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Wigner function on a cylinder

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Wignerova funkce na válci

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Declaration

I hereby declare that the thesis entitled "Wigner function on a cylinder" has been composed solely by myself under the guidance of doc. Mgr. Ladislav Mišta Ph.D. by using resources, which are referred to in the list of literature. I agree with the further usage of this document according to the requirements of the Department of Optics.

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Abstract	Phase space Wigner function of a linear harmonic oscillator is a useful concept which provides a visual image of the corresponding quantum state and serves as a powerful computational tool which may simplify calculations. While these roles of the Wigner function are well understood for the linear harmonic oscillator it is not so with other fundamental quantum systems. One of such systems is the system of a quantum rigid rotor. The non-trivial cylindrical topology of its phase space creates a freedom of choice in the phase of the displacement operator, giving rise to different definitions of the Wigner function for this system. The purpose of this bachelor thesis is to identify the best Wigner function for the considered system, which would possess the maximum number of required properties. To achieve this goal, we first summarize the properties any Wigner function should have, then we review currently known forms of the Wigner function. The required properties produce a rather complex set of equations. Surprisingly, we are able to solve it. As a result, we get a new Wigner function, which is, unlike the previous cases, equipped with all the required properties and at the same time it exhibits a simpler structure. Interestingly, by relaxing the requirement of reality of the Wigner function, we arrive to an even more computationally friendly but complex-valued quasiprobability distribution. Our results provide new computational tools for the fundamental system of a quantum rigid rotor and pave the way towards development of the quantum mechanical formalism in its curved phase space.
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Abstrakt	Wignerova funkce lineárního harmonického oscilátoru je užitečný pojem, který nám poskytuje vizuální obraz příslušného kvantového stavu a slouží také jako mocný výpočetní nástroj, který může zjednodušit výpočty. I když jsou tyto role Wignerovy funkce dobře pochopeny pro lineární harmonický oscilátor, u jiných základních kvantových systémů tomu tak není. Jedním z takových systémů je systém kvantového tuhého setrvačníku. Netriviální válcová topologie jeho fázového prostoru vytváří volnost ve volbě fáze posunovacího operátoru, z čehož vyplývají různé definice Wignerovy funkce pro tento systém. Cílem této bakalářské práce je indentifikovat nejlepší Wignerovu funkci pro uvažovaný systém, která by měla maximální počet požadovaných vlastností. K dosažení tohoto cíle nejprve shrneme vlastnosti, které by měla mít každá Wignerova funkce, a poté prohlédneme v současnosti známé tvary Wignerovy funkce. Požadované vlastnosti vedou k poměrně složité soustavě rovnic. Překvapivě jsme soustavu schopni vyřešit. Výsledkem je nová Wignerova funkce, která je na rozdíl od předchozích případů vybavena všemi požadovanými vlastnostmi a zároveň vykazuje jednodušší strukturu. Zajímavé je, že zmírněním požadavku na reálnost Wignerovy funkce dospějeme k ještě výpočetně přívětivějšímu, ale komplexnímu kvazirozdělení pravděpodobnosti. Naše výsledky poskytují nové výpočetní nástroje pro základní systém kvantového tuhého setrvačníku a otevírají cestu k rozvoji kvantově mechanického formalismu v jeho zakřiveném fázovém prostoru.
Klíčová slova	kvanová mechanika, fázový prostor, Wignerova funkce, kvantový moment hybnosti, kvantový tuhý setrvačník
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Contents

Introduction

In quantum mechanics one often uses the wave picture, based on complex-valued wave functions, to describe the possible states of a considered system. A more illustrative way is to use the phase space formalism to represent the system. A phase space is a space of eigenvalues of the canonical operators fully describing the system and the possible states are represented by quasiprobability distributions. The simplest phase space is obtained for systems characterized by canonically conjugate position and momentum operators, $\hat{x}$ and $\hat{p}$, respectively, which is a plane. There also exist uniquely defined quasiprobability distributions for this system, including, e.g. the *Wigner function* $W(x,p)$ [1]. But there are other systems, for example, a hybrid system described by an operator $\hat{A}$ with integer eigenvalues $a \in \mathbb{Z}$ and an operator $\hat{B}$ whose eigenvalue is an arbitrary real number $b \in \mathbb{R}$. How would this change the corresponding phase space and would the Wigner function $W(a,b)$ be uniquely defined? This thesis aims to investigate a particular instance of the hybrid system, where $\hat{A}$ is the z-th component of the angular momentum operator and $\hat{B}$ is a complex exponential of an angle. The thesis will also investigate their phase space and possible constructions of their Wigner function.

The phase space formulation of quantum mechanics was fully developed by H. Groenewold [2] and J. Moyal [3], building on earlier works of H. Weyl [4], E. Wigner [1]. One tries to find a mapping relating operators from the Hilbert space to functions on the phase space. The advantage of the quantum phase space is that we can give quantum states a "visual" representation by the so-called quasiprobability distribution, which is a quantum analogy of a classical probability distribution in the classical phase space of statistical mechanics [5–12]. The term "quasiprobability" reflects the fact that unlike classical probability distributions the distributions representing quantum states may loose some of the properties of an ordinary probability distribution, e.g. it can be negative.

This phase space representation was developed in the context of the system characterized by the position $\hat{x}$ and momentum $\hat{p}$ operators, obeying the commutation rule $[\hat{x},\hat{p}] = i\hbar\mathbb{1}$. As we said above, their phase space is a plane $\mathbb{R}_2 = \{[x,p]; x,p \in \mathbb{R}\}$. Their Wigner function of a density matrix $\hat{\rho}$ is defined as

$$W(x,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} \left\langle x - \frac{x'}{2}\right|\hat{\rho}\left|x + \frac{x'}{2}\right\rangle e^{\frac{ipx'}{\hbar}}\, dx' \,, \tag{1}$$

which can be viewed as a Fourier transform of the off-diagonal elements of the density matrix $\hat{\rho}$ in the eigenbasis $\{|x\rangle\}_{x\in\mathbb{R}}$ of the position operator $\hat{x}$. Notice, that for a given $\hat{\rho}$, the Wigner function $W(x,p)$ is uniquely defined. A fundamental quantum system described by such operators is the system of a point mass, the well-known linear harmonic oscillator[1] with one degree of freedom. This system can be used to

[1]Here and throughout this thesis we mean the quantum linear harmonic oscillator.

model light [13], collective spin of an atomic ensemble [14], motional degree of freedom of a trapped ion [15], mechanical mode of a movable cavity mirror [16], etc.

However, with the increasing control of elementary constituents of matter, the interest in a more complex fundamental quantum system has risen. Namely, for many quantum systems description based on a model of a point mass is insufficient. This is because this model does not adequately describe the behavior connected with the rotation of the system. The concept of a rigid body was developed to fill this gap, as to be a closer approximation of real bodies, which also takes into account their geometric shape. The simplest model of such a body is a system composed of just two point masses held at a constant (rigid) distance apart, the so-called *rigid rotor*. This textbook problem can be solved analytically [17] and provides explanations of basic phenomena connected with the rotation of bodies. For example, single-photon orbital angular momentum [18], azimuthal evolution of optical beams [19] and molecular rotations [20] can be modeled by this system. Conceivable applications include orientation-dependent metrology [21–23], ultracold chemistry [24, 25], highly sensitive torque sensors [26, 27], realizations of a quantum heat engine [28] and levitated nanomagnets [29], to cite a few. If we are interested in the spectra of a given model, then the *quantum rigid rotor* presents a suitable description of such a system.

Quantum rigid rotor represents one of the fundamental canonical systems of quantum mechanics, similarly as the potential dam or linear harmonic oscillator. The predictions we get from this system are in agreement with real observations. Compared to the linear harmonic oscillator, some aspects of the quantum rigid rotor are not as well developed. One of them being the description of its complementary variables and with them, closely related, their phase space formulation. In analogy with the linear harmonic oscillator the momentum variable p is replaced by the variable L, as the component of angular momentum along the axis orthogonal to the plane of rotation. One may be temped to put the angular position ϕ in the place of position x, however this may entail many pitfalls and calls for a very subtle analysis [30, 31]. The most satisfactory approach avoiding these difficulties is based on the use of an unitary exponential operator $\hat{E}$, or equivalently Hermitian sine $\hat{S}$ and cosine $\hat{C}$ operators, as the complementary observables to $\hat{L}$ [32]. In this thesis we follow the latter approach to develop the phase space method corresponding to the variables E and L.

Building on the latter ideas one can introduce the concept of the Wigner function in two ways. One is based on the group-theoretical methods [33] and leads to rather complex expressions. The other one is more intuitive and simpler and utilises analogies with the linear harmonic oscillator [34, 35]. In the second approach, in contrast with the linear harmonic oscillator, one can adopt different phase conventions leading to different Wigner functions, which are generally equipped with different properties. So far, the optimal phase convention, which would lead to the Wigner function possessing the maximum amount of properties, has not been found.

In the present thesis we seek this phase convention and compare the resulting Wigner functions with the existing ones. Lastly we analyze the behavior of the various Wigner functions and provide their visualization.

Chapter 1

Introduction of the complementary variables

In this chapter we introduce the complementary variables of the quantum rigid rotor, as well as the states minimizing their uncertainty relations.

1.1 Complementary variables of the quantum rigid rotor

The basic complementary observables of the quantum rigid rotor have been found to be the Hermitian angular momentum operator $\hat{L}$ and the unitary operator $\hat{E}$, which satisfy the following commutation relation:

$$\left[\hat{E},\, \hat{L}\right] = \hat{E}\,, \tag{1.1}$$

of the Euclidean algebra $\mathfrak{e}(2)$ of the Euclidean group E(2) of the rigid motions in the plane [32]. In the ϕ-representation the operators $\hat{L}$ and $\hat{E}$ (in units of the reduced Planck constant $\hbar$) read as

$$\hat{L} = -i\frac{\partial}{\partial\phi}\,, \quad \hat{E} = e^{-i\phi}\,. \tag{1.2}$$

Notice, that the commutation relation (1.1) is not as simple as the commutation relation of the position $\hat{x}$ and momentum $\hat{p}$ operators $[\hat{x}, \hat{p}] = i\mathbb{1}$ (in units of the reduced Planck constant $\hbar$), where (up to a constant) these operators commute to the identity operator. However the commutation relation (1.1) is not of the complex form $\left[\hat{A}, \hat{B}\right] = \hat{C}$, where $\hat{A} \neq \hat{B} \neq \hat{C}$, since our operators commute to another operator that we already know (the exponential operator $\hat{E}$). In this sense the commutation relation (1.1) is in the next class of difficulty after the commutation relation $[\hat{x}, \hat{p}] = i\mathbb{1}$.

To find the eigenstates of the angular momentum operator $\hat{L}$ we need to solve the eigenvalue problem

$$-i\frac{\partial}{\partial\phi}\Psi_n(\phi) = n\Psi_n(\phi)\,, \tag{1.3}$$

whose solution is, in addition, required to be 2π-periodic, i.e., to satisfy

$$\Psi_n(\phi + 2\pi) = \Psi_n(\phi)\,.$$

By solving the differential equation we find the normalized eigenfunctions of the operator $\hat{L}$ in the form

$$\Psi_n(\phi) = \langle\phi|n\rangle = \frac{1}{\sqrt{2\pi}} e^{in\phi} \,, \tag{1.4}$$

where $n \in \mathbb{Z}$ are integer eigenvalues of $\hat{L}$. Further, the eigenfunctions (1.4) are orthonormal with respect to the scalar product

$$\langle\psi_1|\psi_2\rangle = \int_0^{2\pi} d\phi \psi_1^*(\phi)\psi_2(\phi) \,, \tag{1.5}$$

i.e.,

$$\langle\Psi_l|\Psi_k\rangle = \int_0^{2\pi} d\phi \Psi_l^*(\phi)\Psi_k(\phi) = \delta_{l,k} \,, \tag{1.6}$$

where $\delta_{l,k}$ is the Kronecker delta

$$\delta_{m,n} = \begin{cases} 1 \,, & \text{if} \quad m = n \,, \\ 0 \,, & \text{if} \quad m \neq n \,; \quad \text{for} \quad m, n \in \mathbb{Z} \,, \end{cases} \tag{1.7}$$

and the eigenfunctions (1.4) comprise an orthonormal basis in the Hilbert space $L_2(0, 2\pi)$ of all square-integrable functions on the interval $[0, 2\pi)$ with the scalar product (1.5), which is therefore the state space of the considered system.

