad i_p = 1, i_j \in \{0,1\}.$$

$\mathbf{n+1}$. Then for $n+1$ we get $n\tau \le t \le (n+1)\tau$ and there, according the Theorem 3.3.1, for the solution of equation (3.22) on this interval holds

$$X_{n+1}(t) = e^{A_0(t-n\tau)}X_n(n\tau) + \int_{n\tau}^t e^{A_0(t-s)}A_1X_n(s-\tau)ds$$

After substitution $X_n(t)$ we have

$$X_{n+1}(t) = e^{A_0(t-n\tau)}\left[\sum_{m=0}^{n-1} e^{A_0(n\tau-m\tau)}f_m(m\tau) + f_n(n\tau)\right]$$

$$+\int_{n\tau}^{t} e^{A_0(t-s)}A_1\left[\sum_{m=0}^{n-1} e^{A_0(s-\tau-m\tau)}f_m(m\tau)+f_n(s-\tau)\right]ds$$

$$=\sum_{m=0}^{n} e^{A_0(t-m\tau)}f_m(m\tau)$$

$$+\int_{n\tau}^{t} e^{A_0(t-s)}A_1e^{A_0(s-n\tau)}ds\sum_{m=0}^{n-1} e^{A_0((n-1)-m)\tau}f_m(m\tau)+\int_{n\tau}^{t} e^{A_0(t-s)}A_1f_n(s-\tau)ds$$

So we need to check whether

$$f_{n+1}(t)=\int_{n\tau}^{t} e^{A_0(t-s)}A_1e^{A_0(s-n\tau)}ds\sum_{m=0}^{n-1} e^{A_0((n-1)-m)\tau}f_m(m\tau)+\int_{n\tau}^{t} e^{A_0(t-s)}A_1f_n(s-\tau)ds.$$

Let

$$I_1=\int_{n\tau}^{t} e^{A_0(t-s)}A_1e^{A_0(s-n\tau)}ds\sum_{m=0}^{n-1} e^{A_0((n-1)-m)\tau}f_m(m\tau),$$

$$I_2=\int_{n\tau}^{t} e^{A_0(t-s)}A_1f_n(s-\tau)ds,$$

So

$$f_{n+1}(t)=I_1+I_2.$$

First integral is easy to solve and equals

$$I_1=\sum_{k_{n+1}=0}^{\infty} A_0^{k_{n+1}}A_1\sum_{k_n=0}^{\infty} A_0^{k_n}\frac{(t-n\tau)^{k_{n+1}+k_n+1}}{(k_{n+1}+k_n+1)!}\sum_{m=0}^{n-1} e^{A_0((n-1)-m)\tau}f_m(m\tau).$$

Using the view of function $f_p(\cdot)$ and the definition of the matrix exponential, function I_1 can be presented as follows

$$I_1=\sum_{\sum i_j=1}^{p}\prod_{j=p}^{1}\left(\sum_{k_j=0}^{\infty} A_0^{k_j}A_1^{i_j}\right)\frac{(t-(p-1)\tau)^{K(p)}}{K(p)!}\prod_{s=p-1}^{1}\frac{\tau^{(1-i_{s+1})K(s)}}{(1-i_{s+1})K(s)!},$$

where $i_{n+1}=1, i_n=0, i_{n-1}=0$.
Second integral can be written as follows

$$I_2=\int_{n\tau}^{t}\sum_{k_{n+1}=0}^{\infty} A_0^{k_{n+1}}A_1\sum_{\substack{\sum i_j=2\\ i_n=1}}^{n}\prod_{j=n}^{1}\left(\sum_{k_j=0}^{\infty} A_0^{k_j}A_1^{i_j}\right)\frac{(t-s)^{k_{n+1}}(s-n\tau)^{K(n)}}{k_{n+1}!K(n)!}$$

$$\times\prod_{s=n-1}^{1}\frac{\tau^{(1-i_{s+1})K(s)}}{(1-i_{s+1})K(s)!}ds$$

Again using the view of function $f_p(\cdot)$ and the definition of the matrix exponential, function I_2 can be presented as follow

$$I_2 = \sum_{\sum i_j=2}^{n+1} \prod_{j=n+1}^{1} \left(\sum_{k_j=0}^{\infty} A_0^{k_j} A_1^{i_j} \right) \frac{(t-n\tau)^{k_{n+1}+i_{n+1}K(n)}}{(k_{n+1}+i_{n+1}K(n))!} \prod_{s=n-1}^{2} \frac{\tau^{(1-i_{s+1})K(s)}}{(1-i_{s+1})K(s)!}$$

where $i_{n+1} = 1, i_n = 1$. And because $k_{n+1} + i_{n+1}K(n) = K(n+1)$, we can write

$$I_1 + I_2 = f_{n+1}(t),$$

so

$$X_{n+1}(t) = \sum_{m=0}^{n} e^{A_0(t-m\tau)} f_m(m\tau) + f_{n+1}(t),$$

where

$$f_p(t) = \sum_{\sum i_j=1}^{p} \prod_{j=p}^{1} \left(\sum_{k_j=0}^{\infty} A_0^{k_j} A_1^{i_j} \right) \frac{(t-(p-1)\tau)^{K(p)}}{K(p)!} \prod_{s=p-1}^{1} \frac{\tau^{(1-i_{s+1})K(s)}}{(1-i_{s+1})K(s)!},$$

$$K(v) = k_v + i_v(1 + k_{v-1} + i_{v-1}(1 + ... + i_2(1 + k_1 + i_1)...)), \quad i_p = 1, i_j \in \{0, 1\}.$$

The theorem is proved. $\square$

Example 3.3.1

Let us have the differential equation of 3^{rd} degree with a constant delay:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-1),$$

where

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau = 1.$$

Using definition of the matrix exponential we can write

$$e^{A_0 t} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{t^1}{1!} + \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 \frac{t^2}{2!} + \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^3 \frac{t^3}{3!} + ...$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{t^1}{1!} + \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{t^2}{2!} + \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{t^3}{3!} + ...$$

$$= \sum_{i=0}^{\infty} \begin{pmatrix} \frac{t^i}{i!} & 0 & 0 \\ t\frac{t^i}{i!} & \frac{t^i}{i!} & 0 \\ 0 & 0 & \frac{t^i}{i!} \end{pmatrix} = \begin{pmatrix} e^t & 0 & 0 \\ te^t & e^t & 0 \\ 0 & 0 & e^t \end{pmatrix}.$$

According to (3.24) for $0 \le t \le 1$ there is

$$X_1(t) = \begin{pmatrix} e^t & 0 & 0 \\ te^t & e^t & 0 \\ 0 & 0 & e^t \end{pmatrix} + \sum_{k_1=0}^{\infty} \begin{pmatrix} 1 & 0 & 0 \\ k_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \frac{t^{k_1+1}}{(k_1+1)!}$$

$$= \begin{pmatrix} 3e^t - 1 & 2te^t - 2e^t + 2 & 3te^t - 3e^t + 2 \\ 3te^t - e^t + 1 & 2te^t - e^t + 1 & 3te^t - e^t + 1 \\ 0 & 0 & 2e^t \end{pmatrix}.$$

Analogically according to (3.24) for $1 \le t \le 2$ there is

$$X_2(t) = \begin{pmatrix} e^t + e^{t-1}(2t-1) & 2e^{t-1} & 2e^{t-1} \\ (t-1)e^t + (2-t)e^{t-1} & e^t + (2t-1)e^{t-1} & 2e^t + (2t-1)e^{t-1} \\ 0 & 0 & 2e^t \end{pmatrix}$$

$$+ \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \begin{pmatrix} 1+2k_1 & 2 & 3 \\ k_2 + (2k_2+1)k_1 & 2k_2+1 & 3k_2+2 \\ 0 & 0 & 1 \end{pmatrix} \left(1 + \frac{t-1}{k_1+k_2+2}\right) \frac{(t-1)^{k_1+k_2+1}}{(k_1+k_2+1)!}.$$

And finally according to the form for $X_k(t)$ for $(k-1) \le t \le k$ there is

$$X_k(t) = \sum_{m=0}^{k-1} \begin{pmatrix} e^t & 0 & 0 \\ te^t & e^t & 0 \\ 0 & 0 & e^t \end{pmatrix} f_m(m) + f_k(t),$$

$$f_p(t) = \sum_{\sum i_j = 1}^{p} \prod_{j=p}^{1} \sum_{k_j=0}^{\infty} \begin{pmatrix} 1 & 2i_j & 2i_j^2 + i_j \\ k_j & 2k_j i_j + 1 & k_j(2i_j^2 + i_j) + 2i_j \\ 0 & 0 & 1 \end{pmatrix}$$

$$\times \frac{(t-(p-1))^{K(p)}}{K(p)!} \prod_{s=p-1}^{1} \frac{1}{(1-i_{s+1})K(s)!},$$

$$K(v) = k_v + i_v(1 + k_{v-1} + i_{v-1}(1 + ... + i_2(1 + k_1 + i_1)...)), \quad i_p = 1, i_j \in \{0,1\}.$$

□

Theorem 3.3.3 *The solution of the Cauchy problem for system* (3.20) *in case* $f(t) \equiv 0$ *with initial conditions* (3.21) *has the form*

$$x(t) = \mathrm{X}_0(t)\varphi(-\tau) + \int_{-\tau}^{0} \mathrm{X}_0(t-\tau-s)\varphi'(s)ds,$$

where $\mathrm{X}_0(t)$ *is the fundamental solutions matrix* (3.24).

Proof. Solution of the system (3.20), which satisfies the initial conditions $x(t) \equiv \varphi(t)$, $-\tau \le t \le 0$, can be described in the form

$$x(t) = \mathrm{X}_0(t)c + \int\limits_{-\tau}^{0} \mathrm{X}_0(t-\tau-s)y'(s)ds, \tag{3.25}$$

where c is a vector of unknown constant, $y(t)$ is an unknown continuously differentiable vector-function and $\mathrm{X}_0(t)$ is the matrix defined in (3.24). Since the matrix $\mathrm{X}_0(t)$ is a solution of system (3.20), then, for any c and $y(t)$ expression (3.25) is also a solution of system (3.20). We choose c and $y(t)$ such that the initial conditions is in the following form

$$x(t) = \mathrm{X}_0(t)c + \int\limits_{-\tau}^{0} \mathrm{X}_0(t-\tau-s)y'(s)ds \equiv \varphi(t).$$

Let put $t = -\tau$. From the definition of the matrix delayed exponential (2.1.11) and from (3.24) there follows that

$$\mathrm{X}_0(-\tau) = I, \quad \mathrm{X}_0(-2\tau - s) = \begin{cases} \Theta, & -\tau < s \le 0, \\ I, & s = -\tau. \end{cases}$$

So we have $\varphi(-\tau) = c$, and formula (3.25) takes the form

$$x(t) = \mathrm{X}_0(t)\varphi(-\tau) + \int\limits_{-\tau}^{0} \mathrm{X}_0(t-\tau-s)y'(s)ds.$$

Since $-\tau \le t \le 0$, let us break the interval in two parts. Getting

$$\varphi(t) = \varphi(-\tau) + \int\limits_{-\tau}^{t} \mathrm{X}_0(t-\tau-s)y'(s)ds + \int\limits_{t}^{0} \mathrm{X}_0(t-\tau-s)y'(s)ds.$$

In the first integral $-\tau \le s \le t$, so $-\tau \le t-\tau-s \le t$ and the late matrix exponential equals

$$\mathrm{X}_0(t-\tau-s) \equiv I, \quad -\tau \le s \le t.$$

In the second integral $t \le s \le 0$, so $t-\tau \le t-\tau-s \le -\tau$ and late matrix exponential is equal

$$\mathrm{X}_0(t-\tau-s) = \begin{cases} \Theta, & 0 \le s < t, \\ I, & s = t. \end{cases}$$

Hence in the interval $-\tau \le t \le 0$ we get

$$\varphi(-\tau) + \int\limits_{-\tau}^{t} y'(s)ds = \varphi(t). \tag{3.26}$$

We get

$$\varphi(-\tau) + y(t) - y(-\tau) = \varphi(t). \tag{3.27}$$

Solving system of equations (3.26), (3.27), we obtain that $y(t) = \varphi(t)$. Substituting this in (3.25), we obtain the statement of the theorem. □

Theorem 3.3.4 *The solution of the heterogeneous system* (3.20)*, that satisfies zero initial conditions, has the form*

$$x(t) = \int_0^t \mathrm{X}_0(t-\tau-s)f(s)ds, \quad t \geq 0, \tag{3.28}$$

where $\mathrm{X}_0(t)$ *is the fundamental solutions matrix* (3.24)*.*

Proof. Since $\mathrm{X}_0(t)$ is the solution of the homogeneous system (3.22), using the method of variation of arbitrary constant, the solution of the heterogeneous system will have the form

$$x(t) = \int_0^t \mathrm{X}_0(t-\tau-s)c(s)ds,$$

where $c(s)$, $0 \leq s \leq t$ is an unknown vector-function. According to Leibniz integral rule differential of the expression will be

$$\dot{x}(t) = \mathrm{X}_0(t-\tau-s)c(s)\big|_{s=t} + \int_0^t \frac{\partial \mathrm{X}_0'(t-\tau-s)}{\partial t}c(s)ds,$$

and after substitution $\mathrm{X}_0(t)$ as a fundamental solutions matrix (3.24) we get

$$\dot{x}(t) = \left[\sum_{m=0}^{k-1} e^{A_0(t-\tau-s-m\tau)}f_m(m\tau) + f_k(t-\tau-s)\right] c(s)\Bigg|_{s=t}$$

$$+ \int_0^t \left[A_0 \sum_{m=0}^{k-1} e^{A_0(t-\tau-s-m\tau)}f_m(m\tau) + f_k'(t-\tau-s)\right] c(s)ds.$$