Independently of the representation, we can express the eigenvalue equation (1.3) as

$$\hat{L} \, |n\rangle = n \, |n\rangle \ , \tag{1.8}$$

and the orthonormality condition (1.6) by the equation

$$\langle l|k\rangle = \delta_{l,k} \, . \tag{1.9}$$

Finally, the basis $\{|n\rangle\}_{n\in\mathbb{Z}}$ satisfies the following completeness condition

$$\sum_{n\in\mathbb{Z}} |n\rangle\langle n| = \mathbb{1} \, . \tag{1.10}$$

Similarly, we can write an eigenvalue equation for the unitary operator $\hat{E}$

$$\hat{E} \, |\phi\rangle = e^{-i\phi} \, |\phi\rangle \ , \tag{1.11}$$

where $|\phi\rangle$ can be written as

$$|\phi\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} e^{-in\phi} \, |n\rangle \ , \quad \phi \in [0, 2\pi) \ , \tag{1.12}$$

which can be viewed as the discrete Fourier transform of the angular momentum eigenstates. They satisfy the normalization condition

$$\langle\phi|\phi'\rangle = \delta_{2\pi}(\phi - \phi') \,, \tag{1.13}$$

where

$$\delta_{2\pi}(\phi) = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} e^{ik\phi} = \sum_{k\in\mathbb{Z}} \delta(\phi - 2k\pi)\,, \tag{1.14}$$

is the 2π-periodic delta function (or Dirac comb). Notice, these eigenstates (1.12) are not normalizable, however they satisfy the following resolution of the identity

$$\int_0^{2\pi} d\phi\, |\phi\rangle\langle\phi| = \mathbb{1}\,. \tag{1.15}$$

Using the commutation relation (1.1) one can show that

$$\hat{L}\hat{E}\,|n\rangle = (n-1)\hat{E}\,|n\rangle\ , \tag{1.16}$$

which implies $\hat{E}\,|n\rangle$ is an eigenstate of $\hat{L}$ corresponding to the eigenvalue $n-1$ and therefore

$$\hat{E}\,|n\rangle = |n-1\rangle\ . \tag{1.17}$$

Likewise it can be found that

$$\hat{E}^\dagger\,|n\rangle = |n+1\rangle\ . \tag{1.18}$$

Rather than working with the unitary operator $\hat{E}$, it is convenient to work with its real and imaginary part, defined as

$$\hat{S} = \frac{\hat{E}^\dagger - \hat{E}}{2i}\,, \quad \hat{C} = \frac{\hat{E}^\dagger + \hat{E}}{2}\,, \tag{1.19}$$

which are the Hermitian sine and cosine operators [36] and satisfy the following commutation relations:

$$\left[\hat{S},\,\hat{L}\right] = i\hat{C}\,, \quad \left[\hat{C},\,\hat{L}\right] = -i\hat{S}\,, \quad \left[\hat{S},\,\hat{C}\right] = 0\,. \tag{1.20}$$

The first commutation relation reveals that the operators $\hat{L}$ and $\hat{S}$ are incompatible and it implies the uncertainty relations

$$\langle(\Delta\hat{L})^2\rangle\ \langle(\Delta\hat{S})^2\rangle \geq \frac{1}{4}|\,\langle\hat{C}\rangle|^2\,. \tag{1.21}$$

In order to capture the richness of the system under study, we consider a more generic unitary operator

$$\hat{E}_\alpha = e^{i\alpha}\hat{E}\,, \tag{1.22}$$

which is the operator $\hat{E}$ shifted by an arbitrary angle α. The commutation relation (1.1) now reads as

$$\left[\hat{E}_\alpha,\,\hat{L}\right] = \hat{E}_\alpha\,. \tag{1.23}$$

Rephrased in terms of the Hermitian operators

$$\hat{S}_\alpha = \frac{\hat{E}_\alpha^\dagger - \hat{E}_\alpha}{2i}\,, \quad \hat{C}_\alpha = \frac{\hat{E}_\alpha^\dagger + \hat{E}_\alpha}{2}\,, \tag{1.24}$$

the commutation relations (1.20) are then

$$\left[\hat{S}_\alpha,\,\hat{L}\right] = i\hat{C}_\alpha\,, \quad \left[\hat{C}_\alpha,\,\hat{L}\right] = -i\hat{S}_\alpha\,, \quad \left[\hat{S}_\alpha,\,\hat{C}_\alpha\right] = 0\,, \tag{1.25}$$

and the uncertainty relations of our interest are given by

$$\langle(\Delta\hat{L})^2\rangle\ \langle(\Delta\hat{S}_\alpha)^2\rangle \geq \frac{1}{4}|\,\langle\hat{C}_\alpha\rangle|^2\,. \tag{1.26}$$

1.1.1 Complmentarity

It can be shown that the observables $\hat{S}_\alpha$ and $\hat{L}$ are complementary. Since the eigenstate $|\phi\rangle$ of the exponential operator $\hat{E}_\alpha$ is also an eigenstate of the Hermitian operator $\hat{S}_\alpha$ (by construction), from the Eq. (1.12), we see that the probability of measuring the value n of the angular momentum along the axis orthogonal to the plane of rotation on a system in a state $|\phi\rangle$ is

$$|\langle n|\phi\rangle|^2 = \frac{1}{2\pi}\,, \quad \forall n \in \mathbb{Z}\,, \forall \phi \in [0, 2\pi)\,, \tag{1.27}$$

which is a constant and therefore the bases $\{|n\rangle\}_{n\in\mathbb{Z}}$ and $\{|\phi\rangle\}_{\phi\in[0,2\pi)}$ are mutually unbiased. Meaning, if we measure the state to have precisely the angular momentum n, then we cannot make any predictions about its angular position ϕ, i.e. the probability distribution is constant as a function of ϕ, and vice versa. Thus the operators $\hat{S}_\alpha$ (and, by construction, $\hat{E}_\alpha$) and $\hat{L}$ are complementary.

1.2 Minimum uncertainty states (MUS)

One can show [32, 33, 37] that the states minimizing the inequality (1.26) are, in the L-representation, given by

$$|n,\alpha\rangle = \frac{1}{\sqrt{I_0(2\kappa)}} \sum_{l\in\mathbb{Z}} e^{i(n-l)\alpha} I_{n-l}(\kappa)\,|l\rangle\,, \tag{1.28}$$

where $\kappa \geq 0$ represents the spread of the angular position ϕ and $I_n(z)$ is the modified Bessel function (see Appendix A for its definition and properties)

$$I_n(z) = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} e^{z\cos\phi + in\phi}\,, \text{ for } \quad n \in \mathbb{Z}\,, z \in \mathbb{C}\,. \tag{1.29}$$

In the ϕ-representation the minimum uncertainty states (1.28) reads as

$$\begin{aligned} |n,\alpha\rangle &= \int_0^{2\pi} d\phi\, \psi_{n,\alpha}(\phi)\,|\phi\rangle\,, \\ \psi_{n,\alpha}(\phi) = \langle\phi|n,\alpha\rangle &= \frac{e^{\kappa\cos(\phi-\alpha)+in\phi}}{\sqrt{2\pi I_0(2\kappa)}}\,, \end{aligned} \tag{1.30}$$

where the generating function

$$\sum_{m\in\mathbb{Z}} I_m(z)\exp(im\phi) = \exp(z\cos\phi)\,, \tag{A.10}$$

has been used (see Appendix A).

The MUS $|n,\alpha\rangle$ yield von Mises distribution for the angular position ϕ, i.e.

$$|\langle\phi|n,\alpha\rangle|^2 = \frac{e^{2\kappa\cos(\phi-\alpha)}}{2\pi I_0(2\kappa)}\,, \tag{1.31}$$

and due to this, the states will be referred to as the *von Mises states*.

The von Mises states resolve the identity as

$$\sum_{n\in\mathbb{Z}} \int_{\alpha_0}^{\alpha_0+2\pi} \frac{d\alpha}{2\pi} \, |n,\alpha\rangle\langle n,\alpha| = \mathbb{1} \,, \tag{1.32}$$

where α_0 is an arbitrary angle.

In what follows we show that the von Mises states allow us to develop a phase space description for the angular momentum L and angular position ϕ, which closely resembles the phase space description for quadrature operators based on standard coherent states.

Chapter 2

Mathematical tools

Having introduced the quantum system under study, we develop some mathematical tools we are going to be using in the pursuit of the phase space description of this system.

2.1 The Fourier transform

The key mathematical tool used by us will be the Fourier transform of an operator (or a function) $A(n, \alpha)$ of an integer n, and an angle α [34]

$$(\mathcal{F}\hat{A})(l, \phi) = \sum_{n \in \mathbb{Z}} \int_{\alpha_0}^{\alpha_0 + 2\pi} \frac{d\alpha}{2\pi} e^{i(l\alpha - \phi n)} \hat{A}(n, \alpha) \,, \tag{2.1}$$

where α_0 is an arbitrary angle. Notice that, since n is an integer, this Fourier transform is always 2π-periodic. Making use of the filtration property of the 2π-periodic delta function (1.14) on the interval of length 2π, one can easily show the following analogy of the Parseval formula for commuting operators $\hat{A}(n, \alpha)$ and $\hat{B}(n, \alpha)$:

$$\sum_{l \in \mathbb{Z}} \int_{\alpha_0}^{\alpha_0 + 2\pi} (\mathcal{F}\hat{A})(l, \phi)(\mathcal{F}\hat{B})^\dagger(l, \phi) \, d\phi = \sum_{n \in \mathbb{Z}} \int_{\alpha_0}^{\alpha_0 + 2\pi} \hat{A}(n, \alpha)\hat{B}^\dagger(n, \alpha) \, d\alpha \,. \tag{2.2}$$

where the symbol † stands for the Hermitian conjugate. From the knowledge of the convolution theorem for the Fourier transform we can derive the following equality for commuting operators $\hat{A}(n, \alpha)$ and $\hat{B}(n, \alpha)$

$$[\mathcal{F}(\hat{A}\hat{B})](n, \alpha) = \sum_{l \in \mathbb{Z}} \int_{\alpha_0}^{\alpha_0 + 2\pi} \frac{d\phi}{2\pi} (\mathcal{F}\hat{A})(n - l, \alpha - \phi)(\mathcal{F}\hat{B})(l, \phi) = (\mathcal{F}\hat{A}) * (\mathcal{F}\hat{B})(n, \alpha) \,, \tag{2.3}$$

where α_0 is an arbitrary angle. Finally, we can also derive the formula for the double Fourier transform

$$[\mathcal{F}(\mathcal{F}\hat{A})](l, \phi) = \hat{A}(l, \phi_0) = \hat{A}(l, \phi - 2k\pi) \,, \tag{2.4}$$

where we used the fact, that an arbitrary angle $\phi \in \mathbb{R}$ can be decomposed as $\phi = \phi_0 + 2k\pi$, where $\phi_0 \in [\alpha_0, \alpha_0 + 2\pi)$ and $k \in \mathbb{Z}$. Clearly, the Fourier transform (2.1) is its own inverse, if only if $\hat{A}(n, \alpha)$ is 2π-periodic. Since for a 2π-periodic $\hat{A}(n, \alpha)$ the Fourier transform (2.1) is independent of α_0, for simplicity we set $\alpha_0 = 0$ in what follows.

2.2 Commuting extensions of angular momentum and angular position operators

The starting point of our considerations is a composite system consisting of the signal system s and the ancillary system a with the Hilbert state space $\mathcal{H}_s \otimes \mathcal{H}_a$. For this system we introduce the total angular momentum operator $\hat{\mathcal{L}}$ and the angular difference operator $\hat{\mathcal{E}}$ by the formulas [38]

$$\hat{\mathcal{L}} = \hat{L}_s + \hat{L}_a \,, \quad \hat{\mathcal{E}} = \hat{E}_s \hat{E}_a^\dagger \,. \tag{2.5}$$

These operators commute, $[\hat{\mathcal{L}}, \hat{\mathcal{E}}] = 0$, and therefore possess a common eigenbasis

$$|N, \Phi\rangle_{sa} = \frac{1}{\sqrt{2\pi}} \sum_{l\in\mathbb{Z}} e^{-il\Phi} \, |l + N\rangle_s \, |-l\rangle_a \,, \tag{2.6}$$

which satisfy the eigenvalue equations

$$\begin{aligned} \hat{\mathcal{L}} \, |N, \Phi\rangle_{sa} &= N \, |N, \Phi\rangle_{sa} \,, \\ \hat{\mathcal{E}} \, |N, \Phi\rangle_{sa} &= e^{-i\Phi} \, |N, \Phi\rangle_{sa} \,, \\ \hat{\mathcal{E}}^\dagger \, |N, \Phi\rangle_{sa} &= e^{i\Phi} \, |N, \Phi\rangle_{sa} \,. \end{aligned} \tag{2.7}$$

The states (2.6) are therefore the eigenstates of $\hat{\mathcal{L}}$ and $\hat{\mathcal{S}} = (\hat{\mathcal{E}}^\dagger - \hat{\mathcal{E}})/2i$ corresponding to eigenvalues N and $\sin\Phi$, respectively. The states (2.6) further satisfy the orthogonality relation

$$\langle M, \Psi | N, \Phi \rangle = \delta_{M,N} \delta_{2\pi}(\Psi - \Phi) \,, \tag{2.8}$$

and fulfill the following completeness condition

$$\sum_{N\in\mathbb{Z}} \int_0^{2\pi} d\Phi \, |N, \Phi\rangle_{sa} \langle N, \Phi| = \mathbb{1}_{sa} \,. \tag{2.9}$$

Note that due to the obvious 2π-periodicity of the operator $|N, \Phi\rangle_{sa} \langle N, \Phi|$ the resolution of identity can be written as

$$\sum_{N\in\mathbb{Z}} \int_{\alpha_0}^{\alpha_0+2\pi} d\Phi \, |N, \Phi\rangle_{sa} \langle N, \Phi| = \mathbb{1}_{sa} \,, \tag{2.10}$$

where α_0 is an arbitrary angle.

These states will be essential in developing the phase space representation of the system of $\hat{L}$ and $\hat{E}$.

Chapter 3

Phase space representation

In this chapter we finally develop the phase space representation of our system of $\hat{L}$ and $\hat{E}$. Preforming the Fourier transform of the MUS leads us to a crucial phase decomposition and introduction of the displacement operator. Next we introduce the notion of phase space distributions, the relation of the Q-function, Wigner function and the P-function. Lastly, we present the required properties, that our Wigner function should have.

3.1 Displacement operator

The phase space representation relies on the Fourier transform of the projectors onto the eigenstates (2.6), i.e.

$$2\pi \left(\mathcal{F} \left|N,\theta\right\rangle_{sa} \left\langle N,\theta\right|\right)(l,\phi) = \hat{\mathcal{E}}^{-l} e^{-i\hat{\mathcal{L}}\phi} = \hat{E}_s^{-l} e^{-i\hat{L}_s\phi} \hat{E}_a^{l} e^{-i\hat{L}_a\phi}\,, \tag{3.1}$$

where $\left|N,\theta\right\rangle_{sa}$ is the eigenstate (2.6) corresponding the eigenvalues N and $\sin\theta$. The first equality follows directly from the application of the operator $\hat{\mathcal{E}}^{-l}e^{-i\hat{\mathcal{L}}\phi}$ to the resolution of identity (2.9) for the eigenstates $\left|N,\theta\right\rangle_{sa}$. Let us now average both sides of the equation (3.1) over the von Mises vacuum state $\left|0,0\right\rangle_a$ of the ancillary system a with the spread parameter κ. This gives

$$\left(\mathcal{F} \left|N,\theta\right\rangle_{s} \left\langle N,\theta\right|\right)(l,\phi) = o(l,\phi)\hat{E}_s^{-l} e^{-i\hat{L}_s\phi}\,, \tag{3.2}$$

where $\left|N,\theta\right\rangle_s$ is the signal von Mises state and

$$o(l,\phi) = {}_a\langle 0,0|\, \hat{E}_a^{l} e^{-i\hat{L}_a\phi} \left|0,0\right\rangle_a \stackrel{1}{=} \langle 0,0|{-l},\phi\rangle \stackrel{2}{=} e^{-il\frac{\phi}{2}} \frac{I_l\left[2\kappa\cos\left(\frac{\phi}{2}\right)\right]}{I_0\left(2\kappa\right)}\,. \tag{3.3}$$

To get the left-hand side (LHS) of Eq. (3.2) we used ${}_a\left\langle 0,0|N,\theta\right\rangle_{sa} = \left|N,\theta\right\rangle_s/\sqrt{2\pi}$, where $\left|N,\theta\right\rangle_s$ is the signal von Mises state. The equality 1 in Eq. (3.3) is a consequence of the formula

$$\hat{E}^{-l} e^{-i\hat{L}\phi} \left|0,0\right\rangle = \left|l,\phi\right\rangle\,, \tag{3.4}$$

which follows from the relations

$$
\begin{aligned}
e^{-i\hat{L}\phi}\left|n,\alpha\right\rangle &= e^{-in\phi}\left|n,\alpha+\phi\right\rangle \,, \\
\hat{E}^{-l}\left|n,\alpha\right\rangle &= \left|n+l,\alpha\right\rangle \,,
\end{aligned}
\tag{3.5}
$$

where $|n,\alpha\rangle$ is the von Mises state (1.28). As for the equality 2 in Eq. (3.3) it follows from the overlap formula

$$
\langle n',\alpha'|n,\alpha\rangle = e^{i(n-n')\left(\frac{\alpha+\alpha'}{2}\right)}\frac{I_{n-n'}\left[2\kappa\cos\left(\frac{\alpha-\alpha'}{2}\right)\right]}{I_0\left(2\kappa\right)}\,, \tag{3.6}
$$

derived in the Supplemental material of [38].

By setting $N{=}n$, $\theta{=}\alpha$ in (3.2) and inserting the right-hand side (RHS) of Eq. (3.3) into the RHS of Eq. (3.2) we finally get the following expression for the Fourier transform of the von Mises states:

$$
\left(\mathcal{F}\left|n,\alpha\right\rangle_s\left\langle n,\alpha\right|\right)(l,\phi) = e^{-il\frac{\phi}{2}}\frac{I_l\left[2\kappa\cos\left(\frac{\phi}{2}\right)\right]}{I_0\left(2\kappa\right)}\hat{E}_s^{-l}e^{-i\hat{L}_s\phi}\,. \tag{3.7}
$$

With respect to what follows it is now convenient to decompose the phase factor $\exp(-il\phi/2)$ as

$$
e^{-il\frac{\phi}{2}} = e^{i\gamma_j(l,\phi)}e^{i\delta_j(l,\phi)}, \tag{3.8}
$$

i.e.,

$$
\gamma_j(l,\phi)+\delta_j(l,\phi) = -l\frac{\phi}{2}+2k\pi, \quad k\in\mathbb{Z}\,. \tag{3.9}
$$

This allows us to express Eq. (3.7) in the following form

$$
\left(\mathcal{F}\left|n,\alpha\right\rangle\!\left\langle n,\alpha\right|\right)(l,\phi) = o_j(l,\phi)\hat{D}_j(l,\phi)\,, \tag{3.10}
$$

where

$$
o_j(l,\phi) = e^{i\gamma_j(l,\phi)}\frac{I_l\left[2\kappa\cos\left(\frac{\phi}{2}\right)\right]}{I_0\left(2\kappa\right)}\,, \tag{3.11}
$$

$$
\hat{D}_j(l,\phi) = e^{i\delta_j(l,\phi)}\hat{E}^{-l}e^{-i\hat{L}\phi}\,, \tag{3.12}
$$

and the system index s has been dropped for simplicity. Note that

$$
o_j(l,\phi) = e^{i\left[\gamma_j(l,\phi)+l\frac{\phi}{2}\right]}\left\langle 0,0|{-l},\phi\right\rangle\,, \tag{3.13}
$$

and it is (up to the phase factor $\exp\left[i\left(\gamma_j+l\frac{\phi}{2}\right)\right]$) the overlap of the von Mises state $|{-l},\phi\rangle$ with the "vacuum" von Mises state $|0,0\rangle$. Further, making use of Eq. (3.4), we see that

$$
\hat{D}_j(l,\phi)\left|0,0\right\rangle = e^{i\delta_j(l,\phi)}\left|l,\phi\right\rangle\,, \tag{3.14}
$$

and thus the operator $\hat{D}_j(l,\phi)$ can be called as the displacement operator [34, 39] in analogy with the displacement operator $\hat{D}(\alpha) = \exp\left(\alpha\hat{a}^\dagger-\alpha^*\hat{a}\right)$ of the harmonic oscillator generating a coherent state $|\alpha\rangle$ from the vacuum state $|0\rangle$, i.e. $\hat{D}(\alpha)\left|0\right\rangle = |\alpha\rangle$.

Let us look closer at some other properties of the displacement operator. Firstly,

$$\hat{D}_j^\dagger(l,\phi) = e^{-i\delta_j(l,\phi)} e^{i\hat{L}\phi} \hat{E}^l = \hat{D}_j^{-1}(l,\phi) \tag{3.15}$$

and the operator (3.12) is unitary. Secondly, the displacement operator (3.12) also exhibits the following completeness property:

$$\mathrm{Tr}\left[\hat{D}_j^\dagger(n,\alpha)\hat{D}_j(l,\phi)\right] = 2\pi\delta_{l,n}\delta_{2\pi}(\alpha-\phi) \tag{3.16}$$

and thus it comprises an operator basis. Also notice the following property

$$\hat{E}^l e^{i\hat{L}\phi} = e^{il\phi} e^{i\hat{L}\phi} \hat{E}^l\,, \tag{3.17}$$

which will be useful, later, in developing the required properties of the Wigner functions $W_j(n,\alpha)$.

3.2 Phase space distributions

The Fourier transform of the projector onto the von Mises states (3.10) plays a central role in our approach of developing (development of) the phase space methods for angular momentum L and angular position ϕ. The phase space methods rely on the introduction of a phase space for the considered system. In the present case it is a set of equidistant rings (each ring is one unit away from the other) on a surface of a cylinder of radius one (Ref. [32] and in particular Fig. 1. of Ref. [38]), i.e. the set $\mathcal{S}_1 \times \mathbb{Z}$ (here $\mathcal{S}_1$ denotes the unit circle and $\mathbb{Z}$ the set of integers). The formalism of phase space quasiprobability distributions has been developed in the context of the linear harmonic oscillator. For this system, we distinguish three main quasiprobability distributions, including the Q-function [40], Wigner function [1] and P-function [41, 42]. In analogy with the quasiprobability distributions of the linear harmonic oscillator, we can introduce similar distributions for the angular momentum L and the angular position ϕ. In the existing literature the main attention has been paid to the Wigner function, which was constructed using group-theoretical methods in Refs. [33, 43] and employing analogies with the linear harmonic oscillator in Refs. [34, 39, 44]. Building on the latter ideas we can introduce an analogy of the Q-function, for a density matrix $\hat{\rho}$, by the following formula [38]:

$$Q^{\hat{\rho}}(n,\alpha) = \frac{\langle n,\alpha|\hat{\rho}|n,\alpha\rangle}{2\pi}\,, \tag{3.18}$$

where $|n,\alpha\rangle$ is the von Mises state. The Q-function is normalized as

$$\sum_{n\in\mathbb{Z}} \int_0^{2\pi} d\alpha Q^{\hat{\rho}}(n,\alpha) = 1. \tag{3.19}$$

Like in the standard probability theory, we also introduce the respective characteristic function of the Q-function as the Fourier transform of the Q-function,

$$C_{Q^{\hat{\rho}}}(l,\phi) = (\mathcal{F}Q^{\hat{\rho}})(l,\phi). \tag{3.20}$$

To further express the RHS we can use the Fourier transform of the von Mises state (3.10).

Namely, by averaging the Fourier transform over the rescaled density matrix $\hat{\rho}/(2\pi)$ we get

$$C_{Q^{\hat{\rho}}}(l,\phi) = (\mathcal{F}Q^{\hat{\rho}})(l,\phi) = \frac{1}{2\pi}\,\mathrm{Tr}\left[\hat{\rho}\left(\mathcal{F}|n,\alpha\rangle\langle n,\alpha|\right)(l,\phi)\right] = o_j(l,\phi)C_{W_j^{\hat{\rho}}}(l,\phi)\,, \quad (3.21)$$

where we introduced the characteristic function

$$C_{W_j^{\hat{\rho}}}(l,\phi) = \frac{1}{2\pi}\,\mathrm{Tr}\left[\hat{\rho}\hat{D}_j(l,\phi)\right]. \quad (3.22)$$

Recalling the relationship among the characteristic functions of the Q-function, Wigner function and P-function of a linear harmonic oscillator [45]

$$C_{Q^{\hat{\rho}}}(\alpha) = e^{-\frac{|\alpha|^2}{2}}C_W^{\hat{\rho}}(\alpha) = e^{-|\alpha|^2}C_P^{\hat{\rho}}(\alpha)\,, \quad (3.23)$$

we see that the function (3.22) can be interpreted as the characteristic function of the Wigner function, which is given by the Fourier transform

$$W_j^{\hat{\rho}}(n,\alpha) = (\mathcal{F}C_{W_j^{\hat{\rho}}})(n,\alpha)\,. \quad (3.24)$$

However, unlike the case of the linear harmonic oscillator, here for each choice of the phase $\delta_j(l,\phi)$ we get a different Wigner function $W_j(n,\alpha)$ which may possess different properties. Note also, that for the considered system the "overlap" $o_j(l,\phi)$, Eq. (3.13), plays the same role as the overlap $\langle\alpha|0\rangle = \exp(-|\alpha|^2/2)$ of the vacuum state $|0\rangle$ and the coherent state $|\alpha\rangle$ of the linear harmonic oscillator, for the system of the linear harmonic oscillator. We can rewrite Eq. (3.24) with the help of Eq. (3.22) as the following mean

$$W_j^{\hat{\rho}}(n,\alpha) = \frac{1}{2\pi}\,\mathrm{Tr}\left[\hat{\rho}\hat{\mathcal{W}}_j(n,\alpha)\right]\,, \quad (3.25)$$

of the operator

$$\hat{\mathcal{W}}_j(n,\alpha) = (\mathcal{F}\hat{D}_j)(n,\alpha) = \sum_{l\in\mathbb{Z}}\int_0^{2\pi}\frac{d\phi}{2\pi}e^{i(n\phi-\alpha l)}\hat{D}_j(l,\phi)\,, \quad (3.26)$$

which is the Fourier transform of the displacement operator. Also note that

$$\hat{\mathcal{W}}_j(0,0) = \sum_{l\in\mathbb{Z}}\int_0^{2\pi}\frac{d\phi}{2\pi}\hat{D}_j(l,\phi)\,, \quad (3.27)$$

and

$$\hat{\mathcal{W}}_j(n,\alpha) = \hat{D}_j(n,\alpha)\hat{\mathcal{W}}_j(0,0)\hat{D}_j^\dagger(n,\alpha)\,, \quad (3.28)$$

where the property (3.17) has been used. The operator (3.28) also exhibits the completeness property

$$\mathrm{Tr}\left[\hat{\mathcal{W}}_j^\dagger(n,\alpha)\hat{\mathcal{W}}_j(l,\phi)\right] = 2\pi\delta_{l,n}\delta_{2\pi}(\alpha-\phi) \quad (3.29)$$

and thus it forms an operator basis.