After substitution in (3.28) we get

$$\mathrm{X}_0(-\tau)c(t) + \int_0^t \left[A_0\mathrm{X}_0(t-\tau-s) + f_k'(t-\tau-s) - A_0 f_k(t-\tau-s)\right] c(s)ds \quad =$$

$$= A_0\left[\int_0^t \mathrm{X}_0(t-\tau-s)c(s)ds\right] + A_1\left[\int_0^{t-\tau} \mathrm{X}_0(t-2\tau-s)c(s)\right] + f(t).$$

Hence $\mathrm{X}_0(-\tau) = I$ and $f_k'(t-\tau-s) - A_0 f_k(t-\tau-s) = \mathrm{X}_0(t-2\tau-s)$, we get

$$c(t) + A_1\int_{t-\tau}^t \mathrm{X}_0(t-2\tau-s)c(s)ds = f(t),$$

$$X_0(t-2\tau-s)=\begin{cases}\Theta, & t-\tau<s\le t,\\ I, & s=t-\tau,\end{cases}$$

then we get $c(t)=f(t)$. Hence follows the dependence (3.28). $\square$

Theorem 3.3.5 *The solution of heterogeneous system* (3.20)*, which satisfies the initial solutions* (3.21) *has the form*

$$x(t)=X_0(t)\varphi(-\tau)+\int\limits_{-\tau}^{0}X_0(t-\tau-s)\varphi'(s)ds+\int\limits_{0}^{t}X_0(t-\tau-s)f(s)ds, \qquad (3.29)$$

where $X_0(t)$ *is the fundamental solutions matrix* (3.24)*.*

Proof. The proof of the theorem follows from theorems 3.3.3 and 3.3.4. $\square$

Example 3.3.2
Let us have the differential equation of 3^{rd} degree with a constant delay:

$$\dot{x}(t)=A_0x(t)+A_1x(t-1)+f(t),$$

where

$$A_0=\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix},\quad A_1=\begin{pmatrix}1&2&3\\0&1&2\\0&0&1\end{pmatrix},$$

with initial conditions $\varphi(t)=(-1,1,0)^T$, $-1\le t\le 0$ and function of disturbance is $f(t)=(0,t,-t)^T$. So we have $n=3$, $A_0A_1\ne A_1A_0$, $\tau=1$. Using the fundamental solutions matrix from the Example 3.3.1 we can use the formule (3.29). Then for $(k-1)\tau\le t\le k\tau$ follows

$$x(t)=X_0(t)\varphi(-\tau)+\int\limits_{-\tau}^{0}X_0(t-\tau-s)\varphi'(s)ds+\int\limits_{0}^{t}X_0(t-\tau-s)f(s)ds$$

$$=X_0(t)\varphi(-1)+\int\limits_{-1}^{0}X_0(t-1-s)\varphi'(s)ds+\int\limits_{0}^{t}X_0(t-1-s)f(s)ds$$

$$=\left(\sum_{m=0}^{k-1}\begin{pmatrix}e^t&0&0\\te^t&e^t&0\\0&0&e^t\end{pmatrix}f_m(m)+f_k(t)\right)\begin{pmatrix}-1\\1\\0\end{pmatrix}$$

$$+\int\limits_{-1}^{0}\left(\sum_{m=0}^{k-1}\begin{pmatrix}e^{t-1-s}&0&0\\(t-1-s)e^{t-1-s}&e^{t-1-s}&0\\0&0&e^{t-1-s}\end{pmatrix}f_m(m)+f_k(t-1-s)\right)\times 0ds$$

$$+\int_0^t \left(\sum_{m=0}^{k-1}\begin{pmatrix} e^{t-1-s} & 0 & 0\\ (t-1-s)e^{t-1-s} & e^{t-1-s} & 0\\ 0 & 0 & e^{t-1-s}\end{pmatrix} f_m(m) + f_k(t-1-s)\right)\begin{pmatrix}0\\ s\\ -s\end{pmatrix} ds$$

$$=\sum_{m=0}^{k-1}\begin{pmatrix} e^t & 0 & 0\\ te^t & e^t & 0\\ 0 & 0 & e^t\end{pmatrix} f_m(m)\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix} + f_k(t)\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix} + 0$$

$$+\int_0^t \sum_{m=0}^{k-1}\begin{pmatrix} e^{t-1-s} & 0 & 0\\ (t-1-s)e^{t-1-s} & e^{t-1-s} & 0\\ 0 & 0 & e^{t-1-s}\end{pmatrix} f_m(m)\begin{pmatrix}0\\ s\\ -s\end{pmatrix} ds$$

$$+\int_0^t f_k(t-1-s)\begin{pmatrix}0\\ s\\ -s\end{pmatrix} ds$$

where

$$f_p(t) = \sum_{\sum i_j=1}^{p}\prod_{j=p}^{1}\sum_{k_j=0}^{\infty}\begin{pmatrix} 1 & 2i_j & 2i_j^2+i_j\\ k_j & 2k_ji_j+1 & k_j(2i_j^2+i_j)+2i_j\\ 0 & 0 & 1\end{pmatrix}$$

$$\times\frac{(t-(p-1))^{K(p)}}{K(p)!}\prod_{s=p-1}^{1}\frac{1}{(1-i_{s+1})K(s)!},$$

$$K(v) = k_v + i_v(1+k_{v-1}+i_{v-1}(1+...+i_2(1+k_1+i_1)...)), \quad i_p = 1, i_j \in \{0,1\}.$$

After subtitution $f_p(t)$ into the last formule, we get

$$x(t) = \sum_{m=0}^{k-1}\sum_{\sum i_j=1}^{m}\prod_{j=m}^{1}\sum_{k_j=0}^{\infty}\frac{1}{K(m)!}$$

$$\times\Bigg[\begin{pmatrix}-1+2i_j\\ (-1+2i_j)\\ 0\end{pmatrix} te^t + \begin{pmatrix}-1+2i_j\\ -k_j+2k_ji_j\\ 0\end{pmatrix} e^t + \begin{pmatrix}0\\ 3i_j-2i_j^2\\ 0\end{pmatrix} te^{t-1}$$

$$+\begin{pmatrix}3i_j-2i_j^2\\ 6i_j^2-7i_j+k_ji_j+1-2k_ji_j^2\\ -1\end{pmatrix} e^{t-1} + \begin{pmatrix}-3i_j+2i_j^2\\ 2k_ji_j^2-4i_j-4i_j^2-k_ji_j-1\\ 1\end{pmatrix} te^{-1}$$

$$+\begin{pmatrix}-3i_j+2i_j^2\\ 7i_j-6i_j^2-k_ji_j-1+2k_ji_j^2\\ 1\end{pmatrix} e^{-1}+\Bigg]\prod_{s=m-1}^{1}\frac{1}{(1-i_{s+1})K(s)!}$$

$$+\sum_{\sum i_j=1}^{k}\prod_{j=k}^{1}\sum_{k_j=0}^{\infty}\Bigg[\begin{pmatrix}i_j-2i_j^2\\ k_ji_j+1-2k_ji_j^2+2i_j\\ -1\end{pmatrix}\frac{-(K(k)t+2t+K(k)+1)(-k)^{K(k)+1}}{(K(k)+2)!}$$

$$+\begin{pmatrix}i_j-2i_j^2\\ k_ji_j+1-2k_ji_j^2+2i_j\\ -1\end{pmatrix}\frac{(K(k)+1)(t-k)^{K(k)+2}}{(K(k)+2)!}$$

$$+\begin{pmatrix} -1+2i_j \\ -k_j+2k_ji_j \\ 0 \end{pmatrix} \frac{(t-(k-1))^{K(k)}}{K(k)!}\Bigg] \prod_{s=k-1}^{1} \frac{1}{(1-i_{s+1})K(s)!},$$

$$K(v) = k_v + i_v(1 + k_{v-1} + i_{v-1}(1 + ... + i_2(1 + k_1 + i_1)...)), \quad i_p = 1, i_j \in \{0, 1\}.$$

$\square$

4 STABILITY OF THE SYSTEM WITH DELAY

4.1 Stability research

Let us consider the equation

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau), \quad t \geq 0, \tag{4.1}$$

with the initial condition

$$x(t) \equiv \varphi(t), \quad -\tau \leq t \leq 0,$$

where $x(t) = (x_1(t), x_2(t), .., x_n(t))^T$ is a vector of states of the system, A_0, A_1 are constant matrices of dimensions $(n \times n)$, $\varphi(t)$ is vector of function, $\tau > 0$ is a constant delay.

In this section, we will investigate the stability of the delayed equation (4.1) with Lyapunov's second method. First we construct the Lyapunov's functional in the form:

$$V(x) = x^T(t) H x(t), \tag{4.2}$$

where H is a symmetric, positive definite matrix. Then

$$\lambda_{min}(H)||x||^2 \leq x^T H x \leq \lambda_{max}(H)||x||^2 \text{ for all } x,$$

where $||\cdot||$ is a norm. Razumichin's condition [84] for $s < t$ is

$$\lambda_{min}(H)||x(s)||^2 \leq V(x(s)) < V(x(t)) \leq \lambda_{max}(H)||x(t)||^2.$$

From that condition there follows:

$$||x(s)|| < \sqrt{\frac{\lambda_{max}(H)}{\lambda_{min}(H)}}||x(t)|| \tag{4.3}$$

Now we will find the derivative of the Lyapunov's functional (4.2) according to (4.1):

$$\frac{dV(x(t))}{dt} = \dot{x}^T(t) H x(t) + x^T(t) H \dot{x}(t)$$

$$= \left[A_0 x(t) + A_1 x(t-\tau)\right]^T H x(t) + x^T(t) H \left[A_0 x(t) + A_1 x(t-\tau)\right]$$

$$= x^T(t) A_0^T H x(t) + x^T(t-\tau) A_1^T H x(t) + x^T(t) H A_0 x(t) + x^T(t) H A_1 x(t-\tau)$$

$$= x^T(t)\left[A_0^T H + H A_0\right] x(t) + x^T(t-\tau) A_1^T H x(t) + x^T(t) H A_1 x(t-\tau).$$

Let us put $A_0^T H + H A_0 = -C$. Now we can rewrite this as follows:

$$\frac{dV(x(t))}{dt} = -x^T(t) C x(t) + x^T(t-\tau) A_1^T H x(t) + x^T(t) H A_1 x(t-\tau). \tag{4.4}$$

According to (4.3), then we can bound the derivative of the Lyapunov's functional as follows:

$$\frac{dV(x(t))}{dt} \leq -\lambda_{min}(C)||x(t)||^2 + 2|HA_1|\sqrt{\frac{\lambda_{max}(H)}{\lambda_{min}(H)}}||x(t)||^2,$$

$$\frac{dV(x(t))}{dt} \leq -\left[\lambda_{min}(C) - 2|HA_1|\sqrt{\frac{\lambda_{max}(H)}{\lambda_{min}(H)}}\right]||x(t)||^2, \tag{4.5}$$

where $|\cdot|$ is a matrix norm. From this and from the Second Lyapunov's Theorem there follows:

Theorem 4.1.1 *If there exists a symmetric, positive definite matrix H such that*

$$\lambda_{min}(C) - 2|HA_1|\sqrt{\frac{\lambda_{max}(H)}{\lambda_{min}(H)}} > 0,$$

then the zero solution $y(t) \equiv 0$ of a system (4.1) *is asymptotically stable for any $\tau > 0$.*

Proof. We set

$$W(x) := -\left[\lambda_{\min}(C) - 2|HA_1|\sqrt{\frac{\lambda_{\max}(H)}{\lambda_{\min}(H)}}\right]||x(t)||^2.$$

Using the statement of the Theorem 4.1.1 we conclude that $W(x) < 0$ if $x \neq 0$ and, in view of (4.5), we have

$$\frac{dV(x(t))}{dt} \leq W(x) < 0.$$

We see that inequality from the Second Lyapunov's theorem (see [74]) holds and asymptotic stability of the zero solution is a consequence of Theorem 4.1.1. □

Theorem 4.1.2 *Let system* (4.1) *is asymptotically stable, there we have the following evaluation of convergence of solution:*

$$||x(t)|| \leq -\left[\lambda_{min}(C) - 2|HA_1|\sqrt{\frac{\lambda_{max}(H)}{\lambda_{min}(H)}}\right]^{-1} \frac{dV(x(t))}{dt},$$

where $V(x(t)) = x^T(t)Hx(t)$ is Lyapunov's functional.