By performing the Fourier transform of the formula (3.20) and using Eq. (2.3) one gets

$$Q^{\hat{\rho}}(n,\alpha) = (\mathcal{F}o_j) * \left(\mathcal{F}C_{W_j^{\hat{\rho}}}\right)(n,\alpha) = \left(k_j * W_j^{\hat{\rho}}\right)(n,\alpha)\,, \tag{3.30}$$

where

$$k_j(n,\alpha) = (\mathcal{F}o_j)(n,\alpha) = \sum_{l\in\mathbb{Z}} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i(n\phi-\alpha l)} o_j(l,\phi)\,, \tag{3.31}$$

is the kernel of the convolution relating the Q-function and the Wigner function. Again, for each $\gamma_j(l,\phi)$ we get a different kernel of the convolution, which can be a simple function for some choice of $\gamma_j(l,\phi)$ but a complicated function for some other choice. Before moving to various choices of the phases let us introduce an analogy of the P-function. Notice that due to the completeness condition (3.16) we can express any density matrix $\hat{\rho}$ as

$$\hat{\rho} = \sum_{l\in\mathbb{Z}} \int_0^{2\pi} d\phi \rho(l,\phi) \hat{D}_j^{\dagger}(l,\phi)\,, \tag{3.32}$$

where

$$\rho(l,\phi) = \frac{1}{2\pi} \operatorname{Tr}\left[\hat{\rho}\hat{D}_j(l,\phi)\right]\,. \tag{3.33}$$

By comparing the last equation with Eq. (3.22) we see that

$$\rho(l,\phi) = C_{W_j^{\hat{\rho}}}(l,\phi)\,, \tag{3.34}$$

thus

$$\hat{\rho} = \sum_{l\in\mathbb{Z}} \int_0^{2\pi} d\phi C_{W_j^{\hat{\rho}}}(l,\phi) \hat{D}_j^{\dagger}(l,\phi)\,. \tag{3.35}$$

Inserting $\left[o_j^*(l,\phi)\right]^{-1} o_j^*(l,\phi) = 1$ into the integrand we obtain

$$\hat{\rho} = \sum_{l\in\mathbb{Z}} \int_0^{2\pi} d\phi \left[o_j^*(l,\phi)\right]^{-1} C_{W_j^{\hat{\rho}}}(l,\phi) \left[o_j(l,\phi)\hat{D}_j(l,\phi)\right]^{\dagger}\,, \tag{3.36}$$

and using (2.2) we get the P-representation of any density matrix

$$\hat{\rho} = \sum_{n\in\mathbb{Z}} \int_0^{2\pi} d\alpha P^{\hat{\rho}}(n,\alpha) \left|n,\alpha\right\rangle\!\left\langle n,\alpha\right|\,, \tag{3.37}$$

where we introduced the P-function as the Fourier transform

$$P^{\hat{\rho}}(n,\alpha) = \left(\mathcal{F}C_{P^{\hat{\rho}}}\right)(n,\alpha)\,, \tag{3.38}$$

of the corresponding characteristic function $C_{P^{\hat{\rho}}}(l,\phi)$ defined by

$$C_{W_j^{\hat{\rho}}}(l,\phi) = o_j^*(l,\phi) C_{P^{\hat{\rho}}}(l,\phi)\,. \tag{3.39}$$

Applying the Fourier transform to both sides of the latter equality and using the formula (2.3) we arrive at the following relation between the P-function and the Wigner function:

$$W_j^{\hat{\rho}}(n,\alpha) = \left[\left(\mathcal{F}o_j^*\right)(m,\beta) * P^{\hat{\rho}}(m,\beta)\right](n,\alpha)\,. \tag{3.40}$$

Hence making use of the P-function of the von Mises state $|n, \alpha\rangle$,

$$P^{|n,\alpha\rangle}(m, \beta) = \delta_{n,m}\delta_{2\pi}(\alpha - \beta) , \tag{3.41}$$

we can express the Wigner function of the state as

$$W_j^{|n,\alpha\rangle}(m, \beta) = \frac{1}{2\pi} \left(\mathcal{F}o_j^*\right)(m - n, \beta - \alpha) . \tag{3.42}$$

Because of the definition of the Fourier transform (2.1) it is easy to show that

$$(\mathcal{F}\hat{A}^*)(l, \phi) = [(\mathcal{F}\hat{A})(-l, -\phi)]^* , \tag{3.43}$$

this implies

$$\left(\mathcal{F}o_j^*\right)(n, \alpha) = [(\mathcal{F}o_j)(-n, -\alpha)]^* = k_j^*(-n, -\alpha) , \tag{3.44}$$

and the Wigner function of the von Mises state is connected to the kernel as follows

$$W_j^{|n,\alpha\rangle}(m, \beta) = \frac{1}{2\pi} k_j^*(n - m, \alpha - \beta) , \tag{3.45}$$

or in the other direction, the kernel can be recovered from the Wigner function like

$$k_j(n, \alpha) = 2\pi \left(W_j^{|n+m,\alpha+\beta\rangle}(m, \beta)\right)^* . \tag{3.46}$$

From (3.30) we can immediately see that

$$Q^{\hat{\rho}}(n, \alpha) = \sum_{l\in\mathbb{Z}} \int_0^{2\pi} d\phi \left(W_j^{|n,\alpha\rangle}(l, \phi)\right)^* W_j^{\hat{\rho}}(l, \phi) , \tag{3.47}$$

and the Q-function of a density matrix $\hat{\rho}$ is an overlap of the complex conjugate of the Wigner function of the von Mises state $W_j^{|n,\alpha\rangle}$ and the Wigner function $W_j^{\hat{\rho}}$ of the density matrix $\hat{\rho}$. Finally from Eq. (3.40) we get

$$W_j^{\hat{\rho}}(n, \alpha) = \sum_{l\in\mathbb{Z}} \int_0^{2\pi} d\phi W_j^{|-n,-\alpha\rangle}(-l, -\phi) P^{\hat{\rho}}(l, \phi) , \tag{3.48}$$

which is an analogous relation between the Wigner function and the P-function.

3.3 Required properties of the Wigner function

The choice of the phases $\gamma_j(l, \phi)$ and $\delta_j(l, \phi)$ are dictated by the following properties which the Wigner function should have.

1. *Reality.* The Wigner function (3.24) should be real, i.e.

$$W_j^*(n, \alpha) = W_j(n, \alpha) . \tag{3.49}$$

   In order to get a real Wigner function the operator (3.26) must be Hermitian, which requires fulfilment of the following condition:

$$\hat{D}^\dagger(-l, 2\pi - \phi) = \hat{D}(l, \phi). \tag{3.50}$$

   This condition is satisfied if

$$\delta_j(-l, 2\pi - \phi) = -\delta_j(l, \phi) - l\phi + 2m\pi , \quad m \in \mathbb{Z} . \tag{3.51}$$

   Clearly, only special choices of $\delta_j(l, \phi)$ satisfy this condition.

2. *Marginal distributions.* One of the useful properties of the Wigner function is that the integration (summation) of the Wigner function (3.24), of a pure state density matrix $\hat{\rho}$, with respect to α (n) should possess the probability distribution of n (α), i.e.

$$\int_0^{2\pi} W_j^{\hat{\rho}}(n,\alpha) d\alpha = \mathrm{Tr}\,[\hat{\rho}\,|n\rangle\langle n|] \equiv p(n)\,,$$
$$\sum_{n\in\mathbb{Z}} W_j^{\hat{\rho}}(n,\alpha) = \mathrm{Tr}\,[\hat{\rho}\,|\alpha\rangle\langle\alpha|] \equiv q(\alpha)\,. \tag{3.52}$$

The former equality requires

$$\delta_j(0,\phi) = 2n\pi\,, \quad n\in\mathbb{Z}\,, \tag{3.53}$$

and the latter equality requires

$$\delta_j(l,0) = 2r\pi\,, \quad r\in\mathbb{Z}\,. \tag{3.54}$$

3. *Normalization.* The Wigner function (3.24) should be normalized as

$$\sum_{n\in\mathbb{Z}} \int_0^{2\pi} d\alpha W_j^{\hat{\rho}}(n,\alpha) = 1\,. \tag{3.55}$$

The normalization condition holds, whenever

$$\delta_j(0,0) = 2s\pi\,, \quad s\in\mathbb{Z}\,. \tag{3.56}$$

Notice, that if either of the conditions (3.53), (3.54) is satisfied, the condition (3.56) is automatically obeyed and the Wigner function is properly normalized.

4. *Periodicity.* This is a new property which is not encountered for the Wigner function of the linear harmonic oscillator. From the definition of the Fourier transform (2.1) it follows that the Wigner function (3.24), the Wigner operator (3.26) and the kernel (3.31) are always 2π-periodic for any choice of $\gamma_j(l,\phi)$ and $\delta_j(l,\phi)$. However, there are choices for which the displacement operator $\hat{D}_j(l,\phi)$ and the overlap $o_j(l,\phi)$ are not 2π-periodic. This, for example, causes the Wigner function (3.24) as well as the kernel k_j (3.31) to be real only on some interval of length 2π, but not any interval of that length. Moreover, the obtained formulas are complicated. For this reason, it makes sense to require 2π-periodicity of $\hat{D}_j(l,\phi)$ and $o_j(l,\phi)$.

   The displacement operator $\hat{D}_j(l,\phi)$ is 2π-periodic if

$$\delta_j(l,\phi+2\pi) = \delta_j(l,\phi) + 2t\pi\,, \quad t\in\mathbb{Z}\,, \tag{3.57}$$

   and the overlap $o_j(l,\phi)$ is 2π-periodic if

$$\gamma_j(l,\phi+2\pi) + l\pi = \gamma_j(l,\phi) + 2u\pi\,, \quad u\in\mathbb{Z}\,. \tag{3.58}$$

5. *Simplicity.* This is a rather vague but practical property. Namely, below we will see that some choice of phases $\gamma_j(l,\phi)$ and $\delta_j(l,\phi)$ leads to considerably simpler expressions for the Wigner functions of basic states like $|n\rangle$, $|\phi\rangle$, $|n,\alpha\rangle$, as well as the kernel $k_j(n,\alpha)$, whereas some other choice yields more complicated expressions making calculations with them involved and cumbersome.

6. *Relation to the parity operator.* In the case of the linear harmonic oscillator the Wigner function is the mean of the displaced parity operator [46]. For the present system the parity operator is given by

$$\hat{P} = \sum_{l\in\mathbb{Z}} |-l\rangle\langle l| = \int_{\alpha_0}^{\alpha_0+2\pi} |-\phi\rangle\langle\phi|\; d\phi\,, \tag{3.59}$$

or equivalently

$$\hat{P} = \frac{1}{2}\left(\sum_{l\in\mathbb{Z}} |-l\rangle\langle l| + \int_{\alpha_0}^{\alpha_0+2\pi} |-\phi\rangle\langle\phi|\; d\phi\right)\,, \tag{3.60}$$

where α_0 is an arbitrary angle and $\hat{P}$ is the parity operator about the origin of the phase space cylinder, i.e. the point $n = 0$, $\alpha = 0$.

Since $\hat{P}$ is the parity operator it has the following property

$$\hat{P} = \hat{P}^{\dagger} = \hat{P}^{-1}\,. \tag{3.61}$$

whose actions on the operators $\hat{L}$ and $\hat{E}$ are

$$\begin{aligned} \hat{P}\hat{L}\hat{P} &= -\hat{L}\,, \\ \hat{P}\hat{E}\hat{P} &= \hat{E}^{\dagger}\,. \end{aligned} \tag{3.62}$$

Notice, that from Eq. (3.25) we can conclude that the operator $\hat{\mathcal{W}}_j(n,\alpha)$ comprising an operator basis is equivalent to being able to write the Wigner function $W_j^{\hat{\rho}}(n,\alpha)$ as the mean (3.25) of the operator $\hat{\mathcal{W}}_j(n,\alpha)$. However, if we set the operator $\hat{\mathcal{W}}_j(n,\alpha)$ to be the displaced parity operator of our system, i.e.

$$\hat{\mathcal{W}}_j(n,\alpha) = \hat{P}_{n,\alpha} = \hat{D}_j(n,\alpha)\hat{P}\hat{D}_j^{\dagger}(n,\alpha)\,, \tag{3.63}$$

we do not get an operator that comprises an operator basis [35]. Thus the Wigner function $W_j^{\hat{\rho}}(n,\alpha)$ cannot be written as the mean value of the displaced parity operator (3.63) for our system.

Chapter 4

Results

This chapter contains original results of the present thesis, that rely on the decomposition introduced in the Section 3.1. First we summarize the most important choices of phases $\gamma_j(l,\phi)$ and $\delta_j(l,\phi)$, which already appeared in the literature and discuss their strengths and weaknesses. Next we propose our own decomposition, which is the optimal compromise between the fulfilment of the required properties and simplicity of the resulting expressions. We also show that by relaxing the reality condition we arrive to a particularly simple phase space distribution, which coincides with the Kirkwood quasiprobability distribution function [47, 48]. Finally, we analyze the behavior of the Wigner functions corresponding to these choices, as well as provide a visualisation of the real Wigner functions.

Before doing that, for convenience, we list once again the required properties of the Wigner function, discussed more deeply in Section 3.3.

4.1 Recapitulation of the required properties

1. *Decomposition of the original phase*

$$\gamma_j(l,\phi) + \delta_j(l,\phi) = -l\frac{\phi}{2} + 2k\pi, \quad k \in \mathbb{Z}\,. \tag{3.9}$$

2. *Reality*

$$\delta_j(-l, 2\pi - \phi) = -\delta_j(l,\phi) - l\phi + 2m\pi\,, \quad m \in \mathbb{Z}\,. \tag{3.51}$$

3. *Marginal distributions*

$$\delta_j(0,\phi) = 2n\pi\,, \quad n \in \mathbb{Z}\,, \tag{3.53}$$

$$\delta_j(l,0) = 2r\pi\,, \quad r \in \mathbb{Z}\,. \tag{3.54}$$

4. *Normalization*

$$\delta_j(0,0) = 2s\pi\,, \quad s \in \mathbb{Z}\,. \tag{3.56}$$

5. *Periodicity of $\hat{D}_j(l,\phi)$*

$$\delta_j(l,\phi+2\pi) = \delta_j(l,\phi) + 2t\pi\,, \quad t \in \mathbb{Z}\,. \tag{3.57}$$

6. *Periodicity of $o_j(l,\phi)$*

$$\gamma_j(l,\phi+2\pi) + l\pi = \gamma_j(l,\phi) + 2u\pi\,, \quad u \in \mathbb{Z}\,. \tag{3.58}$$

4.2 Different choices of phases

In what follows we analyze different choices of the phases $\gamma_j(l,\phi)$ and $\delta_j(l,\phi)$ and discuss the properties of the corresponding Wigner functions $W_j(n,\alpha)$ and the kernels $k_j(n,\alpha)$, as well as provide explicit expressions of the Wigner functions for basic states: $|n\rangle$, $|\phi\rangle$, $|n,\alpha\rangle$. All the discussed phase choices fulfil Eqs. (3.53), (3.54) and therefore the corresponding Wigner functions posses correct marginal distributions and are properly normalized. For these reasons below we only discuss reality, periodicity and simplicity properties of the studied functions.

Before doing that, notice that the fulfilment of Eqs. (3.53), (3.54) makes the Wigner functions of the angular momentum eigenstate $W_j^{|n\rangle}(m,\beta)$ and angular position eigenstate $W_j^{|\phi\rangle}(m,\beta)$ independent of j, i.e. (for detailed computation see Appendix B.1)

$$W^{|n\rangle}(m,\beta) = \frac{1}{2\pi}\delta_{m,n}\,, \tag{4.1}$$

$$W^{|\phi\rangle}(m,\beta) = \frac{1}{2\pi}\delta_{2\pi}(\beta-\phi)\,. \tag{4.2}$$

Note that $W^{|n\rangle}(m,\beta)$ is independent of β and the Wigner function is normalized, reflecting the normalization of $|n\rangle$. The Wigner function $W^{|\phi\rangle}(m,\beta)$, on the other hand, is independent of m and is not normalized due to the fact that $|\phi\rangle$ is not normalizable.

Now for the different choices of phases.

1) Let

$$\gamma_1(l,\phi) = 0 \quad \text{and} \quad \delta_1(l,\phi) = -l\frac{\phi}{2}\,.$$

This choice has been investigated thoroughly in Refs. [34, 35, 39]. Notice that for this choice the conditions (3.57) and (3.58) are not satisfied, thus both $\hat{D}_1(l,\phi)$ and $o_1(l,\phi)$ are not 2π-periodic. Also Eq. (3.51) is not satisfied and the Wigner function is not real on an arbitrary interval of length 2π. However the reality of the Wigner function can be saved in the following way.

Let us explicitly write down the displacement operator

$$\hat{D}_1(l,\phi) = e^{-il\frac{\phi}{2}}\hat{E}^l e^{-i\hat{L}\phi}\,, \tag{4.3}$$

note that

$$\hat{D}_1^\dagger(l,\phi) = \hat{D}_1(-l,-\phi)\,, \tag{4.4}$$

where the property (3.17) has been used. From this and Eq. (3.22) it follows that

$$C^*_{W_1}(l,\phi) = C_{W_1}(-l,-\phi)\,, \tag{4.5}$$

and by the definition of the Wigner function (3.24) we can write

$$W_1(n,\alpha) = (\mathcal{F}C_{W_1})(n,\alpha) = \sum_{l\in\mathbb{Z}}\int_{-\pi}^{\pi}\frac{d\phi}{2\pi}e^{i(n\phi-\alpha l)}C_{W_1}(l,\phi)\,. \tag{4.6}$$

Further

$$W_1^*(n,\alpha) = [(\mathcal{F}C_{W_1})(n,\alpha)]^* = \left(\mathcal{F}C_{W_1}^*\right)(-n,-\alpha)\,, \tag{4.7}$$

where we used Eq. (3.43), and finally using Eq. (4.5) we get

$$\left(\mathcal{F}C_{W_1}^*\right)(-n,-\alpha) = \sum_{l\in\mathbb{Z}}\int_{-\pi}^{\pi}\frac{d\phi}{2\pi}e^{-i(n\phi-\alpha l)}C_{W_1}(-l,-\phi) = W_1(n,\alpha)\,, \tag{4.8}$$

and the Wigner function $W_1(n,\alpha)$ is real.

Notice that in Eq. (4.6) and Eq. (4.8) it was essential to restrict ourselves to a symmetric interval $(-\pi,\pi]$, so the Wigner function $W_1(n,\alpha)$ is real only on the interval $(-\pi,\pi]$. This restriction of the interval will be present for any integration with this choice of phases $\gamma_1(l,\phi)$ and $\delta_1(l,\phi)$.

With the help of Eq. (3.26) we compute the Wigner operator [35, 39]

$$\hat{\mathcal{W}}_1(n,\alpha) = \sum_{p\in\mathbb{Z}} e^{-i2p\alpha}\,|n+p\rangle\langle n-p| + \frac{1}{\pi}\sum_{p,q\in\mathbb{Z}}\frac{(-1)^{n-q}}{q-n+\frac{1}{2}}e^{-i(2p+1)\alpha}\,|p+q+1\rangle\langle q-p|\;. \tag{4.9}$$

I. Rigas *et al.* [35, 39] have found the overlap of the von Mises state $|n,\alpha\rangle$ with the momentum eigenstate $|k\rangle$ to be

$$\langle k|n,\alpha\rangle = \frac{e^{-\left[ik\alpha+\frac{(k-n)^2}{2}\right]}}{\sqrt{\vartheta_3(0,\frac{1}{e})}}\,, \tag{4.10}$$

where $\vartheta_3(\xi,q)$ denotes the third Jacobi theta function [49]

$$\vartheta_3(\xi,q) = \sum_{n\in\mathbb{Z}} e^{i2n\xi}q^{n^2}\,. \tag{4.11}$$

For the properties of $\vartheta_3(\xi,q)$ see Appendix C of Ref. [32].

The Wigner function of the von Mises state $|n,\alpha\rangle$ takes the form

$$W_1^{|n,\alpha\rangle}(m,\beta) = W_+^{|n,\alpha\rangle}(m,\beta) + W_-^{|n,\alpha\rangle}(m,\beta)\,, \tag{4.12}$$

where [35, 39]

$$W_+^{|n,\alpha\rangle}(m,\beta) = \frac{e^{-(m-n)^2}}{2\pi\vartheta_3(0,\frac{1}{e})}\vartheta_3\left(\beta-\alpha,\frac{1}{e}\right)\,, \tag{4.13}$$

and [35, 39]

$$W_-^{|n,\alpha\rangle}(m,\beta) = \frac{e^{i(\alpha-\beta)-\frac{1}{2}}}{2\pi^2\vartheta_3(0,\frac{1}{e})}\vartheta_3\left(\beta-\alpha+\frac{i}{2},\frac{1}{e}\right)\sum_{l\in\mathbb{Z}}(-1)^{l-m+n}\frac{e^{-l^2-l}}{l+n-m+\frac{1}{2}}\,. \tag{4.14}$$

Using the Eq. (3.46) we find the kernel of this Wigner function as

$$k_1(n,\alpha) = k_+(n,\alpha) + k_-(n,\alpha)\,, \tag{4.15}$$

where

$$k_+(n,\alpha) = \frac{e^{-n^2}}{\vartheta_3(0,\frac{1}{e})}\vartheta_3\left(\alpha,\frac{1}{e}\right)\,, \tag{4.16}$$

and

$$k_-(n,\alpha) = \frac{e^{-i\alpha-\frac{1}{2}}}{\pi\vartheta_3(0,\frac{1}{e})}\vartheta_3\left(\frac{i}{2}+\alpha,\frac{1}{e}\right)\sum_{l\in\mathbb{Z}}(-1)^{l+n}\frac{e^{-l^2-l}}{l+n+\frac{1}{2}}\,. \tag{4.17}$$

Another way of approaching this is through the overlap

$$\langle k|n,\alpha\rangle = \frac{e^{i(n-k)\alpha}}{\sqrt{I_0\,(2\kappa)}}I_{n-k}\,(\kappa)\,, \tag{4.18}$$

where we simply computed the overlap of the angular momentum eigenstate $|k\rangle$ and the von Mises state $|n,\alpha\rangle$ (1.28).

The Wigner function (4.12) then takes the form (for detailed computation see Appendix B.2)

$$W_+^{|n,\alpha\rangle}(m,\beta) = \frac{I_{2(m-n)}\,[2\kappa\cos(\alpha-\beta)]}{2\pi I_0\,(2\kappa)}\,, \tag{4.19}$$

and

$$W_-^{|n,\alpha\rangle}(m,\beta) = \frac{1}{2\pi^2 I_0\,(2\kappa)}\sum_{q\in\mathbb{Z}}\frac{(-1)^{m-q}}{q-m+\frac{1}{2}}I_{2(q-n)+1}\,[2\kappa\cos(\beta-\alpha)]\,. \tag{4.20}$$

and the corresponding kernels are

$$k_+(n,\alpha) = \frac{I_{2n}\,[2\kappa\cos\alpha]}{I_0\,(2\kappa)}\,, \tag{4.21}$$

and

$$k_-(n,\alpha) = \frac{1}{\pi I_0\,(2\kappa)}\sum_{(l=q-m)\in\mathbb{Z}}\frac{(-1)^l}{l+\frac{1}{2}}I_{2(l-n)+1}\,(2\kappa\cos\alpha)\,. \tag{4.22}$$

Notice, that from Eq. (4.19) and Eq. (4.20) we can immediately check, that the Wigner function $W_1^{|n,\alpha\rangle}(m,\beta)$ is real. But let us reiterate, due to the restriction of the interval to $(-\pi,\pi]$ the Wigner function $W_1^{|n,\alpha\rangle}(m,\beta)$ is real only on this interval.

Either way, as one can see, the Wigner function takes on a rather complicated form. Even though this Wigner function is real it is real only on a specific interval of length 2π. Also the expressions are pretty complicated, which can be seen from Eqs. (4.19) and (4.20).

2) One can try to find the best choice of the phases $\gamma_j(l,\phi)$ and $\delta_j(l,\phi)$, for which the corresponding Wigner function $W_j(m,\beta)$ possesses the maximum amount of required properties. This requires solving the, rather complex, set of equations listed in Section 4.1. Interestingly it can be done and the result can be the following choice:

$$\gamma_2(l,\phi) = \left(\frac{l}{2}-\left\lfloor\frac{l}{2}\right\rfloor\right)\phi \quad \text{and} \quad \delta_2(l,\phi) = \left\lfloor-\frac{l}{2}\right\rfloor\phi\,, \tag{4.23}$$

where we introduced the floor function $\lfloor l/2 \rfloor$ of $l/2$ defined as

$$\left\lfloor \frac{l}{2} \right\rfloor = \begin{cases} \frac{l}{2}\,, & \text{if} \quad l \quad \text{is even}\,, \\ \frac{l-1}{2}\,, & \text{if} \quad l \quad \text{is odd}\,, \end{cases} \tag{4.24}$$

which satisfies the identity

$$\left\lfloor \frac{l}{2} \right\rfloor - \left\lfloor \frac{-l}{2} \right\rfloor = l\,. \tag{4.25}$$

Further, using (3.31) we get the kernel (for detailed computation see Appendix B.3)

$$k_2(n,\alpha) = \frac{1}{I_0\left(2\kappa\right)}\left[I_{2n}\left(2\kappa\cos\alpha\right) + I_{2n+1}\left(2\kappa\cos\alpha\right)\right]. \tag{4.26}$$

Through Eq. (3.45) one immediately gets to the Wigner function of the von Mises state

$$W_2^{|n,\alpha\rangle}(m,\beta) = \frac{1}{2\pi I_0\left(2\kappa\right)}\left\{I_{2(m-n)}\left[2\kappa\cos(\beta-\alpha)\right] + I_{2(m-n)-1}\left[2\kappa\cos(\beta-\alpha)\right]\right\}. \tag{4.27}$$

Notice, due to the fact that $I_n\left(z\right) \in \mathbb{R}$ for $z \in \mathbb{R}$, the Wigner function is real on any interval of length 2π, which stems from the satisfaction of the reality condition (3.51).