Proof. Inequality following from (4.5) with condition that

$$\lambda_{min}(C) - 2|HA_1|\sqrt{\frac{\lambda_{max}(H)}{\lambda_{min}(H)}} > 0.$$

□

Example 4.1
Let us have the system of differential equations with a constant delay:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix} = \begin{pmatrix} -5 & 1 & -1 \\ 0 & -5 & 0 \\ 1 & 0 & -5 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t-1) \\ x_2(t-1) \\ x_3(t-1) \end{pmatrix},$$

initial conditions $\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \end{pmatrix}$ for $-1 \leq t \leq 0$. So we have $\tau = 1$.
Let us construct the Lyapunov's functional with the matrix

$$H = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix}.$$

Then we have:

$$C = -A_0^T H - H_0 A = -\begin{pmatrix} -5 & 0 & 1 \\ 1 & -5 & 0 \\ -1 & 0 & -5 \end{pmatrix}^T \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix}$$

$$-\begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} -5 & 1 & -1 \\ 0 & -5 & 0 \\ 1 & 0 & -5 \end{pmatrix} = \begin{pmatrix} 40 & -13 & 0 \\ -13 & 42 & -11 \\ 0 & -11 & 40 \end{pmatrix}.$$

Finally, we have

$$\lambda_{min}(C) = 23.9413,$$

$$\lambda_{min}(H) = 2.5858, \ \lambda_{max}(H) = 5.4142,$$

so:

$$\lambda_{min}(C) - 2|HA|\sqrt{\frac{\lambda_{max}(H)}{\lambda_{min}(H)}} = 3.6831 > 0,$$

and according to Theorem 4.1.1 we made a conclusion that solution of the system is asymptotically stable. And the Lyapunov's functional is

$$V(x) = x^T(t) \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix} x(t) =$$

$$= 4x_1^2(t) - 2x_1(t)x_2(t) + 4x_2^2(t) - 2x_2(t)x_3(t) + 4x_3^2(t).$$

So we can bounded $||x(t)||$ using the result of the Theorem 4.1.2. According to (4.4) we can calculate

$$\frac{dV(x(t))}{dt} = -x^T(t)Cx(t) + x^T(t-\tau)A_1^T Hx(t) + x^T(t)HA_1x(t-\tau)$$

$$= -\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}^T \begin{pmatrix} 40 & -13 & 0 \\ -13 & 42 & -11 \\ 0 & -11 & 40 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

$$+ \begin{pmatrix} x_1(t-1) \\ x_2(t-1) \\ x_3(t-1) \end{pmatrix}^T \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}^T \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

$$+ \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}^T \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t-1) \\ x_2(t-1) \\ x_3(t-1) \end{pmatrix}$$

$$= -\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}^T \begin{pmatrix} 40 & -13 & 0 \\ -13 & 42 & -11 \\ 0 & -11 & 40 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

$$+ \begin{pmatrix} x_1(t-1) \\ x_2(t-1) \\ x_3(t-1) \end{pmatrix}^T \begin{pmatrix} -3 & -3 & 1 \\ 0 & 1 & -4 \\ 1 & -4 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

$$+ \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}^T \begin{pmatrix} -3 & 0 & 1 \\ -3 & 1 & -4 \\ 1 & -4 & 1 \end{pmatrix} \begin{pmatrix} x_1(t-1) \\ x_2(t-1) \\ x_3(t-1) \end{pmatrix}.$$

For convergence of solution of given equation we have the following inequality:

$$\begin{aligned} ||x(t)||^2 \leq -0.2715 \big(& -40x_1^2(t) + 26x_1(t)x_2(t) - 42x_2^2(t) + 22x_2(t)x_3(t) - 40x_3^2(t) \\ & -6x_1(t)x_1(t-1) - 3x_1(t-1)x_2(t) + 2x_1(t-1)x_3(t) \\ & +2x_2(t-1)x_2(t) - 8x_2(t-1)x_3(t) + 2x_3(t-1)x_1(t) \\ & -8x_3(t-1)x_2(t) + 2x_3(t-1)x_3(t) - 3x_2(t-1)x_1(t) \big). \end{aligned}$$

□

5 CONTROLLABILITY OF THE SYSTEM WITH DELAY

5.1 Controllability in the system with same matrices

Let us have the control system of differential matrix equation

$$\dot{x}(t) = Ax(t) + Ax(t-\tau) + Bu(t), \quad t \geq 0, \tag{5.1}$$

with initial conditions

$$x(t) = \varphi(t), \quad -\tau \leq t \leq 0$$

where $x = (x_1(t), ..., x_n(t))^T$ is a vector of states of the system,
$u(t) = (u_1(t), ..., u_r(t))^T$ is a vector of control functions,
A, B are constant matrices of dimensions $(n \times n)$, $(n \times r)$ respectively,
$\tau > 0$ is a constant delay.

Theorem 5.1.1 *For relatively controllability of linear system with delay* (5.1) *is necessary and sufficient that* $rank(S) = n$*, where*

$$S = \{B \ AB \ A^2B \ ... \ A^{n-1}B\},$$

hence S *is a matrix constructed by augmenting matrices* B, AB,..., $A^{n-1}B$.

Proof: sufficiency. Let us assume the rank of the matrix S is n and prove that in this case the vector-functions $\omega_i(t)$, $i = 1, .., n$

$$\omega(t) = \begin{pmatrix} \omega_1(t) \\ ... \\ \omega_n(t) \end{pmatrix} = \mathrm{X}_0(t)B$$

are linear independent for $0 \leq t \leq t_1$ (here $\mathrm{X}_0(t)$ is the fundamental matrix of solutions of equation (3.3)). In this case, according to the Theorem 2.1.6, the system 5.1 will be relatively controllable. We should prove that if $rank(S) = n$, then there is no such constant vector $l = (l_1, ..., l_n)$ ($||l|| > 0$), that following identity is true

$$l \cdot \mathrm{X}_0(\xi)B \equiv \theta, \quad 0 \leq \xi \leq t_1, \tag{5.2}$$

where θ is zero vector. This statement is equal the following: if there exist some interval $0 \leq \xi \leq t_1$ and some vector l, for which the equality 5.2 is true, then the rank of the matrix S is less then n.

To prove let us differentiate identity (5.2) $(n-1)$ times on a variable ξ:

$$l \cdot \frac{d}{d\xi}\mathrm{X}_0(\xi)B \equiv \theta,$$

$$l \cdot \frac{d^2}{d\xi^2} \mathrm{X}_0(\xi) B \equiv \theta,$$

$$\dots$$

$$l \cdot \frac{d^{n-1}}{d\xi^{n-1}} \mathrm{X}_0(\xi) B \equiv \theta.$$

As far as $\mathrm{X}_0(\xi)$ is the fundamental matrix of solutions of equation (3.3), then we can rewrite last $(n-1)$ identities as follow

$$l \cdot [A\mathrm{X}_0(\xi) + A\mathrm{X}_0(\xi - \tau)] B \equiv \theta,$$

$$l \cdot \left[A^2\mathrm{X}_0(\xi) + 2A^2\mathrm{X}_0(\xi - \tau) + A^2\mathrm{X}_0(\xi - 2\tau)\right] B \equiv \theta,$$

$$\dots$$

$$l \cdot \left[A^{n-1}\mathrm{X}_0(\xi) + (n-1)A^{n-1}\mathrm{X}_0(\xi - \tau) + \dots + A^{n-1}\mathrm{X}_0(\xi - (n-1)\tau)\right] B \equiv \theta.$$

Putting $\xi = 0$ in 5.2 and last $n-1$ identities, we get

$$l \cdot B \equiv \theta,$$

$$l \cdot 2AB \equiv \theta,$$

$$l \cdot 3A^2B \equiv \theta,$$

$$\dots$$

$$l \cdot nA^{n-1}B \equiv \theta.$$

It is possible only if the rank of the matrix S is lower than n. Got a contradiction.

Proof: necessity. Necessity of the theorem statement will be established if it is proved that from the linear independence of the vector-functions $\omega_i(t)$, $i = 1, .., n$ for $0 \leq t \leq t_1$ follows that the rank of the matrix S is equal n.This statement equals following: of the rank of the matrix S is less than n, than the vector-functions $\omega_i(t)$, $i = 1, .., n$ are linear dependent for $0 \leq t \leq t_1$.

Let us assume that the rank of the matrix S is less than n. Then there is exist such n-dimension vector $l = (l_1, ..., l_n)$ ($||l|| > 0$), that

$$l \cdot A^k B \equiv \theta, \quad k = 0, 1, .., n-1.$$

Using the Hamilton-Kelly's formula, we conclude that

$$l \cdot A^k B \equiv \theta \tag{5.3}$$

for all digit k. Using the view of fundamental matrix of solutions we can write

$$l \cdot \omega(t) = l \cdot \left[\sum_{m=0}^{k-1} 2\mathrm{e}^{A(t-m\tau)} \sum_{p=0}^{m} (-1)^{p+m} A^p \frac{(t-m\tau)^p}{p!} + (-I)^k \right] B.$$

And if in the last equality vector l is put from identity 5.3, we obtain

$$l \cdot \omega(t) \equiv \theta,$$

and this mean linear dependence of vectors $\omega_i(t)$, $t = 1, .., n$. Got a contradiction.□

Example 5.1.1
Let us have the differential equation of 3^{rd} degree with a constant delay:

$$\dot{x}(t) = Ax(t) + Ax(t-1) + Bu(t),$$

where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As we see $\tau = 1$ and $n = 3$. We want to know whether this system is relatively controllable so let us check the necessary and sufficient condition. We will find the matrix S:

$$S = \{B\ AB\ ...\ A^{n-1}B\} = \{B\ AB\ A^2B\} = \begin{pmatrix} 1 & 1 & 0 & 2 & 2 & 0 & 3 & 3 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We have, $rank(S) = 2$, so the system is not relatively controllable. □

Example 5.1.2
Let us have the differential equation of 3^{rd} degree with a constant delay:

$$\dot{x}(t) = Ax(t) + Ax(t-1) + Bu(t),$$

where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

As we see $\tau = 1$ and $n = 3$. We want to know whether this system is relatively controllable so let us check the necessary and sufficient condition. We will find the matrix S:

$$S = \{B\ AB\ ...\ A^{n-1}B\} = \{B\ AB\ A^2B\} = \begin{pmatrix} 0 & 0 & 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

We have, $rank(S) = 3$, so the system is relatively controllable. □

5.2 Controllability in the system with commutative matrices

Let us consider the control system of differential matrix equation

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + Bu(t), \quad t \geq 0, \tag{5.4}$$

with initial conditions

$$x(t) = \varphi(t), \quad -\tau \leq t \leq 0$$

where $x = (x_1(t), ..., x_n(t))^T$ is a vector of states of the system,
$u(t) = (u_1(t), ..., u_r(t))^T$ is a vector of control functions,
A_0, A_1 are commutative constant matrices of dimensions $(n \times n)$,
B is constant matrix of dimension $(n \times r)$,
$\tau > 0$ is a constant delay.

Theorem 5.2.1 [105] *For relatively controllability of the linear stationary system with delay* (5.4) *it is sufficient that for* $(k-1)\tau \leq t \leq k\tau$ *the* $rank(S_k) = n$*, where*

$$S_k = \{B \quad e^{-A_0\tau} A_1 B \quad e^{-2A_0\tau} A_1^2 B \quad ... \quad e^{-(k-1)A_0\tau} A_1^{k-1} B\},$$

hence S_k *is a matrix constructed by augmenting matrices* B, $e^{-A_0\tau} A_1 B$, $e^{-2A_0\tau} A_1^2 B$, ..., $e^{-(k-1)A_0\tau} A_1^{k-1} B$.

Proof. Let system (5.4) be relatively controllable. Then for any $\varphi(t)$, x_1 and t_1 there exist a control $u^*(t)$ such that for a closed system (5.4) there exists a solution $x^*(t)$, which satisfies boundary conditions $x(t) \equiv \varphi(t), -\tau \leq t \leq 0$. The representation of the Cauchy problem for the heterogeneous equation as the sum is as follows:

$$x(t) = \mathrm{X}_0(t)\varphi(-\tau) + \int_{-\tau}^{0} \mathrm{X}_0(t-\tau-s)\varphi'(s)ds + \int_{0}^{t} \mathrm{X}_0(t-\tau-s)Bu(s)ds.$$

where $\mathrm{X}_0(t)$ is fundamental matrix of solutions of the equation (3.12), $D = e^{-A_0\tau}A_1$.
When control is $u^*(t)$ in time moment $t = t_1$ we get

$$x_1 = \mathrm{X}_0(t_1)\varphi(-\tau) + \int_{-\tau}^{0} \mathrm{X}_0(t_1-\tau-s)\varphi'(s)ds + \int_{0}^{t_1} \mathrm{X}_0(t_1-\tau-s)Bu^*(s)ds. \tag{5.5}$$

Denote

$$x_1 - \mathrm{X}_0(t_1)\varphi(-\tau) - \int_{-\tau}^{0} \mathrm{X}_0(t_1-\tau-s)\varphi'(s)ds = \mu \tag{5.6}$$

And using the representation of $\mathrm{X}_0(t)$ from (3.14) we get (D was defined in Theorem 3.2.2)

$$\int_{0}^{t_1} e^{A_0(t_1-\tau-s)} e_{\tau}^{D(t_1-\tau-s)} Bu^*(s)ds = \int_{-\tau}^{t_1-\tau} e^{A_0\xi} e_{\tau}^{D\xi} Bu^*(t_1-\tau-\xi)d\xi$$

$$= \int_{-\tau}^{0} e^{A_0\xi} Bu^*(t_1-\tau-\xi)d\xi + \int_{0}^{\tau} e^{A_0\xi}\left[I + D\frac{\xi}{1!}\right] Bu^*(t_1-\tau-\xi)d\xi$$

$$+ \int_{\tau}^{2\tau} e^{A_0\xi}\left[I + D\frac{\xi}{1!} + D^2\frac{(\xi-\tau)^2}{2!}\right] Bu^*(t_1-\tau-\xi)d\xi + ...$$

$$+ \int_{(k-2)\tau}^{t_1-\tau} e^{A_0\xi}\left[I + D\frac{\xi}{1!} + D^2\frac{(\xi-\tau)^2}{2!} + ... + D^{k-1}\frac{(\xi-(k-2)\tau)^{k-1}}{(k-1)!}\right] Bu^*(t_1-\tau-\xi)d\xi$$

$$= \int_{-\tau}^{t_1-\tau} e^{A_0\xi} Bu^*(t_1-\tau-\xi)d\xi + \int_{0}^{t_1-\tau} e^{A_0\xi} D\frac{\xi}{1!} Bu^*(t_1-\tau-\xi)d\xi$$

$$+ \int_{\tau}^{t_1-\tau} e^{A_0\xi} D^2\frac{(\xi-\tau)^2}{2!} Bu^*(t_1-\tau-\xi)d\xi + \int_{2\tau}^{t_1-\tau} e^{A_0\xi} D^3\frac{(\xi-2\tau)^3}{3!} Bu^*(t_1-\tau-\xi)d\xi + ...$$

$$+ \int_{(k-3)\tau}^{t_1-\tau} e^{A_0\xi} D^{k-2}\frac{(\xi-(k-3)\tau)^{k-2}}{(k-2)!} Bu^*(t_1-\tau-\xi)d\xi$$

$$+ \int_{(k-2)\tau}^{t_1-\tau} e^{A_0\xi} D^{k-1}\frac{(\xi-(k-2)\tau)^{k-1}}{(k-1)!} Bu^*(t_1-\tau-\xi)d\xi = (h).$$