Computing the Wigner operator (3.27) one arrives at the following expression (for detailed computation see Appendix B.4)

$$\hat{\mathcal{W}}_2(0,0) = (\mathbb{1} + \hat{E})\hat{P}\,, \tag{4.28}$$

and by (3.28) we get

$$\hat{\mathcal{W}}_2(n,\alpha) = \hat{D}_2(n,\alpha)(\mathbb{1} + \hat{E})\hat{P}\hat{D}_2^\dagger(n,\alpha)\,, \tag{4.29}$$

where $\hat{P}$ is the parity operator (3.59). The Wigner operator (4.29) coincides (up to the definition of the operator $\hat{E}$) with the Wigner operator obtained, with the help of heuristic arguments, in Ref. [48].

As we can see, it is possible to get a real Wigner function with relatively simple expressions for basic states $|n\rangle$, $|\phi\rangle$ and $|n,\alpha\rangle$. Granted, this requires solving the set of equations in Section 4.1, but the reward is the simplicity of the resulting expressions.

Let us note that I. Rigas *et al.* [35] also tried for a choice of $\delta(l,\phi)$, that would create a Wigner function of a simpler expression than (4.13) and (4.14). They discussed only the displacement operator $\hat{D}_j(l,\phi)$ and the phase choice they made was

$$\widetilde{\delta}_2(l,\phi) = -l\frac{\phi}{2} + \left\lceil \frac{1-(-1)^l}{4} \right\rceil \phi\,, \tag{4.30}$$

which is equivalent to making the following choice of phases

$$\widetilde{\gamma}_2(l,\phi) = \left(\left\lfloor \frac{l}{2} \right\rfloor - \frac{l}{2}\right)\phi = -\gamma_2(l,\phi) \quad \text{and} \quad \widetilde{\delta}_2(l,\phi) = -\left\lfloor \frac{l}{2} \right\rfloor \phi\,. \tag{4.31}$$

This choice leads to the kernel (for detailed computation see Appendix B.5)

$$\widetilde{k}_2(n,\alpha) = \frac{1}{I_0\,(2\kappa)}\left[I_{2n}\,(2\kappa\cos\alpha) + I_{2n-1}\,(2\kappa\cos\alpha)\right], \tag{4.32}$$

which is very similar to $k_2(n,\alpha)$ (4.26).

With the help of Eq. (3.45) we find the Wigner function for the von Mises state to be

$$\widetilde{W}_2^{|n,\alpha\rangle}(m,\beta) = \frac{1}{2\pi I_0\,(2\kappa)}\left\{I_{2(m-n)}\,[2\kappa\cos(\beta-\alpha)] + I_{2(m-n)+1}\,[2\kappa\cos(\beta-\alpha)]\right\}, \tag{4.33}$$

which is again very similar to $W_2^{|n,\alpha\rangle}(m,\beta)$ (4.27).

The Wigner operator $\hat{\widetilde{\mathcal{W}}}_2(0,0)$ turns out to be (for detailed computation see Appendix B.6)

$$\hat{\widetilde{\mathcal{W}}}_2(0,0) = (\mathbb{1} + \hat{E}^\dagger)\hat{P}, \tag{4.34}$$

and by (3.28) we get

$$\hat{\widetilde{\mathcal{W}}}_2(n,\alpha) = \hat{D}(n,\alpha)(\mathbb{1} + \hat{E}^\dagger)\hat{P}\hat{D}^\dagger(n,\alpha), \tag{4.35}$$

where $\hat{P}$ is the parity operator (3.59). Let us note that the Wigner operator (4.35) coincides (up to the factor $1/(2\pi)$ and the definition of the displacement operator $\hat{D}(l,\phi)$ (3.12)) with the Wigner operator (53) of Ref. [35].

The similarity of the kernels (compare Eqs. (4.26) and (4.32)), Wigner functions of the von Mises state (compare Eqs. (4.27) and (4.33)) and the Wigner operators (compare Eqs. (4.29) and (4.35)) begs us to seek a deeper connection between $W_2^{|n,\alpha\rangle}(m,\beta)$ and $\widetilde{W}_2^{|n,\alpha\rangle}(m,\beta)$. Indeed, one can show that (for detailed computation see Appendix B.7)

$$\widetilde{W}_2^{|n,\alpha\rangle}(m,\beta) = W_2^{|-n,-\alpha\rangle}(-m,-\beta). \tag{4.36}$$

3) It is interesting to relax the reality condition (3.51), in order to find a choice of phases, which would create the simplest forms of k_3, W_3 and $\hat{\mathcal{W}}_3$, resulting in very simple calculations with these expressions.

With this in mind let

$$\gamma_3(l,\phi) = -l\frac{\phi}{2}, \quad \text{and} \quad \delta_3(l,\phi) = 0.$$

One can check that all the conditions listed in Section 4.1 are satisfied except for (3.51), i.e. both $\hat{D}_3(l,\phi)$ and $o_3(l,\phi)$ are 2π-periodic, but the Wigner function is generally complex-valued. Which is exactly what we were aiming for.

The kernel turns out to be (for detailed computation see Appendix B.8)

$$k_3(n,\alpha) = \frac{I_n\,(\kappa)}{I_0\,(2\kappa)}e^{\kappa\cos\alpha - in\alpha}. \tag{4.37}$$

Using Eq. (3.45) we get the Wigner function for the von Mises state $|n,\alpha\rangle$ in the form

$$W_3^{|n,\alpha\rangle}(m,\beta) = \frac{I_{m-n}\,(\kappa)}{2\pi I_0\,(2\kappa)}e^{\kappa\cos(\beta-\alpha)+i(m-n)(\beta-\alpha)}. \tag{4.38}$$

Computing the Wigner operator (3.26) one gets (for detailed computation see Appendix B.9)

$$\hat{\mathcal{W}}_3(n,\alpha) = 2\pi \left|\alpha\right\rangle \left\langle\alpha|n\right\rangle \left\langle n\right| \,, \tag{4.39}$$

note, that this Wigner operator is not hermitian, therefore, via Eq. (3.25), the corresponding Wigner function $W_3^{\hat{\rho}}(m,\beta)$ is generally complex-valued. However observe, that the expressions for the kernel (4.37) and the Wigner function (4.38) are the simplest among the ones we presented.

To demonstrate the simplicity of this complex-valued Wigner function we swiftly compute the Wigner functions of the angular momentum eigenstate $|n\rangle$

$$\begin{aligned} W_3^{|n\rangle}(m,\beta) &= \frac{1}{2\pi}\left\langle n|\hat{\mathcal{W}}_3(m,\beta)|n\right\rangle = \left\langle n|\beta\right\rangle \left\langle\beta|m\right\rangle \left\langle m|n\right\rangle = \\ &= \frac{1}{2\pi} e^{i(m-n)\beta}\delta_{m,n} = \frac{1}{2\pi}\delta_{m,n} \,, \end{aligned}$$

and the angular position eigenstate $|\phi\rangle$

$$\begin{aligned} W_3^{|\phi\rangle}(m,\beta) &= \left\langle\phi|\beta\right\rangle \left\langle\beta|m\right\rangle \left\langle m|\phi\right\rangle = \frac{1}{2\pi}\delta_{2\pi}(\phi-\beta)e^{im(\beta-\phi)} = \\ &= \frac{1}{2\pi}\delta_{2\pi}(\phi-\beta) \,. \end{aligned}$$

Which agrees with the statement we said at the beginning of this section. Imagine, that for some reason we would not have the knowledge of Eqs. (4.1) and (4.2), then computing these equations would be much more difficult with the operator (4.9) or (4.29). More importantly this tells us, that even though the Wigner operator (4.39) is not hermitian and the Wigner function is generally complex-valued, we can obtain Wigner functions that are real.

Let us note that because of the form of the operator (4.39) this complex-valued function $W_3(m,\beta)$ can be in fact identified [48] with the Kirkwood quasiprobability distribution function [47].

We have already seen the P-function of the von Mises state $|n,\alpha\rangle$, Eq. (3.41), and several Wigner functions of this state, (4.12), (4.27) and (4.38)). For completeness we present the Q-function of the von Mises state, which due to the definition (3.18) and with the help of the overlap formula (3.6) turns out to be

$$Q^{|n,\alpha\rangle}(m,\beta) = \frac{I_{n-m}^2\left[2\kappa\cos\left(\frac{\alpha-\beta}{2}\right)\right]}{2\pi I_0^2(2\kappa)} \,, \tag{4.40}$$

and, using Eqs. (3.52) with the help of Eqs. (4.18) and (1.31), the marginal distributions are

$$p^{|n,\alpha\rangle}(m) = \frac{I_{n-m}^2(\kappa)}{I_0^2(2\kappa)} \,, \tag{4.41}$$

$$q^{|n,\alpha\rangle}(\beta) = \frac{e^{2\kappa\cos(\alpha-\beta)}}{2\pi I_0^2(2\kappa)} \,. \tag{4.42}$$

Of course there are other possible choices of $\gamma_j(l,\phi)$ and $\delta_j(l,\phi)$, however the choice is limited if one wants to preserve the properties of the Wigner function. The periodicity of $\hat{D}_j(l,\phi)$, Eq. (3.57), and $o_j(l,\phi)$, Eq. (3.58), requires the dependence on ϕ to be linear and ϕ always has to be multiplying l. The dependence on l also has to be linear, since a nonlinear dependence on l creates many computational difficulties very early on. For example some quadratic phase choices in l could be $\gamma(l,\phi) = l^2\phi/2$ and $\delta(l,\phi) = -l(l+1)\phi/2$ or $\gamma(l,\phi) = -l^2\phi/2$ and $\delta(l,\phi) = l(l-1)\phi/2$, which satisfy all the required conditions, but create computational difficulties. Also notice, if we want to satisfy the marginal distributions conditions, Eqs. (3.53) and (3.54), there cannot be any absolute terms in the expressions of $\gamma_j(l,\phi)$ and $\delta_j(l,\phi)$.

The possibility to go from a uniquely defined Q-function to a uniquely defined P-function via different Wigner functions is depicted in Figure 4.1.

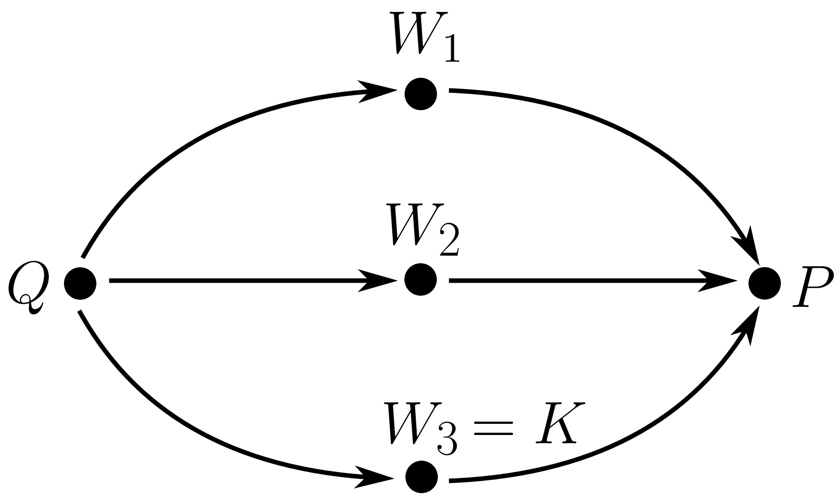


Figure 4.1: Graphical representation of several paths from a uniquely defined Q-function through different Wigner functions W_1, W_2 and W_3 to a uniquely defined P-function. Some paths may lead through other quasiprobability distributions (e.g. Kirkwood quasiprobability distribution function K). Other possible choices of phases $\gamma_j(l,\phi)$ and $\delta_j(l,\phi)$ exist, leading to other Wigner functions and paths, which are not depicted.

4.2.1 Overview of the phase choices

Let us collect previous results in neat tables. As we mentioned at the start of Section 4.2, the Wigner functions of the angular momentum eigenstate $|n\rangle$ and angular position eigenstate $|\phi\rangle$ are independent of j so we will not include them in the tables. Inspection of the Tab. 4.1 and Tab. 4.2 reveals that the simplest Wigner function is $W_3^{|n,\alpha\rangle}(m,\beta)$, but it is complex-valued. On the other hand the Wigner function $W_1^{|n,\alpha\rangle}(m,\beta)$ is real, but very complicated to compute with. The optimal choice seems to be $W_2^{|n,\alpha\rangle}(m,\beta)$.