Using $e^{A_0\xi} = \sum_{i=0}^{\infty} A_0^i \frac{\xi^i}{i!}$ let us denoted

$$\psi_1(i) = \int_{-\tau}^{t_1-\tau} \frac{\xi^i}{i!} \cdot u^*(t_1-\tau-\xi)d\xi;$$

$$\psi_2(i) = \int_{0}^{t_1-\tau} \frac{\xi^i}{i!} \cdot \frac{\xi-\tau}{1!} \cdot u^*(t_1-\tau-\xi)d\xi;$$

$$\ldots;$$

$$\psi_{k-1}(i) = \int_{(k-3)\tau}^{t_1-\tau} \frac{\xi^i}{i!} \cdot \frac{(\xi-(k-3)\tau)^{k-2}}{(k-2)!} \cdot u^*(t_1-\tau-\xi)d\xi;$$

$$\psi_k(i) = \int_{(k-2)\tau}^{t_1-\tau} \frac{\xi^i}{i!} \cdot \frac{(\xi-(k-2)\tau)^{k-1}}{(k-1)!} \cdot u^*(t_1-\tau-\xi)d\xi.$$

And using (5.6) correlation (5.5) get the form

$$\sum_{i=0}^{\infty} A_0^i B\psi_1(i) + \sum_{i=0}^{\infty} A_0^i DB\psi_2(i) + \ldots + \sum_{i=0}^{\infty} A_0^i D^{k-1} B\psi_k(i) = \mu. \qquad (5.7)$$

Since $D = e^{-A_0\tau}A_1$, we rewrite expression (5.7) as

$$\sum_{i=0}^{\infty} A_0^i B\psi_1(i) + \sum_{i=0}^{\infty} A_0^i e^{-A_0\tau} A_1 B\psi_2(i) + \ldots + \sum_{i=0}^{\infty} A_0^i e^{-A_0(k-1)\tau} A_1^{k-1} B\psi_k(i) = \mu \quad (5.8)$$

Since for any matrix A_0, $e^{A_0\xi} \neq \Theta$, where Θ is a zero matrix and since the system is relatively controllable, (5.8) has a solution for any vector μ. If $k < n$, then the system is over defined and not always has a solution. Therefore, for controllability it is necessary $t_1 \geq (k-1)\tau \geq (n-1)\tau$. From the Hamilton-Kelly's formula there follows that any power A_0^i, $i \geq n$ of matrix A_0 can be expressed by linear combination of matrices $I, A_0, A_0^2, ..., A_0^{n-1}$. Therefore if $k \geq n$ system (5.8) can be substituted

$$B\overline{\psi_1(t_1)} + e^{-A_0\tau} A_1 B\overline{\psi_2(t_1)} + \ldots + e^{-A_0(k-1)\tau} A_1^{k-1} B\overline{\psi_k(t_1)} = \mu \qquad (5.9)$$

where $\overline{\psi_j(t_1)}$, $j = 1, 2, .., k$ - some functions of variable t_1. And if (5.9) has solution for any μ, then $rank S_k = n$, where

$$S_k = \{B \quad e^{-A_0\tau} A_1 B \quad e^{-2A_0\tau} A_1^2 B \quad ... \quad e^{-(k-1)A_0\tau} A_1^{k-1} B\}.$$

Sufficient condition is proved. □

Remark 5.2.2 *Using the Hamilton-Kelly's formula, we notice that every matrix* A_1^s, $s > n-1$ *can be presented as linear combination of matrices* A_1^j, $j = 0, .., n-1$, *so when* $k \geq n-1$ *matrix* S_k *became*

$$S_k = S_n = \{B \quad e^{-A_0\tau} A_1 B \quad e^{-2A_0\tau} A_1^2 B \quad ... \quad e^{-(n-1)A_0\tau} A_1^{n-1} B\}.$$

Example 5.2.1
Let us have the differential equation of 3^{rd} degree with a constant delay:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-1) + Bu(t),$$

where

$$A_0 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As we see $\tau = 1, n = 3$ and $A_0 A_1 = A_1 A_0$. We want to know whether this system is relatively controllable in the moment of time $t_1 = 3$. Let us check the sufficient condition. We will find the matrix S_3:

$$S_3 = \left\{B \quad e^{-A_0\tau} A_1 B \quad e^{-2A_0\tau} A_1^2 B\right\} = \left\{B \quad (e^{-1}I) A_1 B \quad (e^{-2}I) A_1^2 B\right\}$$

$$= \begin{pmatrix} 1 & 0 & 0 & e^{-1} & 2e^{-1} & 0 & e^{-2} & 4e^{-2} & 0 \\ 0 & 1 & 0 & 0 & e^{-1} & 0 & 0 & e^{-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Sufficient condition is not implemented so we can not conclude if the system is relatively controllable. □

Example 5.2.2
Let us have the differential equation of 3^{rd} degree with a constant delay:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-1) + Bu(t),$$

where

$$A_0 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

As we see $\tau = 1, n = 3$ and $A_0 A_1 = A_1 A_0$. We want to know whether this system is relatively controllable in the moment of time $t_1 = 3$. Let us check the sufficient condition. First, we will find the matrix S_3:

$$S_3 = \left\{ B \quad e^{-A_0\tau} A_1 B \quad e^{-2A_0\tau} A_1^2 B \quad \right\} = \left\{ B \quad (e^{-1}I) A_1 B \quad (e^{-2}I) A_1^2 B \right\} =$$

$$= \begin{pmatrix} 1 & 0 & e^{-1} & 2e^{-1} & e^{-2} & 6e^{-2} \\ 0 & 0 & 0 & e^{-1} & 0 & 2e^{-2} \\ 0 & 1 & 0 & e^{-1} & 0 & e^{-2} \end{pmatrix}.$$

We have $rank(S_3) = 3$, so the system is relatively controllable for $t_1 = 3$. □

5.3 Controllability in the system with general matrices

Let us consider the control system of differential matrix equation

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + Bu(t), \quad t \geq 0, \tag{5.10}$$

with initial conditions

$$x(t) = \varphi(t), \quad -\tau \leq t \leq 0$$

where $x = (x_1(t), ..., x_n(t))^T$ is a vector of states of the system,
$u(t) = (u_1(t), ..., u_r(t))^T$ is a vector of control functions,
A_0, A_1, B are constant matrices of dimensions $(n \times n)$, $(n \times n)$, $(n \times r)$ respectively,
$\tau > 0$ is a constant delay.

Conjecture 5.3.1 *Because of the view* (3.13) *of fundamental matrix of solutions of equation* (5.10) *and way of construction of the solution of the heterogeneous equation (*3.29) *with the initial condition $x(t) \equiv \varphi(t)$, $-\tau \leq t \leq 0$, solution of the vector-problem* (5.10) *on the time interval $(p-1)\tau \leq t \leq p\tau$ can be written in form*

$$x(t) = \psi_0(t) + \sum_{\sum i_j = 0}^{p} \prod_{j=p}^{1} \left(\sum_{k_j=0}^{\infty} A_0^{k_j} A_1^{i_j} \right) B\psi_p(t,u), \quad i_j \in \{0,1\},$$

where

$$\psi_0(t) = \mathrm{X}_0(t)\varphi(-\tau) + \int_{-\tau}^{0} \mathrm{X}_0(t-\tau-s)\varphi'(s)ds,$$

$\mathrm{X}_0(t)$ *is fundamental matrix of solutions of the equation* (3.22),

$$\psi_p(t,u) = \int_{k\tau}^{t} \frac{(t-(p-1)\tau)^{K(p)}}{K(p)!} u(s)ds \cdot \prod_{s=p-1}^{1} \frac{\tau^{(1-i_{s+1})K(s)}}{(1-i_{s+1})K(s)!},$$

$$K(v) = k_v + i_v(1 + k_{v-1} + i_{v-1}(1 + ... + i_2(1 + k_1 + i_1)...)), \quad i_p = 1, i_j \in \{0,1\}.$$

Now we introduce for the equation (5.10) analogue of the characteristic equation

$$Q_i(s) = A_0 Q_{i-1}(s) + A_1 Q_{i-1}(s-\tau), \quad s \geq 0, i = 1, 2, ..$$

$$Q_0(0) = B, \quad Q_0(s) = \Theta, s \neq 0,$$

where Θ is zero matrix. Using the Hamilton-Kelly's formula, we notice that every matrix A_0^s, $s > n-1$ can be presented as linear combination of matrices A^j,

$j = 0, .., n-1$. Function takes for $0 \le s \le p\tau$ the following linear independent values:

	$s=0$	$s=\tau$	...	$s=p\tau$
$Q_0(s)$	B	Θ	...	Θ
$Q_1(s)$	A_0B	A_1B	...	Θ
$Q_2(s)$	A_0^2B	$(A_0A_1+A_1A_0)B$	...	Θ
...	...	...	...	...
$Q_p(s)$	A_0^pB	$(A_0^{p-1}A_1+..+A_1A_0^{p-1})B$	...	A_1^pB
$Q_{p+1}(s)$	$A_0^{p+1}B$	$(A_0^pA_1+..+A_1A_0^p)B$	...	$(A_0A_1^p+..+A_1^pA_0)B$
...	...	...	...	...
$Q_{n-1}(s)$	$A_0^{n-1}B$	$(A_0^{n-2}A_1+..+A_1A_0^{n-2})B$	...	$(A_0^{n-p-1}A_1^p+..+A_1^pA_0^{n-p-1})B$
$Q_n(s)$	-	$(A_0^{n-1}A_1+..+A_1A_0^{n-1})B$	...	$(A_0^{n-p}A_1^p+..+A_1^pA_0^{n-p})B$
...	...	...	...	...
$Q_{n+p-1}(s)$	-	-	...	$A_0^{n-1}A_1^pB$

Let us denote

$$Q = \{Q_0 \;\; Q_1 \;\; Q_2 \;\; ... \;\; Q_{n+p-1}\}$$
$$= \{Q_0(0) \;\; Q_1(0) \;\; Q_1(\tau) \;\; Q_2(0) \;\; Q_2(\tau) \;\; Q_2(2\tau) \;\; ... \;\; Q_{n+p-1}(p\tau)\},$$

or

$$Q = \{B \;\; A_0B \;\; A_1B \;\; A_0^2B \;\; (A_0A_1+A_1A_0)B \;\; A_1^2B \;\; A_0^3B$$
$$(A_0^2A_1+A_0A_1A_0+A_1A_0^2)B \;\; (A_0A_1^2+A_1A_0A_1+A_1^2A_0)B \;\; A_1^3B \;\; ... \;\; A_0^{n-1}A_1^pB\}.$$

Theorem 5.3.2 *For relatively controllability of a linear stationary system with delay* (5.10) *it is sufficient that for* $(p-1)\tau \le t \le p\tau$ *will* $rank(Q) = n$, *where*

$$Q = \{B \;\; A_0B \;\; A_1B \;\; A_0^2B \;\; (A_0A_1+A_1A_0)B \;\; A_1^2B \;\; A_0^3B$$
$$(A_0^2A_1+A_0A_1A_0+A_1A_0^2)B \;\; (A_0A_1^2+A_1A_0A_1+A_1^2A_0)B \;\; A_1^3B \;\; ... \;\; A_0^{n-1}A_1^pB\},$$

hence Q *is a matrix constructed by augmenting matrices* B, A_0B, A_1B, A_0^2B, $(A_0A_1+A_1A_0)B$, A_1^2B, A_0^3B, $(A_0^2A_1+A_0A_1A_0+A_1A_0^2)B$, $(A_0A_1^2+A_1A_0A_1+A_1^2A_0)B$, A_1^3B, ... , $A_0^{n-1}A_1^pB$.