Table 4.1: Reality of the Wigner function $W_j^{|n,\alpha\rangle}(m,\beta)$, periodicity of the operator $\hat{D}_j(l,\phi)$ and overlap $o_j(l,\phi)$ for different choices of $\gamma_j(l,\phi)$ and $\delta_j(l,\phi)$.

j	$\gamma_j(l,\phi)$	$\delta_j(l,\phi)$	real	periodicity of $\hat{D}_j(l,\phi)$	periodicity of $o_j(l,\phi)$
1	0	$-l\phi/2$	✓	×	×
2	$(l/2-\lfloor l/2\rfloor)\,\phi$	$\lfloor -l/2\rfloor\,\phi$	✓	✓	✓
3	$-l\phi/2$	0	×	✓	✓

Table 4.2: Kernels $k_j(n,\alpha)$ and Wigner functions of the von Mises state $|n,\alpha\rangle$ for different choices of $\gamma_j(l,\phi)$ and $\delta_j(l,\phi)$.

j	$k_j(n,\alpha)\cdot I_0(2\kappa)$	$W_j^{|n,\alpha\rangle}(m,\beta)\cdot 2\pi I_0(2\kappa)$
1	complex formula: (4.15)$\cdot I_0(2\kappa)$	complex formula: (4.12)/$[2\pi I_0(2\kappa)]$
2	$I_{2n}(2\kappa\cos\alpha)+I_{2n+1}(2\kappa\cos\alpha)$	$I_{2(m-n)}[2\kappa\cos(\beta-\alpha)]+I_{2(m-n)-1}[2\kappa\cos(\beta-\alpha)]$
3	$I_n(\kappa)\exp(\kappa\cos\alpha-in\alpha)$	$I_{m-n}(\kappa)\exp[\kappa\cos(\beta-\alpha)+i(m-n)(\beta-\alpha)]$

4.3 Behavior of the Wigner functions

In this section we analyze the behavior of our Wigner functions for the von Mises vacuum state $|0,0\rangle$ and provide their visualization.

Because $W_1^{|0,0\rangle}(m,\beta)$ takes on a rather complicated form (also this Wigner function was somewhat analyzed in Ref. [35]) and $W_3^{|0,0\rangle}(m,\beta)$ is complex-valued, therefore it cannot be depicted, we restrict ourselves to the Wigner function $W_2^{|0,0\rangle}(m,\beta)$ and consequently $\widetilde{W}_2^{|0,0\rangle}(m,\beta)$, due to Eq. (4.36). Visualization will be provided for all Wigner functions, we presented, that are real.

Due to the reality of the Wigner function $W_1^{|0,0\rangle}(m,\beta)$ we analyze the functions on the interval $\beta\in(-\pi,\pi]$. This correspond to cutting the phase space cylinder along the line $\phi=\pi$ and unraveling it into a two-dimensional "plane" with $m\in\mathbb{Z}$ on one axis and $\beta\in(-\pi,\pi]$ on the orthogonal axis.

4.3.1 Analysis

For a detailed analysis see Appendix C. Here we list the main takeaways. The Wigner function

$$W_2^{|0,0\rangle}(m,\beta)=\frac{1}{2\pi I_0(2\kappa)}\{I_{2m}(2\kappa\cos\beta)+I_{2m-1}(2\kappa\cos\beta)\}, \tag{4.43}$$

is even in β. From Eq. (4.36) it follows that

$$\widetilde{W}_2^{|0,0\rangle}(m,\beta) = W_2^{|0,0\rangle}(-m,\beta)\,, \quad \forall m \in \mathbb{Z} \quad \text{and} \quad \beta \in \mathbb{R}\,. \tag{4.44}$$

Due to this fact, the results of the analysis of $W_2^{|0,0\rangle}(m,\beta)$ will be the same as $\widetilde{W}_2^{|0,0\rangle}(m,\beta)$, except for where the sign of the index m would matter.

For the case of the spread parameter $\kappa = 0$, from the formula (4.43), using $I_n(0) = \delta_{n,0}$ we get

$$W_2^{|0,0\rangle}(m,\beta) = \frac{1}{2\pi}\delta_{m,0}\,, \tag{4.45}$$

which is the Wigner function of the angular momentum eigenstate, Eq. (4.1) with $n = 0$.

Much more interesting is the case when $\kappa > 0$. For this case the Wigner functions $W_2^{|0,0\rangle}(m,\beta)$, as functions of β, have points of potential extrema at $\beta = \pm\pi$, $\beta = \pm\frac{\pi}{2}$ and $\beta = 0$. There is a global maximum at $\beta = 0$ for every m. At $\beta = \pm\frac{\pi}{2}$ the Wigner function (4.43) takes the form

$$W_2^{|0,0\rangle}\left(m,\frac{\pi}{2}\right) = \frac{1}{2\pi I_0(2\kappa)}\delta_{m,0}\,, \tag{4.46}$$

and except for $W_2^{|0,0\rangle}(0,\beta)$, all the Wigner functions $W_2^{|0,0\rangle}(m,\beta)$ are equal to zero at $\beta = \pm\frac{\pi}{2}$. Further,

$$W_2^{|0,0\rangle}(m,\pm\pi) = \frac{1}{2\pi I_0(2\kappa)}\left[I_{2m}(2\kappa) - I_{2m-1}(2\kappa)\right]\begin{cases} < 0\,, & \text{if} \quad m \in \mathbb{Z}^+\,, \\ > 0\,, & \text{if} \quad m \in \mathbb{Z}_0^-\,. \end{cases} \tag{4.47}$$

Equation (4.47) coupled with Eq. (4.46) tells us that the Wigner functions $W_2^{|0,0\rangle}(m,\beta)$ of non-positive integers m are non-negative on the whole interval $\beta \in (-\pi,\pi]$. Thanks to Eq. (4.44) we immediately see that

$$\widetilde{W}_2^{|0,0\rangle}(m,\pm\pi)\begin{cases} > 0\,, & \text{if} \quad m \in \mathbb{Z}_0^+\,, \\ < 0\,, & \text{if} \quad m \in \mathbb{Z}^-\,, \end{cases} \tag{4.48}$$

and the Wigner functions $\widetilde{W}_2^{|0,0\rangle}(m,\beta)$ of non-negative integers m are non-negative on the whole interval $\beta \in (-\pi,\pi]$.

The Wigner functions $W_2^{|0,0\rangle}(m,0)$ have a global maximum at $m = 0$. As m increases the Wigner functions $W_2^{|0,0\rangle}(m,0)$ descend for $m \in \mathbb{Z}^+$ and ascend for $m \in \mathbb{Z}^-$. The speed of the descend [ascend] of $W_2^{|0,0\rangle}(m,0)$ is at least as fast as a geometric series with the ratio $I_1(2\kappa)/I_0(2\kappa)$ $[I_2(2\kappa)/I_1(2\kappa)]$ for the Wigner functions of positive [negative] integers m, i.e.

$$0 < \frac{W_2^{|0,0\rangle}(m,0)}{W_2^{|0,0\rangle}(0,0)} \leq \left[\frac{I_1(2\kappa)}{I_0(2\kappa)}\right]^m\,,\; 0 < \frac{W_2^{|0,0\rangle}(-m,0)}{W_2^{|0,0\rangle}(0,0)} \leq \left[\frac{I_2(2\kappa)}{I_1(2\kappa)}\right]^m\,, \quad \forall m \in \mathbb{Z}_0^+\,, \tag{4.49}$$

where equality occurs if $m = 0$. This can be seen in Fig. 4.2 and Fig. 4.3. Using Eq. (4.44) we get the following inequalities

$$0 < \frac{\widetilde{W}_2^{|0,0\rangle}(m,0)}{\widetilde{W}_2^{|0,0\rangle}(0,0)} \leq \left[\frac{I_2(2\kappa)}{I_1(2\kappa)}\right]^m\,,\; 0 < \frac{\widetilde{W}_2^{|0,0\rangle}(-m,0)}{\widetilde{W}_2^{|0,0\rangle}(0,0)} \leq \left[\frac{I_1(2\kappa)}{I_0(2\kappa)}\right]^m\,, \quad \forall m \in \mathbb{N}_0\,, \tag{4.50}$$

where equality occurs if $m = 0$, and the descending [ascending] ratios are switched, just as one would expect.

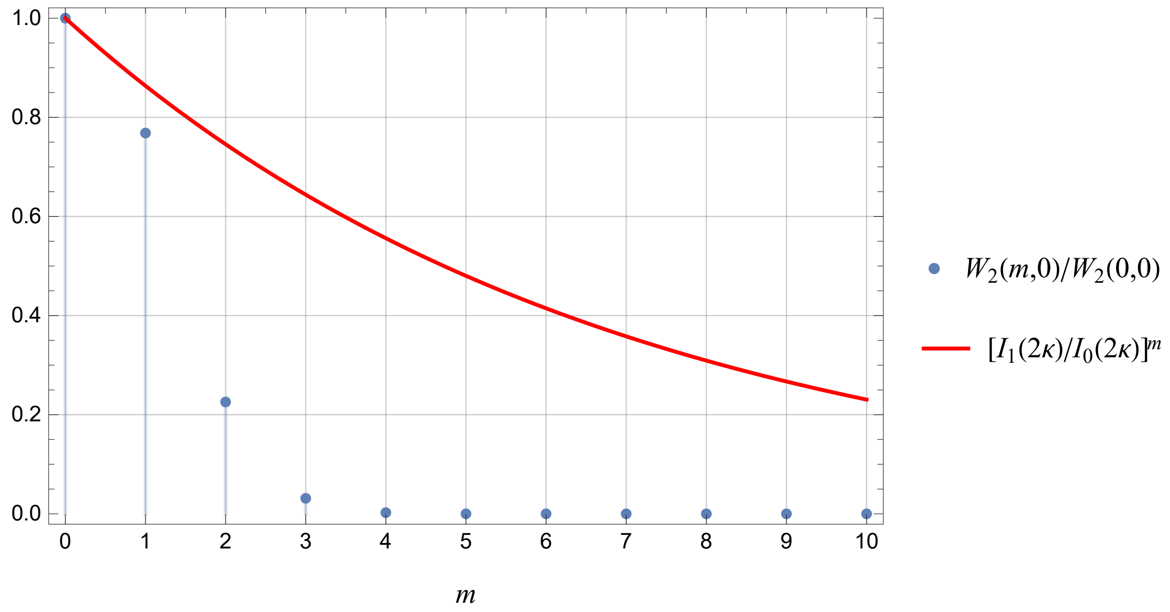


Figure 4.2: The quotient of the Wigner functions $W_2^{|0,0\rangle}(m,0)/W_2^{|0,0\rangle}(0,0)$ decreases at least as fast as $[I_1(2\kappa)/I_0(2\kappa)]^m$ (here shown for $\kappa = 2$).

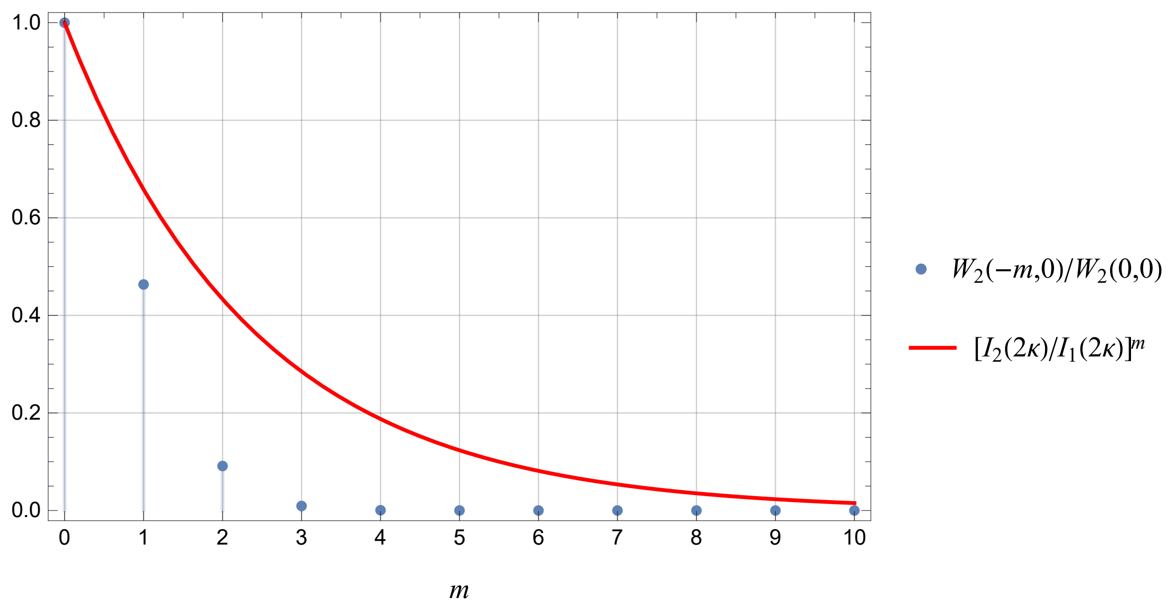


Figure 4.3: The quotient of the Wigner functions $W_2^{|0,0\rangle}(-m,0)/W_2^{|0,0\rangle}(0,0)$ decreases at least as fast as $[I_2(2\kappa)/I_1(2\kappa)]^m$ (here shown for $\kappa = 2$).

4.3.2 Visualization

Here we provide visualization of the real Wigner functions $W_j^{|0,0\rangle}(m,\beta)$, $j=1,2$, of the "vacuum" von Mises state $|0,0\rangle$. For better distinguishability we set $\kappa=2$.