Proof. The system (5.10) is relatively controllable when there exists a control vector $u_0(t)$, such that system for some time pass from the initial state $x_0 = x(t_0) = (x_1^0, ..., x_n^0)^T$ into the direct position $x_1 = x(t_1) = (x_1^1, ..., x_n^1)^T$. This means, when there exists a vector $u_0(t)$, which suits the following equality

$$x(t_1) - x(t_0) = \psi_0(t_1) + \sum_{\sum i_j=0}^{p} \prod_{j=p}^{1} \left(\sum_{k_j=0}^{\infty} A_0^{k_j} A_1^{i_j} \right) B\psi_p(t_1, u_0(t_1)), \quad i_j \in \{0,1\}. \tag{5.11}$$

Since $\psi_0(t_1)$ is a constant vector that depends on initial conditions and is independent of control (by structure), we introduce such variable $\hat{x} = x(t_1) - x(t_0) - \psi_0(t_1)$ and let $\psi_p(t_1, u_0(t_1)) = \psi_p$. In the new notation system (5.11) will be as follows:

$$\hat{x} = \sum_{\sum i_j=0}^{p} \prod_{j=p}^{1} \left(\sum_{k_j=0}^{\infty} A_0^{k_j} A_1^{i_j} \right) B\psi_p, \quad i_j \in \{0,1\}. \tag{5.12}$$

In the system (5.12) we open the sums and regroup terms to get coefficients that characteristic equation produces.

$$\hat{x} = \sum_{\sum i_j=0}^{p} \prod_{j=p}^{1} \left(\sum_{k_j=0}^{\infty} A_0^{k_j} A_1^{i_j} \right) B\psi_p$$

$$= \prod_{j=p}^{1} \left(\sum_{k_j=0}^{\infty} A_0^{k_j} A_1^{i_j} \right) B\psi_{p_{\sum i_j=0}} + \prod_{j=p}^{1} \left(\sum_{k_j=0}^{\infty} A_0^{k_j} A_1^{i_j} \right) B\psi_{p_{\sum i_j=1}} + ...$$

$$+ \prod_{j=p}^{1} \left(\sum_{k_j=0}^{\infty} A_0^{k_j} A_1^{i_j} \right) B\psi_{p_{\sum i_j=p}}$$

As far, as every matrix A_0^s, $s > n-1$ can be presented as linear combination of matrices A_0^j, $j = 0,..,n-1$, then the last equality can be rewtited as

$$\hat{x} = \prod_{j=p}^{1} \left(\sum_{k_j=0}^{n-1} A_0^{k_j} A_1^{i_j} \right) B\overline{\psi}_{p_{\sum i_j=0 \sum k_j=0}} + ... + \prod_{j=p}^{1} \left(\sum_{k_j=0}^{n-1} A_0^{k_j} A_1^{i_j} \right) B\overline{\psi}_{p_{\sum i_j=0 \sum k_j=n-1}}$$

$$+ \prod_{j=p}^{1} \left(\sum_{k_j=0}^{n-1} A_0^{k_j} A_1^{i_j} \right) B\overline{\psi}_{p_{\sum i_j=1 \sum k_j=0}} + ... + \prod_{j=p}^{1} \left(\sum_{k_j=0}^{n-1} A_0^{k_j} A_1^{i_j} \right) B\overline{\psi}_{p_{\sum i_j=1 \sum k_j=n-1}}$$

$$+...$$

$$+ \prod_{j=p}^{1} \left(\sum_{k_j=0}^{n-1} A_0^{k_j} A_1^{i_j} \right) B\overline{\psi}_{p_{\sum i_j=p \sum k_j=0}} + ... + \prod_{j=p}^{1} \left(\sum_{k_j=0}^{n-1} A_0^{k_j} A_1^{i_j} \right) B\overline{\psi}_{p_{\sum i_j=p \sum k_j=n-1}}$$

$$= Q_0(0)\overline{\psi}_{p_{\sum i_j=0 \sum k_j=0}} + ... + Q_{n-1}(0)\overline{\psi}_{p_{\sum i_j=0 \sum k_j=n-1}}$$
$$+Q_1(\tau)\overline{\psi}_{1_{\sum i_j=1 \sum k_j=0}} + ... + Q_n(\tau)\overline{\psi}_{1_{\sum i_j=1 \sum k_j=n-1}} + ...$$
$$+Q_p(p\tau)\overline{\psi}_{p_{\sum i_j=p \sum k_j=0}} + ... + Q_{n+p-1}(p\tau)\overline{\psi}_{p_{\sum i_j=p \sum k_j=n-1}},$$

where $\overline{\psi}_{p_{\sum \cdot \sum \cdot}}$ is new functions, appeared as linear combination of functions $\psi_{p_{\sum \cdot}}$.

Get a system with an finite number of unknowns and the vector of absolute terms in length n. If the rank of the matrix

$$Q = \{Q_0(0) \;\; Q_1(0) \;\; Q_1(\tau) \;\; Q_2(0) \;\; Q_2(\tau) \;\; Q_2(2\tau) \;\; ... \;\; Q_{n+p-1}(p\tau)\}$$

will equal n than the system will have solution. In this case the solution of the system will be the vector that is determined by the vector of absolute terms $\hat{x}$. Since the vector of absolute terms is defined from any finite state of the system (5.10), we conclude that system (5.10) can be moved in any point if the conditions of the theorem is true. This mean that the system (5.10) is relatively controllable if the matrix Q has rank n. □

Remark 5.3.3 *Using the Hamilton-Kelly's formula, we notice that every matrix A_1^s, $s > n-1$ can be presented as linear combination of matrices A_1^j, $j = 0,..,n-1$, so when $p \geq n-1$ we get for $s \geq n\tau$ linear dependent values, and matrix Q became*

$$Q = \{Q_0 \quad Q_1 \quad Q_2 \quad ... \quad Q_{2n-2}\}$$

$$= \{B \quad A_0B \quad A_1B \quad A_0^2B \quad (A_0A_1 + A_1A_0)B \quad A_1^2B \quad A_0^3B$$

$$(A_0^2A_1 + A_0A_1A_0 + A_1A_0^2)B \quad (A_0A_1^2 + A_1A_0A_1 + A_1^2A_0)B \quad A_1^3B \quad ... \quad A_0^{n-1}A_1^{n-1}B\}.$$

Example 5.3.1
Let us have the differential equation of 3^{rd} degree with a constant delay:

$$\dot{x}(t) = A_0x(t) + A_1x(t-1) + Bu(t),$$

where

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As we see $\tau = 1, n = 3$ and $A_0A_1 \neq A_1A_0$. We want to know whether this system is relatively controllable. Let us check the necessary and sufficient condition. First, we will find the matrix Q:

$$Q = \left\{B \quad A_0B \quad A_1B \quad A_0^2B \quad (A_0A_1 + A_1A_0)B \quad A_1^2B \quad ... \quad A_0^2A_1^2B\right\} =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 0 & 1 & 0 & 0 & 4 & 4 & 0 & 1 & 4 & 0 & & 1 & 4 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 0 & 2 & 4 & 0 & 0 & 1 & 0 & ... & 2 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 \end{pmatrix}.$$

Sufficient condition is not implemented so we can not conclude if the system is relatively controllable. □

Example 5.3.2
Let us have the differential equation of 3^{rd} degree with a constant delay:

$$\dot{x}(t) = A_0x(t) + A_1x(t-1) + Bu(t),$$

where

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

As we see $\tau = 1, n = 3$ and $A_0A_1 \neq A_1A_0$. We want to know whether this system is relatively controllable. Let us check the necessary and sufficient condition. First, we will find the matrix Q:

$$Q = \left\{B \quad A_0B \quad A_1B \quad A_0^2B \quad (A_0A_1 + A_1A_0)B \quad A_1^2B \quad ... \quad A_0^2A_1^2B\right\} =$$

$$= \begin{pmatrix} 0 & 3 & 0 & 6 & 10 & & 10 \\ 0 & 2 & 0 & 7 & 4 & ... & 24 \\ 1 & 1 & 1 & 2 & 4 & & 1 \end{pmatrix}.$$

We have $rank(Q) = 3$, so the system is relatively controllable. $\square$

6 CONTROL CONSTRUSTION

6.1 Systems with same matrices

Theorem 6.1.1 [108] *Let us have the control problem with delay with the same matrices* (5.1). *Let* $t_1 \geq (k-1)\tau$ *and the necessary and sufficient condition for controllability is implemented:*

$$rank(S) = rank\left(\{B \; AB \; A^2B \; ... \; A^{n-1}B\}\right) = n.$$

Then the control function can be taken as

$$u(\xi) = [\mathrm{X}_0(t_1-\tau-\xi)B]^T \left[\int_0^{t_1} \mathrm{X}_0(t_1-\tau-s)BB^T[\mathrm{X}_0(t_1-\tau-s)]^T ds\right]^{-1} \mu,$$

$$0 \leq \xi \leq t_1,$$

$$\textit{where} \quad \mu = x_1 - \mathrm{X}_0(t_1)\varphi(-\tau) - \int_{-\tau}^{0} \mathrm{X}_0(t_1-\tau-s)\varphi'(s)ds,$$

and $\mathrm{X}_0(t)$ *is the fundamental matrix of solutions* (3.5) *on time interval* $t \geq (k-1)\tau$.

Proof. Using the Cauchy integral representation we have that the solution of the system (5.1) with initial conditions $x_0(t) \equiv \varphi(t)$, $-\tau \leq t \leq 0$ has the form

$$x(t) = \mathrm{X}_0(t)\varphi(-\tau) + \int_{-\tau}^{0} \mathrm{X}_0(t-\tau-s)\varphi'(s)ds + \int_0^t \mathrm{X}_0(t-\tau-s)Bu(s)ds \qquad (6.1)$$

Using the notations

$$\mu = x(t_1) - \mathrm{X}_0(t_1)\varphi(-\tau) - \int_{-\tau}^{0} \mathrm{X}_0(t_1-\tau-s)\varphi'(s)ds,$$

we obtain: the system (6.1) has a solution $x(t)$ that satisfies the initial conditions $x(t) = \varphi(t)$, $-\tau \leq t \leq 0$, $x(t_1) = x_1$ if and only if the integrated equation

$$\mu =$$

$$\int_0^{t_1} \left(\sum_{m=o}^{k-1} 2e^{A(t_1-s-(m+1)\tau)} \sum_{p=0}^{m} (-1)^{p+m} A^p \frac{(t_1-s-(m+1)\tau)^p}{p!} + (-I)^k\right) Bu(s)ds \quad (6.2)$$

has solution $u(\xi)$, $0 \leq \xi \leq t_1$. We will search for the solution as a linear combination

$$u(\xi) =$$

$$\left[\left(\sum_{m=o}^{k-1} 2e^{A(t_1-\xi-(m+1)\tau)}\sum_{p=0}^{m}(-1)^{p+m}A^p\frac{(t_1-\xi-(m+1)\tau)^p}{p!}+(-I)^k\right)B\right]^T c \qquad (6.3)$$

where $c=(c_1,c_2,.....,c_n)^T$ is an unknown vector. After substitution (6.3) in system (6.2), we get

$$\mu=\left[\int_0^{t_1}\left(\sum_{m=o}^{k-1} 2e^{A(t_1-s-(m+1)\tau)}\sum_{p=0}^{m}(-1)^{p+m}A^p\frac{(t_1-s-(m+1)\tau)^p}{p!}+(-I)^k\right)B\times\right.$$

$$\left.B^T\left(\sum_{m=o}^{k-1} 2e^{A(t_1-s-(m+1)\tau)}\sum_{p=0}^{m}(-1)^{p+m}A^p\frac{(t_1-s-(m+1)\tau)^p}{p!}+(-I)^k\right)^T ds\right]c. \quad (6.4)$$

We will show that system (6.4) has the only solution. From proof of Theorem 5.1.1 we know that $\mathrm{X}_0(t-\tau-s)B$ can be represented by a linear combination independent functions with coefficients B, AB, ..., $A^{n-1}B$. Since $rank(S)=n$, then, when $0\le s\le t_1$, will hold $\mathrm{X}_0(t-\tau-s)B\neq\Theta$. Therefore for any constant vector $l=(l_1,l_2,.....,l_n)^T$, $(||l||>0)$ in $0\le s\le t_1$ will hold

$$||[\mathrm{X}_0(t-\tau-s)B]^T l||^2\neq 0, \quad 0\le s\le t_1.$$

And for any l is true [48]:

$$\int_0^{t_1}\left|\left|B^T\left[\sum_{m=o}^{k-1} 2e^{A(t_1-s-(m+1)\tau)}\sum_{p=0}^{m}(-1)^{p+m}A^p\frac{(t_1-s-(m+1)\tau)^p}{p!}+(-I)^k\right]^T l\right|\right|^2 ds$$

$$=l^T\int_0^{t_1}\mathrm{X}_0(t_1-\tau-s)BB^T[\mathrm{X}_0(t_1-\tau-s)]^T ds\cdot l>0,$$

or the matrix

$$\int_0^{t_1}\mathrm{X}_0(t_1-\tau-s)BB^T[\mathrm{X}_0(t_1-\tau-s)]^T ds$$

is positive definite. Therefore its determinant is nonzero. When solving system (6.4), we obtain

$$c=\left[\int_0^{t_1}\mathrm{X}_0(t_1-\tau-s)BB^T[\mathrm{X}_0(t_1-\tau-s)]^T ds\right]^{-1}\mu.$$

□

Example 6.1.1
Let us have the differential equation of 3^{rd} degree with a constant delay:

$$\dot{x}(t) = Ax(t) + Ax(t-1) + Bu(t),$$

where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have $\tau = 1, n = 3$. It is easy to see that the necessary and sufficient condition for controllability is implemented (becase of full rank of the matrix B, matrix S have full rank too), so the system is controllable.
Let us construct such control function that move system in time moment $t_1 = 2$ in point $x_1 = (1, 1, 1)^T$, using initial condition $x_0(t) = \varphi(t) = (0, 0, 0)^T$, $-1 \leq t \leq 0$. Using the result of the Theorem 6.1.1 we write:

$$u(t) = [\mathrm{X}_0(t_1 - \tau - t)B]^T \left[\int_0^{t_1} \mathrm{X}_0(t_1 - \tau - s)BB^T [\mathrm{X}_0(t_1 - \tau - s)]^T ds \right]^{-1} \mu,$$

$$\mu = x_1 - \mathrm{X}_0(t_1)\varphi(-\tau) - \int_{-\tau}^{0} \mathrm{X}_0(t_1 - \tau - s)\varphi'(s)ds.$$

While $\varphi(t) = (0, 0, 0)^T$, $-1 \leq t \leq 0$ then $\mu = (1, 1, 1)^T$. So, we have

$$u(t) = [\mathrm{X}_0(1 - t)B]^T \left[\int_0^{2} \mathrm{X}_0(1 - s)BB^T \left[\mathrm{X}_0(1 - s)\right]^T ds \right]^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

And finally

$$u(t) = \begin{cases} \begin{pmatrix} (-0.84t + 0.8)e^t - 0.4 \\ (-0.84t + 0.8)e^t - 0.4 \\ (-0.42t^2 + 0.8t - 0.14)e^t + 0.07 \end{pmatrix}, & 0 \leq t \leq 1, \\ \\ \begin{pmatrix} 0.02(t+1)e^t + 0.02(t^2 - t - 1)e^{t-1} + 0.01 \\ 0.02(t+1)e^t + 0.02(t^2 - t - 1)e^{t-1} + 0.01 \\ 0.01(t^2 + 2t + 4)e^t + 0.01(t^3 + t - 6)e^{t-1} \end{pmatrix}, & 1 \leq t \leq 2. \end{cases}$$

□

6.2 Systems with commutative matrices

Theorem 6.2.1 [105] *Let we have the control problem with delay with the commutative matrices* (5.4)*. Let* $t_1 \geq (k-1)\tau$ *and the sufficient conditions for controllability be implemented:*

$$rank(S_k) = rank\left(\{B; e^{-A_0\tau}A_1B; e^{-2A_0\tau}A_1^2B; ...; e^{-(k-1)A_0\tau}A_1^{k-1}B\big)\right\} = n,$$

Then the control function can be taken as

$$u(\xi) = [\mathrm{X}_0(t_1-\tau-\xi)B]^T \left[\int_0^{t_1} \mathrm{X}_0(t_1-\tau-s)BB^T\left[\mathrm{X}_0(t_1-\tau-s)\right]^T ds\right]^{-1}\mu,$$

$$0 \leq \xi \leq t_1,$$

$$where \qquad \mu = x_1 - \mathrm{X}_0(t_1)\varphi(-\tau) - \int_{-\tau}^{0} \mathrm{X}_0(t_1-\tau-s)\varphi'(s)ds,$$

and $\mathrm{X}_0(t)$ *is the fundamental matrix of solutions* (3.14) *on time interval* $t \geq (k-1)\tau$*.*

Proof. Using the Cauchy integral representation, we have that the solution of the system (5.4) with initial conditions $x_0(t) \equiv \varphi(t)$, $-\tau \leq t \leq 0$ has the form

$$x(t) = \mathrm{X}_0(t)\varphi(-\tau) + \int_{-\tau}^{0} \mathrm{X}_0(t-\tau-s)\varphi'(s)ds + \int_0^t \mathrm{X}_0(t-\tau-s)Bu(s)ds \qquad (6.5)$$

Using the notations (3.14) we obtain: the system (6.5) has a solution $x(t)$ that satisfies the initial conditions $x(t) \equiv \varphi(t)$, $-\tau \leq t \leq 0$, $x(t_1) = x_1$ if and only if the integrated equation

$$\int_0^{t_1} e^{A_0(t_1-\tau-s)}e_\tau^{D(t_1-2\tau-s)}Bu(s)ds = \mu \qquad (6.6)$$

has solution $u(\xi)$, $0 \leq \xi \leq t_1$, where D was defined in Theorem 3.2.2. We will search for the solution as a linear combination

$$u(\xi) = \left[e^{A_0(t_1-\tau-\xi)}e_\tau^{D(t_1-2\tau-\xi)}B\right]^T c \qquad (6.7)$$

where $c = (c_1, c_2, ..., c_n)^T$ is unknown vector. After substitution (6.7) in system (6.6), we get

$$\left[\int_0^{t_1} e^{A_0(t_1-\tau-s)}e_\tau^{D(t_1-2\tau-s)}BB^T\left[e^{A_0(t_1-\tau-s)}e_\tau^{D(t_1-2\tau-s)}\right]^T ds\right] c = \mu \qquad (6.8)$$

We will show that system (6.8) has the only solution. Since $t_1 \geq (k-1)\tau$, then using the Kelly's formula, vector $e^{A_0(t-\tau-s)}e_\tau^{D(t-2\tau-s)}B$ for any fixed $0 \leq s \leq t_1$ can be represented by a linear combination of independent function with coefficints $B; e^{-A_0\tau}A_1B; e^{-2A_0\tau}A_1^2B; ...; e^{-(k-1)A_0\tau}A_1^{k-1}B$. Since vectors are linearly independent, then when $0 \leq s \leq t_1$ will hold

$$e^{A_0(t-\tau-s)}e_\tau^{D(t-2\tau-s)}B \neq \Theta.$$

Therefore for any constant vector $l = (l_1, l_2, ..., l_n)^T$, $(||l|| > 0)$ in $0 \leq s \leq t_1$ will hold

$$\left|\left|\left[e^{A_0(t-\tau-s)}e_\tau^{D(t-2\tau-s)}B\right]^T l\right|\right|^2 \neq 0, \quad 0 \leq s \leq t_1.$$

And for any vector l is true [48]:

$$\int_0^{t_1} \left|\left|B^T\left[e^{A_0(t_1-\tau-s)}e_\tau^{D(t_1-2\tau-s)}\right]^T l\right|\right|^2 ds$$

$$= l^T \int_0^{t_1} e^{A_0(t_1-\tau-s)}e_\tau^{D(t_1-2\tau-s)}BB^T\left[e^{A_0(t_1-\tau-s)}e_\tau^{D(t_1-2\tau-s)}\right]^T ds \cdot l > 0,$$

or the matrix

$$\left[\int_0^{t_1} e^{A_0(t_1-\tau-s)}e_\tau^{D(t_1-2\tau-s)}BB^T\left[e^{A_0(t_1-\tau-s)}e_\tau^{D(t_1-2\tau-s)}\right]^T ds\right]$$

is positive definite. Hence its determinant is nonzero. When solving system (6.8) we obtain

$$c = \left[\int_0^{t_1} e^{A_0(t_1-\tau-s)}e_\tau^{D(t_1-2\tau-s)}BB^T[e^{A_0(t_1-\tau-s)}e_\tau^{D(t_1-2\tau-s)}]^T ds\right]^{-1} \mu.$$

□

Example 6.2.1

Let us have the differential equation of 2^{nd} degree with a constant delay:

$$\dot{x}(t) = A_0x(t) + A_1x(t-1) + Bu(t),$$

where

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So we have $\tau = 1, n = 2$ and $A_0A_1 = A_1A_0$. We want to know whether this system is controllable in the moment of time $t_1 = 3$. Let us check the sufficient conditions:

$$rank(S_3) = rank\left(B \quad e^{-A_0\tau}A_1B \quad e^{-2A_0\tau}A_1^2B\right) = 2.$$

It is easy to see that the sufficient conditions for controllability is implemented ($rank(B) = 2$), so the system is controllable in time moment $t_1 = 3$.

Let us construct such control function that move system in time moment $t_1 = 3$ in point $x_1 = (1,1)^T$, using initial condition $x_0(t) \equiv \varphi(t) = (0,0)^T$, $-\tau \le t \le 0$. Using the result of the theorem (6.2.1) we write:

$$u(t) = \left[e^{A_0(t_1-\tau-t)} e_\tau^{D(t_1-2\tau-t)} B\right]^T \times$$

$$\left[\int_0^{t_1} e^{A_0(t_1-\tau-s)} e_\tau^{D(t_1-2\tau-s)} BB^T \left[e^{A_0(t_1-\tau-t)} e_\tau^{D(t_1-2\tau-t)}\right]^T ds\right]^{-1} \mu,$$

$$\mu = x_1 - e^{A_0(t_1)} e_\tau^{D(t_1)} \varphi(-\tau) - \int_{-\tau}^{0} e^{A_0(t_1-\tau-s)} e_\tau^{D(t_1-2\tau-s)} \varphi'(s) ds.$$

So, we have

$$u(t) = \left[e^{2-t} e_1^{D(1-t)}\right]^T \left[\int_0^3 e^{2-s} e_1^{D(1-s)} \left[e^{(2-s)} e_1^{D(1-s)}\right]^T ds\right]^{-1} \mu,$$

$$\mu = (1,1)^T - (e^3 I) e_1^{3D} (0,0)^T - \int_{-1}^{0} e^{(2-s)} (e_1 I)^{D(1-s)} (0,0)^T ds = (1,1)^T.$$

Finally we have

$$u(t) = \begin{cases} \begin{pmatrix} 0.17e^t \\ 0.06e^t \end{pmatrix}, & 0 \le t \le 1, \\[2ex] \begin{pmatrix} (0.00012t + 0.1)e^t \\ 0.0003e^t \end{pmatrix}, & 1 \le t \le 2, \\[2ex] \begin{pmatrix} (-0.00002t^2 + 0.00016t + 0.0005)e^t \\ (-0.00002t^2 - 0.00007t - 0.00011)e^t \end{pmatrix}, & 2 \le t \le 3. \end{cases}$$

□

6.3 Systems with general matrices

Theorem 6.3.1 *Let us have the control problem with delay with general matrices* (5.10)*. Let* $t_1 \geq (k-1)\tau$ *and the sufficient conditions for controllability be implemented:* $det(Q) = n$*, where the matrix* Q *was defined in Theorem 5.3.2. Then the control function can be taken as*

$$u(\xi) = [\mathrm{X}_0(t_1-\tau-\xi)B]^T \left[\int_0^{t_1} \mathrm{X}_0(t_1-\tau-s)BB^T\left[\mathrm{X}_0(t_1-\tau-s)\right]^T ds\right]^{-1} \mu,$$

$$0 \leq \xi \leq t_1,$$

$$\textit{where} \quad \mu = x_1 - \mathrm{X}_0(t_1)\varphi(-\tau) - \int_{-\tau}^{0} \mathrm{X}_0(t_1-\tau-s)\varphi'(s)ds,$$

and $\mathrm{X}_0(t)$ *is the fundamental matrix of solutions* (3.24) *on time interval* $t \geq (k-1)\tau$*.*

Proof. Using the Cauchy integral representation, we have that the solution of the system (5.10) with initial conditions $x_0(t) \equiv \varphi(t)$, $-\tau \leq t \leq 0$ has the form

$$x(t) = \mathrm{X}_0(t)\varphi(-\tau) + \int_{-\tau}^{0} \mathrm{X}_0(t-\tau-s)\varphi'(s)ds + \int_0^t \mathrm{X}_0(t-\tau-s)Bu(s)ds \qquad (6.9)$$

Using the notations (3.24) we obtain: the system (6.9) has a solution $x(t)$ that satisfies the initial conditions $x(t) \equiv \varphi(t)$, $-\tau \leq t \leq 0$, $x(t_1) = x_1$ if and only if the integrated equation

$$\int_0^{t_1} \mathrm{X}_0(t-\tau-s)Bu(s)ds = \mu \qquad (6.10)$$

has solution $u(\xi)$, $0 \leq \xi \leq t_1$. We will search for the solution as a linear combination

$$u(\xi) = \left[\mathrm{X}_0(t-\tau-\xi)\right]^T c \qquad (6.11)$$

where $c = (c_1, c_2, \ldots.., c_n)^T$ is unknown vector. After substitution (6.11) in system (6.10), we get

$$\left[\int_0^{t_1} \mathrm{X}_0(t-\tau-s)BB^T\left[\mathrm{X}_0(t-\tau-s)\right]^T ds\right] c = \mu \qquad (6.12)$$

We will show that system (6.12) has the only solution. According the proof of the Theorem 3.3.5, vector $\mathrm{X}_0(t-\tau-s)B$ for any fixed $0 \leq s \leq t_1$ can be represented by a linear combination of functions $\psi_{i_1 i_2 \ldots i_p}(t_1, u_0(t_1))$ (defined in proof of the Theorem 5.3.2) with coefficients B, A_0B, A_1B, A_0^2B, $(A_0A_1+A_1A_0)B$, A_1^2B, A_0^3B, ..., $A_0^2A_1^2B$. Since vectors are linearly independent, then when $0 \leq s \leq t_1$ will hold

$$\mathrm{X}_0(t-\tau-s)B \neq \Theta.$$

Therefore for any constant vector $l = (l_1, l_2, ..., l_n)^T$, $(||l|| > 0)$ in $0 \le s \le t_1$ will hold

$$\left|\left|[\mathrm{X}_0(t-\tau-s)B]^T\, l\right|\right|^2 \neq 0, \quad 0 \le s \le t_1.$$

And for any l is true [48]:

$$\int_0^{t_1} \left|\left|[\mathrm{X}_0(t-\tau-s)B]^T\, l\right|\right|^2 ds$$

$$= l^T \int_0^{t_1} \mathrm{X}_0(t-\tau-s)BB^T \left[\mathrm{X}_0(t-\tau-s)\right]^T ds \cdot l,$$

or the matrix

$$\left[\int_0^{t_1} \mathrm{X}_0(t-\tau-s)BB^T \left[\mathrm{X}_0(t-\tau-s)\right]^T ds\right]$$

is positive definite. Hence its determinant is nonzero. When solving system (6.12) we obtain

$$c = \left[\int_0^{t_1} \mathrm{X}_0(t-\tau-s)BB^T[\mathrm{X}_0(t-\tau-s)]^T ds\right]^{-1} \mu.$$

□

Example 6.3.1
Let us have the differential equation of 3^{rd} degree with a constant delay:

$$\dot{x}(t) = A_0x(t) + A_1x(t-1) + Bu(t),$$

where

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

with initial conditions $x = (1, 1, 0)^T, -1 \le t \le 0$.

As we see $\tau = 1, n = 3$ and $A_0A_1 \neq A_1A_0$. Let us construct such control function that move system in time moment $t_1 = 0.9$ in point $x_1 = (1, 1, 1)^T$. We already know from Example 5.3.2 that the system is relatively controllable. And accoding to the result of calculation fundamental solution matrix from Example 3.3.1 we get

$$u(s) = [\mathrm{X}_0(t_1-\tau-s)B]^T \left[\int_0^{t_1} \mathrm{X}_0(t_1-\tau-s)BB^T \left[\mathrm{X}_0(t_1-\tau-s)\right]^T ds\right]^{-1} \mu,$$

$$= \left[X_1(-0.1-s) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]^T \left[\int_0^2 X_1(-0.1-s) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^T X_1(-0.1-s)^T ds \right]^{-1} \mu$$

where

$$X_1(t) = \begin{pmatrix} 3e^t - 1 & 2te^t - 2e^t + 2 & 3te^t - 3e^t + 2 \\ 3te^t - e^t + 1 & 2te^t - e^t + 1 & 3te^t - e^t + 1 \\ 0 & 0 & 2e^t \end{pmatrix}$$

and

$$\mu = x_1 - \mathrm{X}_0(t_1)\varphi(-\tau) - \int_{-\tau}^{0} \mathrm{X}_0(t_1 - \tau - s)\varphi'(s)ds$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - X_1(0.9) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -6.89 \\ -7.15 \\ 1 \end{pmatrix}.$$

And the control will be

$$u(t) = 218.49te^{-t} + 72.21e^{-t} - 88.08, \quad 0 \le t \le 0.9.$$

□

7 CONCLUSIONS

In this thesis, a solution of the system of linear differential equation with delay in general form was built. There was presented the view of solutions for the system with same matrices, the system with commutative matrices and the general case matrices. Examples were given to illustrate the proposed solution.

The stability and the asymptotic stability of a solution of a certain class of a differential linear matrix equation with delay was investigated. The Lyapunov's functional has the basic role in the investigation. Example was given to illustrate the proposed method of investigation of the stability of the system.

Necessary and sufficient condition for controllability of differential linear matrix equation with the same matrices with delay was defined and the control was built. Sufficient conditions for controllability of differential equation with commutative matrices and general matrices with delay were also defined and the control was build. Examples were given also to illustrate the proposed controllability criterions and controls were build.

The prove of necessity of conditions from the Theorems 5.2.1 and 5.3.1 remains open problems.

Also open problem remains the construction the control function optimal due some criterion.

As future step to investigated can be consider the differential equation

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + w(t),$$

where $w(t)$ is a stochastic vector ("white noice").

Also it is open problem to construct controllability criterion for the system with non-constant delay

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h(t)), \quad 0 < h(t) < t.$$

REFERENCES

[1] Aizerman M.A., Gantmacher F.R.: *Absolute Stability of Regulator Systems.* Holden-Day, Inc., San Francisco, 1964. 172pp.

[2] Akhmerov R.R., Kamenskii M.I., Potapov A.S., Rodkina A.J., Sadovskii B. N.: *The theory of equations of neutral type.* Results of Sciences and Technology, Mathematical Analysis, Part 19, 1982, 55-126. (In Russian)

[3] Andronov A.A., Pontryagin L.S.: *Coarse Systems.* Doklady AV SSSR, 1937, 15(5),247-250.

[4] Azbelev N.V., Maksimov V.P., Rakhmatulina L.F.: *Introduction to Theory of Functional Differential Equations.* Nauka. 1991. 278. (In Russian)

[5] Azbelev N.V., Simonov P.M.: *Stability of Differential Equations with Aftereffect.* Stability and Control: Theory, Methods and Applications 20. London: Taylor and Francis. xviii, 222 p. (2003).

[6] Barbashin E.A.: *Lyapunov Functions.* Nauka. 1970. 240. (In Russian)

[7] Bellman R.: *Stability Theory of Solutions of Differential Equations.* Foreign Literature PH. 1954. 216.

[8] Bellman R., Cooke K., Kennet L.: *Differential-Difference Equations.* World. 1967. 548.

[9] Birkhoff G.D.: *Dynamical Systems with Two Degrees of Freedom* Trans. Amer. Math. Soc. 18: 199–300. 1917.

[10] Bogoliubov N.N., Mitropolsky Y.A.: *Asymptotic Methods in the Theory of Non-Linear Oscillations.* New York, Gordon and Breach. 1961

[11] Bohl P.: *Über Differentialgleichungen.* J. für Reine und Angewandte Mathematik, 144:284-313, 1913

[12] Boichuk A., Diblík J., Khusainov D.Ya., Růžičková M.: *Boundary value problems for delay differential systems.* Advances in Difference Equations, vol. 2010, Article ID 593834, 20 pages, 2010. doi:10.1155/2010/593834

[13] Boichuk A., Diblík J., Khusainov D.Ya., Růžičková M.: *Fredholm's boundary-value problems for differential systems with a single delay.* Nonlinear Analysis, **72** (2010), 2251–2258. (ISSN 0362-546X)

[14] Carkovs Je., Verniger I., Zasinskii V.: *On stochastic stability of Markov dynamical systems.* Theor. Probability and Math. Statistic, No 75, 2007, 179 - 188.

[15] Chetaev N.G.: *Stability of Motion*. Moscow, Nauka, 1990, 176 p. (In Russian)

[16] Chukwu E.N.: *Stability and Time-optimal Control of Hereditary Systems. With application to the economic dynamics of the US.* 2nd ed. Series on Advances in Mathematics for Applied Sciences. 60. Singapore: World Scientific. xix, 495 p. (2001).

[17] Daletskii J.A., Crane M.G.: *Stability of Differential Equations Solutions in Banach Space.* Nauka. 1970. 534. (In Russian)

[18] Demidovich B.P.: *Lectures on the Mathematical Theory of Stability.* Nauka. 1967. 472. (In Russian)

[19] Diblík J., Khusainov D., Lukáčová J., Růžičková M.: *Control of oscillating systems with a single delay.* Advances in Difference Equations, Volume 2010 (2010), Article ID 108218, 15 pages, doi:10.1155/2010/108218.

[20] Diblík J., Khusainov D., Růžičková M.: *Controllability of linear discrete systems with constant coefficients and pure delay.* SIAM Journal on Control and Optimization, **47**, No 3 (2008), 1140–1149. DOI: 10.1137/070689085, url = http://link.aip.org/link/?SJC/47/1140/1. (ISSN Electronic: 1095-7138, Print: 0363-0129)

[21] Diblík J., Khusainov D., Baštinec J., Ryvolová A.: *Exponential stability and estimation of solutions of linear differential systems with constant delay of neutral type.* In 6. konference o matematice a fyzice na vysokých školách technických s mezinárodní účastí. Brno, UNOB Brno. 2009. p. 139 - 146. ISBN 978-80-7231-667-0.

[22] Diblík J., Baštinec J., Khusainov D., Ryvolová A.: *Estimates of perturbed solutions of neutral type equations.* Žurnal Občisljuvalnoji ta Prikladnoji Matematiki. 2009. 2009(2). p. 108 - 118. ISSN 0868-6912.(in Ukrainian)

[23] Diblík J., Ryvolová A., Baštinec J., Khusainov D.: *Stability and estimation of solutions of linear differential systems with constant coefficients of neutral type.* Journal of Applied Mathematics. 2010. III.(2010)(2). p. 25 - 33. ISSN 1337-6365.

[24] Diblík J., Ryvolová A., Khusainov D., Baštinec J.: *Stability of linear differential system.* In Eight International Conference on Soft Computing Applied in Computer and Economic Environments. Kunovice, EPI Kunovice. 2010. p. 173 - 180. ISBN 978-80-7314-201-8.

[25] Diblík J., Khusainov D., Baštinec J., Ryvolová A.: *Estimates of perturbed solutions of neutral type equations.* Žurnal Občisljuvalnoji ta Prikladnoji Matematiki. 2010. 2009(2(98)). p. 108 - 118. ISSN 0868-6912.

[26] Diblík J., Khusainov D., Baštinec J., Ryvolová A.: *Optimization methods of investigation of stability of systems of neutral type.* In XXVII International Colloquium on the Management of Educational Process aimed at current issues in science, education and creative thinking development, Proceedings. Brno, FEM UNOB. 2010. p. 111 - 120. ISBN 978-80-7231-733-2.

[27] Diblík J., Baštincová A., Baštinec J., Khusainov D., Shatyrko A.: *Estimates of perturbed nonlinear systems of indirect control of neutral type.* Cybernetics and Computer Engineering. 2010. 2010(160). p. 72 - 85. ISSN 0452-9901.

[28] Diblík J., Baštinec J., Khusainov D., Baštincová A.: *Exponential stability and estimation of solutions of linear differential systems of neutral type with constant coefficients.* Boundary Value Problems. 2010. 2010(1). Article ID 956121, doi: 10.1155/2010/956121. p. 1 - 20. ISSN 1687-2762.

[29] Diblík J., Khusainov D., Baštinec J., Baštincová A., Shatyrko A.: *Estimates of perturbation of nonlinear indirect interval regulator system of neutral type.* Journal of Automation and Information Sciences. 2011. 2011 (43)(DOI: 10.1615/JAu). p. 13 - 28. ISSN 1064-2315.

[30] Diblík J., Baštinec J., Khusainov D., Baštincová A.: *Interval stability of linear systems of neutral type.* Žurnal občisljuvalnoji ta prikladnoji matematiki. 2012. 2011(391060). p. 148 - 160. ISSN 0868-6912.

[31] Dzhalladova I.A., Baštinec J., Diblík J., Khusainov D.Ya.: *Estimates of exponential stability for solutions of stochastic control systems with delay.* Hindawi Publishing Corporation. Abstract and Applied Analysis. Volume 2011, Article ID 920412, 14 pages, doi: 10.1155/2011/920412.

[32] Dzhalladova I.A., Khusainov D.Ya.: *Convergence estimates for solutions of a linear neutral type stochastic equation.* Functional Differential Equations, V.18, N?.3-4, 2011. 177-186.

[33] El-Kebir Boukas, Zi-Kuan Liu.: *Deterministic and Stochastic Time Delay Systems.* Birkhauser, Boston-Basel-Berlin. 2002. 423 p.

[34] Elsgolts L.E., Norkin S.B.: *Introduction to the Theory of Differential Equations with Delay Argument.* Nauka. p.296. 1971. (In Russian)

[35] Fedorenko R.P. *Approximate Solution of Optimal Control Problems.* Moskva. Nauka. 1978. 488. (In Russian)

[36] Gabasov R.F., Kirilova F.M.: *Qualitative Theory of Optimal Processes.* Moskva. Nauka. 1971. 508. (In Russian)

[37] Gantmacher F.R.: *Theory of Matrixes.* Moscow, Nauka, 1966, 576 p. (In Russian)

[38] Gelig A.H., Leonov G.A., Jakubovicz V.A.: *The Stability of Nonlinear Systems with a Non-unique Equilibrium State.* Nauka, Moscow, Russia, 1978. (In Russian)

[39] Gelig A.H., Leonov G.A., Fradkov A.L., Eds.: *Nonlinear Systems. Frequency-domain and Matrix Inequalities.* Moscow, Fizmatlit, 2008. (In Russian)

[40] Germanovich O.P.: *Linear Periodic Equation of Neutral Type and Their Applications.* LGU. 1986. 106. (In Russian)

[41] Gershgorin S.A.: *About the limits of eigenvalues in a matrix.* Iz. Ak. nauk USSR, Otd. Fiz-Math. Nauk 7, 749-754, 1931. (In Russian)

[42] Gikhman J.I., Skorokhod A.V.: *Stochastic Differential Equations.* Springer Verlag, 1972.

[43] Gopalsamy K.: *Stability and Oscillations in Delay Differential Equations of Population Dynamics.* Mathematics and its Applications (Dordrecht). 74. Dordrecht etc.: Kluwer Academic Publishers. xii, 501 p. 1992.

[44] Gu K., Kharitonov V.L., Chen J.: *Stability of Time-delay Systems.* Control Engineering. Boston, MA: Birkhäuser. xx, 353 p. 2003.

[45] Hale J.: *Theory of Functional Differential Equations.* World. 1977. 421. (In Russian)

[46] Hale J., Lunel S.M.V.: *Introduction to Functional Differential Equations.* Springer, New York. 447p. 1993.

[47] Hale J., Verduyn L.S.M.: *Introduction to Functional Differential Equations.* Springer-Verlag, New-York-Berlin-Ytidelberg-London-Paris, 1991.

[48] Kalman R.E.: *On the General Theory of Control System.* Proc. First International Congress of IFAC, vol. 2, AN SSSR, 1961. (in Russian)

[49] Kelley W.G., Peterson A.C.: *Difference Equations. An Introduction with Applications.* Academic Press. Inc., Harcourt Brace Jovanovich Publishers, 1991.

[50] Keqin Gu, Kharitonov V.L., Jie Chen: *Stability of Time-Delay Systems.* Birkhäuser, Boston. 2005. 351.

[51] Kharitonov V.L.: *Stability criterion of one type of quasipolynoms of retarded type.* Automatics and Telemechanics. No 2, 1991, 73 - 82. (In Russian)

[52] Khusainov D.Ya., Yunkova E.A.: *Investigation of the stability of linear systems of neutral type by the Lyapunov-function method.* (English. Russian original) Differ. Equations 24, No.4, 424-431 (1988); translation from Differ. Uravn. 24, No.4, 613-621. 1988.

[53] Khusainov D.Ya., Shatyrko A.V.: *The method of Lyapunov's function in investigation of stability of differential-functional systems.* Kiev, 1997, 236 p. (In Russian)

[54] Khusainov D.Ya., Shuklin G.V.: *About relatively controllability of systems with pure delay.* Applied Mechanics. 2005. No.41,2. 118-130. (In Russian)

[55] Kolmanovskii V.B.: *About stability of nonlinear system with delay.* Mathematical Notes Ural University. 1970. 743-751. (In Russian)

[56] Kolmanovskii V.B., Myshkis A.D.: *Introduction to the Theory and Applications of Functional Differential Equations.* Mathematics and its Applications. 463. Dordrecht: Kluwer Academic Publishers. xvi, 648 p. 1999.

[57] Kolmanovskii V.B., Myshkis A.D.: *Applied Theory of Functional Differential Equations.* Mathematics and Its Applications. Soviet Series. 85. Dordrecht: Kluwer Academic Publishers. xv, 234 p. 1992.

[58] Kolmanovskii V.B., Nosov V.R.: *Stability and Periodical Regimes of Controlled Systems with Aftereffect.* Nauka. 1981. 448 p. (In Russian)

[59] Kolmanovskii V.B., Nosov V.R.: *Stability of Functional Differential Equations.* Mathematics in Science and Engineering, Vol. 180. London etc.: Academic Press, Inc. (Harcourt Brace Jovanovich, Publishers). XIV, 217 p. 1986.

[60] Korenevskij D.G.: *Stability of Dynamical Systems under Random Perturbations of Parameters. Algebraic criteria.* Kiev: Naukova Dumka. 208 p. 1989. (In Russian)

[61] Krasovskii N.N.: *Inversion of Theorems of Second Lyapunov's Method and Stability Problems in the First Approximation.* Applied Mathematics and Mechanics. 1956. 255-265. (In Russian)

[62] Krasovskii N.N.: *Some Problems of Theory of Stability of Motion.* Moscow, Fizmatgiz, 1959. (In Russian)

[63] Krasovskii N.N.: *Stability of Motion. Applications of Lyapunov's Second Method to Differential Systems and Equations with Delay.* Translated by J.L. Brenner. Stanford, Calif.: Stanford University Press 1963, VI, 188 p. 1963.

[64] Krasovskii N.N.: *The Theory of Motion Control. Linear Systems.* Nauka. 1968. 475. (In Russian)

[65] Krutko P.D.: *Inverse problems of the control systems dynamic: Non-linear models.* Nauka. 1988. 326. (In Russian)

[66] Kuntsevich V.M., Lychak M.M.: *Synthesis of Automatic Control Systems via Lyapunov Functions.* Nauka. 1977. 400. (In Russian)

[67] Kurbatov V.G.: *Linear Differential-difference Equations.* Voroneg. 1990, 168 p. (In Russian)

[68] La Salle J.P., Lefshetz S.: *Stability By Liapunov's Direct Method, With Applications.* Moscow, Mir, 1964, 168 p. (In Russian)

[69] Leonov G.A.: *Chaotic Dynamics and the Classical Theory of Motion Stability: [monograph].* Igevsk. 2006. 167. (In Russian)

[70] Liu Mei-Gin: *Stability analysis of neutral-type nonlinear delayed systems: An LMI approach,* Journal of Zhejiang University, SCIENCE A, No.7, 237–244 (2006).

[71] Liu Xiu-Xiang, Xu Bugong: *A further note on stability criterion of linear neutral delay-differential systems.* J. Franklin Inst. 343, No. 6, 630-634 (2006).

[72] Liu Xiu-Xiang, Xu Bugong: *Absolute Stability of Nonlinear Control Systems.* Springer Science+Business Media B.V., 2008, 390 p.

[73] Lurie A.I.: *Some Nonlinear Problems of the Theory of Automatic Control.* Moscow, Gostekhizdat, 1951, 251 p. (In Russian)

[74] Lyapunov A.M.: *General Problem of Stability of Motion.* Math. Soc., Kharkov, 1892,(Published in Collected Papers, 2,Ac. Sci. USSR. Moscow-Leningrad, 1956, 2-263). (In Russian)

[75] Malkin I.G.: *Theory of Stability of Motion.* Moscow, Nauka, 1966, 530 p. (In Russian)

[76] Miller R.K.: *Nonlinear Volterra Integral Equations.* W. A. Benjamin. 1971. 468.

[77] Minorsky N.F.: *Self-excited in dynamical systems possessing retarded actions.* j. Appl. Mech. 1942. 9.

[78] Mitropolskii U.A.: *Differential Equations with Delay Argument.* Naukova dumka. 1977. 299. (In Russian)

[79] Myshkis A.D.: *Linear Differential Equations with Delay Argument.* Nauka. 1972. 352. (In Russian)

[80] Norkin S.B.: *Second Order Differential Equations with Delay Argument. Some Questions in the Theory of Vibrations of Systems with Delay.* Nauka. 1965. 354. (In Russian)

[81] Park J.H., Won S.: *A note on stability of neutral delay-differential systems.* J. Franklin Inst. 336, No.3, 543-548 (1999).

[82] Pinney E.: *Ordinary Differential-Difference Equation.* Foreign Literature PH. 1961. 248.

[83] Polak E.: *Optimization: Algorithms and Consistent Approximations.* Springer Series in Applied Mathematical Sciences, Vol. 124, New York, Springer-Verlag. 1997. 782.

[84] Razumihin B.S.: *Stability of Delay Systems.* Nauka. 1988. 112. (In Russian)

[85] Rusch H., Abets P., Laloy M.: *Direct Method of Lyapunov in Stability Theory.* World. 1980. 300.

[86] Ryvolová A.: *Stability properties of solutions of linear differential systems of neutral type.* In Proceedings of the 16th conference STUDENT EEICT. Brno, BUT. 2010. p. 119 - 123. ISBN 978-80-214-4080-7.

[87] Sabitov K.B.: *Functional, Differential and Integral Equations. Textbook for University Students Majoring in "Applied Mathematics and Informatics" and the Direction of "Applied Mathematics and Computer Science".* Hight school. 2005. 702. (In Russian)

[88] Shatyrko A. V., Khusainov D. Ya., Diblík J., Baštinec J., Rivolová A.: *Estimations of perturbed interval nonlinear systems non-direct regulations of neutral type.* Problems of Control and Informatics, 1, 2011. 15-29.

[89] Sysoev V.V.: *System modeling multi-object. Methods of analysis and optimization of complex systems.* Moskow. IFGP Sciences, 1993.1. 80-88. (In Russian)

[90] Tamarkin J.D.: *Some general problems of the theory of ordinary linear differential equations.* Petrograd, Mathematische Zeitschift, vol. 27 (1927), p.1.

[91] Vasiliev F.P. *Optimization Methods.* Moskow. Nauka. 2002. 415. (In Russian)

[92] Velichko D.A.: *Methods of multi-criteria search for optimal choices of equipment and technology for production lines (for example, semiconductor manufacturing).* Diss. Candidate. tehn. Science. Voronezh. 1983. 219s. (In Russian)

[93] Volterra V.: *The Theory of Functional, Integral and Integro-Differential Equations.* Nauka. 1982. 304. (In Russian)

[94] Vorotnikov V.I., Rumyantsev V.V.: *Stability and control on the part of the coordinates of the phase vector of dynamical systems: Theory, Methods and Applications.* ISBN: 5-89176-154-8. Nauchnyy mir. 2001. 321. (In Russian)

[95] Xiaoxin L., Liqiu W., Pei Y.: *Stability of Dynamical Systems.* Monograph Series on Nonlinear Science and Complexity 5. Amsterdam: Elsevier. xii, 706 p. 2007.

[96] Zubov V.I.: *Stability of Motion. Methods of Lyapunov and their Applications.* Moscow, Visshaya shkola, 1973, 217 p. (In Russian)

[97] Baštinec J.; Piddubna G.; Khusainov D.Ya.: *Solution of a certain class of matrix linear delayed system.* In Dynamical System Modeling and Stability Investigation. Kiev, Ukraine, University of Kyiv, UA. 2011. p. 149 - 150. ISBN 966-76-52-00-9.

[98] Baštinec J., Piddubna G.: *Solution of matrix linear delayed system.* In XXIX International Colloquium on the Management on the Educational Process aimed at current issues in science, education and creative thinking development. Brno. 2011. p. 51 - 60. ISBN 978-80-7231-780-6.

[99] Baštinec J., Piddubna G.: *Solution of matrix linear delayed system.* In 7. konference o matematice a fyzice na vysokých školách technických s mezinárodní úcastí. Brno. 2011. p. 48 - 57. ISBN 978-80-7231-815-5.

[100] Baštinec J., Piddubna G.: *Solution of one practice matrix linear delayed system.* In XXX International Colloquium on the Management of Educational Process. Brno. 2012. p. 33 - 38. ISBN 978-80-7231-866-7.

[101] Baštinec J., Piddubna G., Khusainov D.Ya.: *Representation of solutions for linear stationary systems with one delay.* In Functional Differential Equations and Applications 2012. Ariel. 2012. p. 20 - 21.

[102] Baštinec J., Piddubna G.: *Stability of a certain class of matrix differential delayed equation.* In International Conference "Presentation of Mathematics 11", Proceedings. Liberec, Technical University of Liberec. 2011. p. 7 - 14. ISBN 978-80-7372-773-4.

[103] Baštinec J., Piddubna G.: *Solutions and stability of solutions of a linear differential matrix system with delay.* In Proceedings of The IEEEAM/NAUN International Conferences. Tenerife. 2011. p. 94 - 99. ISBN 978-1-61804-058-9.

[104] Baštinec J., Piddubna G., Khusainov D.Ya.: *Stability and stabilization of linear systems with aftereffect.* In XIV International Conference Stability and Oscillations of Nonlinear Control Systems. Moskow. 2012. p. 332 - 334. ISBN 978-5-91450-106-5.

[105] Baštinec J.; Piddubna G.: *Controllability of stationary linear systems with delay.* In 10th International conference APLIMAT. Bratislava, FME STU. 2011. p. 207 - 216. ISBN978-80-89313-51-8.

[106] Baštinec J., Piddubna G., Diblík J., Khusainov D.Ya.: *Solution and controllability of matrix linear differential equation with delay.* In 8th International ISAAC Congress, Moskow. 2011. p. 315 - 315. ISBN 978-5-209-04088-0.

[107] Baštinec J., Piddubna G.: *Solutions and controllability on systems of differential equations with delay.* In Ninth International Conference on Soft Computing Applied in Computer and Economic Environments, ICSC 2011. Kunovice, EPI Kunovice. 2011. p. 115 - 120. ISBN 978-80-7314-221-6.

[108] Baštinec J., Piddubna G.: *Controllability for a certain class of linear matrix systems with delay.* In APLIMAT, 11th International Conference. Bratislava, STU. 2012. p. 93 - 102. ISBN 978-80-89313-58-7.

[109] Baštinec J., Piddubna G.: *Controllability and control construction of linear matrix system with delay.* In XIV International Scientific Conference Devoted to the memory of Academician M. Kravchuk. Kyev, Ukraina, Kyev University. 2012. p. 11 - 12. ISBN 978-617-696-013-3.

[110] Baštinec J., Piddubna G.: *Controllability of matrix linear differential systems with delay.* In Conference on Differential and Difference Equations and Applications, Těrchová. 2012. 2012. p. 36 - 36. ISBN 978-80-554-0543-8.

[111] Baštinec J., Piddubna G.: *Controllability research of linear differential matrix system with delay.* In Modeling, Control and Stability MCS-2012. Simferopol. 2012. p. 24 - 25. ISBN 978-966-491-327-7.

[112] Baštinec J., Piddubna G.: *Controllability for a certain class of linear matrix systems with delay.* Journal of Applied Mathematics. 2013. V(2012)(II). p. 13 - 23. ISSN1̃337-6365.

[113] Baštinec J., Piddubna G.: *Controllability of matrix linear delayed system.* In XVI International Conference "Dynamical system modeling and stability investigation". Kyjev. 2013. p. 342 - 343. ISBN 978-80-7231-923-7.

[114] Baštinec J., Piddubna G.: *Solution and controllability research of one matrix linear delayed system.* In XXXI International Colloquium on the Management of Educational Process. 2013. p. 21 - 26. ISBN 978-80-7231-924-4.

[115] Baštinec J., Piddubna G., Khusainov D.Ya.: *Controllability criterion for linear delayed differential equation.* In WDEA 2013, 6th International Workshop on Differential Equations and Applications. Izmir, Turecko, Izmir University. 2013. p. 13 - 14.

[116] Piddubna G.: *Controllability of Stationary Linear Systems with Delay.* In Student EEICT 2011, Proceedings of the 17th Conference, Vol. 3. Brno, FEEC BUT. 2011. p. 371 - 375. ISBN 978-80-214-4273-3.

[117] Piddubna G.: *Controllability and Control Construction for a Certain Class of Linear Matrix Systems with Delay.* In Student EEICT 2012, Proceedings of the 18th Conference, Vol. 3. Brno, FEEC BUT. 2012. p. 278 - 282. ISBN 978-80-214-4462-1.

[118] Piddubna G.: *Controllability and Control Construction for Linear Matrix Systems with Delay.* In Student EEICT 2013, Proceedings of the 19th Conference, Vol. 3. Brno, FEEC BUT. 2013. p. 154 - 158. ISBN 978-80-214-4695-3.

[119] Piddubna G.: *Controllability criterion for one linear matrix delayed equation.* In Student EEICT 2014, Proceedings of the 20th Conference, Vol. 3. Brno, FEEC BUT. 2014. p. 154 - 158. ISBN 978-80-214-4924-4.