Although we did not analyze the Wigner function $W_1^{|0,0\rangle}(m,\beta)$ mainly because of its complicated expression, Eq. (4.12), for completeness we include the graphs in Fig. 4.4 and Fig. 4.5. We can see many similarities with $W_2^{|0,0\rangle}(m,\beta)$, e.g. they are even functions of β, they have a global maximum at $\beta=0$ for any m (better seen from Fig. 4.5), close to $\beta=\pm\pi$ we see that for odd m the Wigner functions $W_1^{|0,0\rangle}(m,\beta)$ take on negative values and for even integers m the Wigner functions $W_1^{|0,0\rangle}(m,\beta)$ stay non-negative on the whole interval $(-\pi,\pi]$.

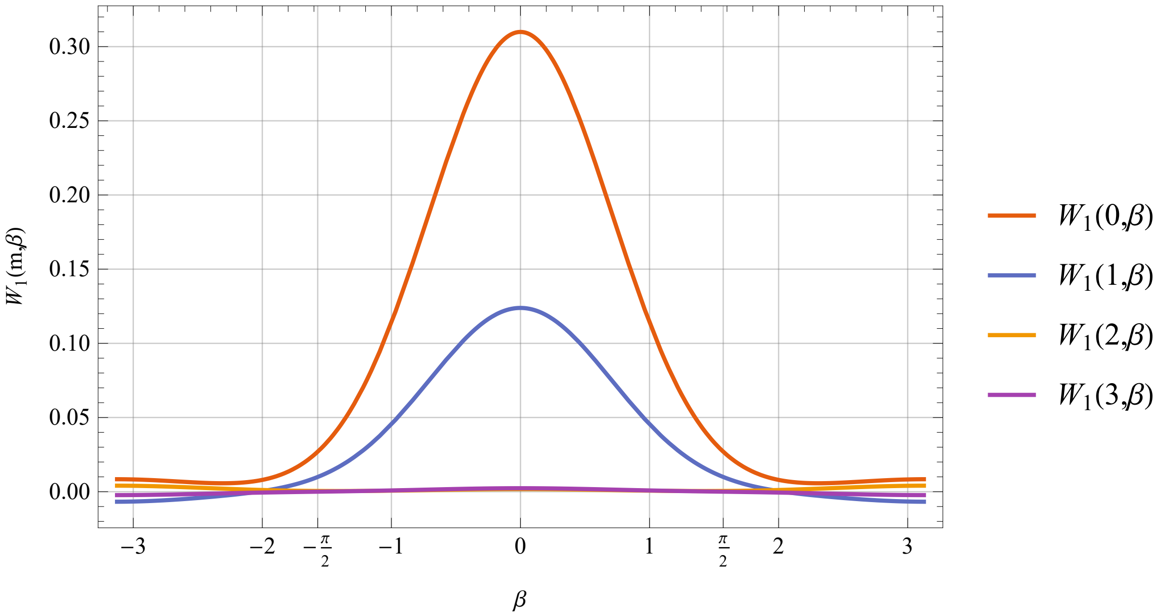


Figure 4.4: Two-dimensional plot of the Wigner functions $W_1^{|0,0\rangle}(m,\beta)$ for various m and $\beta\in(-\pi,\pi)$. Only a few functions are depicted, because other Wigner functions $W_1^{|0,0\rangle}(m,\beta)$ would not be distinguishable.

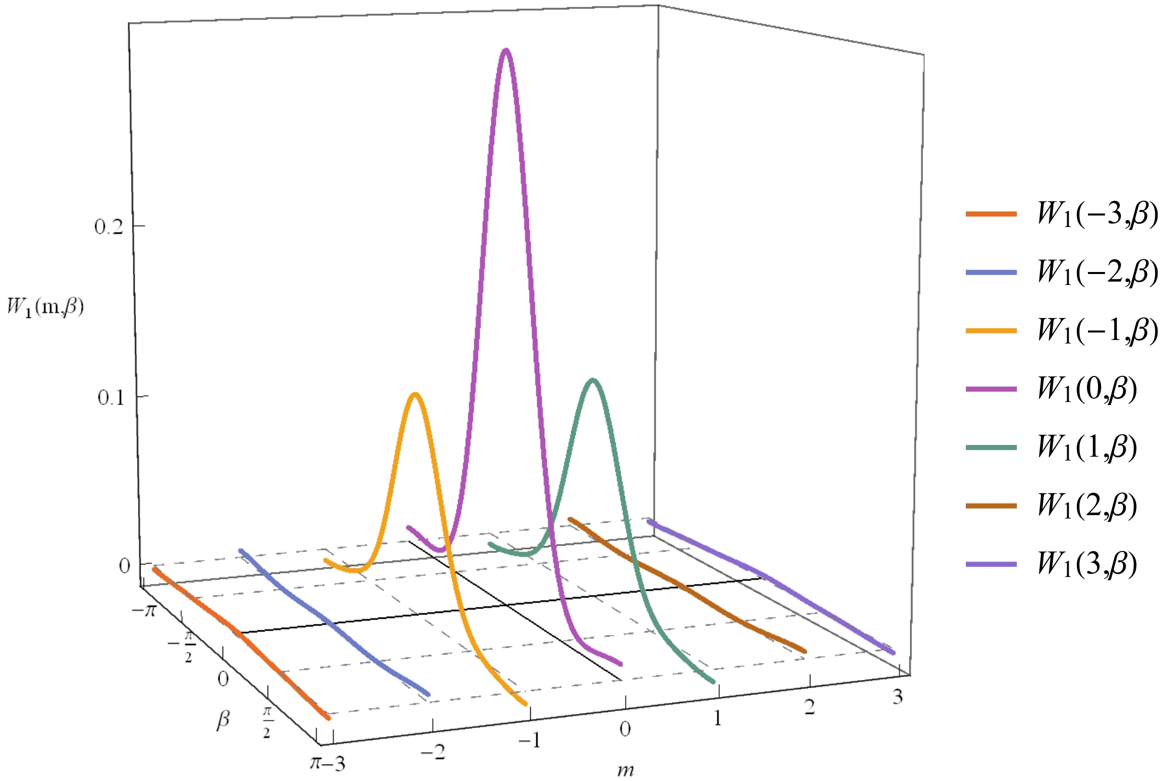
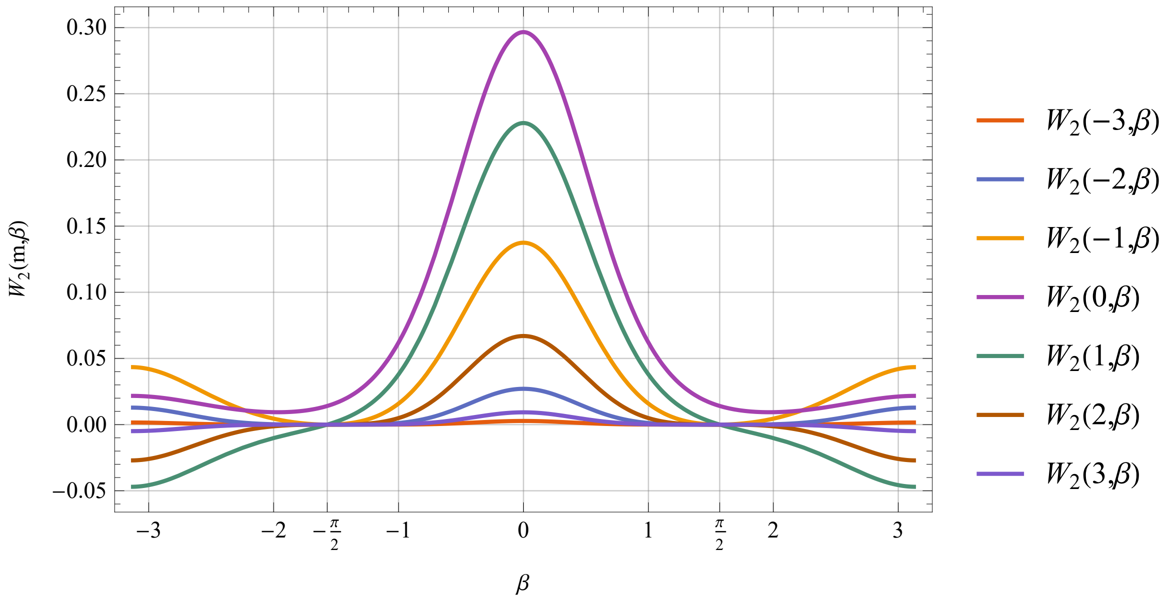
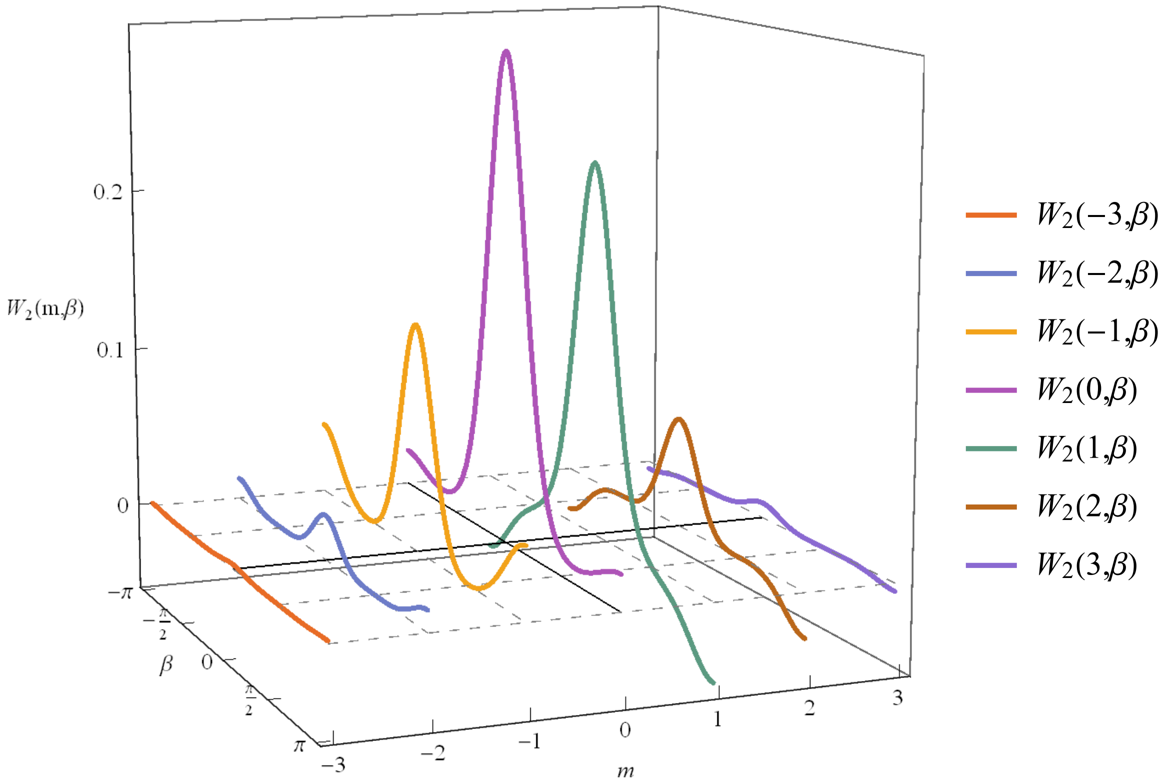


Figure 4.5: Three-dimensional plot of the Wigner functions $W_1^{|0,0\rangle}(m,\beta)$ for various m and $\beta \in (-\pi, \pi]$.

Moving on to $W_2^{|0,0\rangle}(m,\beta)$, which we can see in Fig. 4.6(a), all the Wigner functions $W_2^{|0,0\rangle}(m,\beta)$ have a global maximum at $\beta = 0$. Except for $W_2^{|0,0\rangle}(0,\beta)$ all Wigner functions are equal to zero at $\beta = \pm\frac{\pi}{2}$. All Wigner functions $W_2^{|0,0\rangle}(m,\beta)$ of positive integers m are negative on the interval $\left(-\pi, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$ and all Wigner functions $W_2^{|0,0\rangle}(m,\beta)$ of non-positive integers m are non-negative on the whole interval $\beta \in (-\pi, \pi]$. Further, in Fig. 4.6(b) we see, that the Wigner functions $W_2^{|0,0\rangle}(m,\beta)$, as a functions of m, have a maximum at $m = 0$ and rapidly decrease as m moves further from the origin. All of this is in agreement with our analysis.



(a) 2D plot



(b) 3D plot

Figure 4.6: Two-dimensional plot **(a)** and three-dimensional plot **(b)** of the Wigner functions $W_2^{|0,0\rangle}(m, \beta)$ for various m and $\beta \in (-\pi, \pi]$.

Conclusion

The aim of this thesis was to investigate a system of operators $\hat{E}$ and $\hat{L}$ characterizing a quantum system of the quantum rigid rotor. $\hat{E}$ being the complex exponential of the angular position and $\hat{L}$ the component of the angular momentum along the axis orthogonal to the plane of rotation. To achieve this we used the phase space formalism, where the possible states of a considered system are represented by quasiprobability distributions. In analogy with the linear harmonic oscillator, the Wigner function was chosen as this quasiprobability distribution.

The system of a quantum rigid rotor is not as simple as the system of a linear harmonic oscillator. Namely, because both $\hat{x}$ and $\hat{p}$ have continuous spectra, the phase space of the linear harmonic oscillator is a plane. However, due to the fact that $\hat{L}$ has integer eigenvalues and $\hat{E}$ having complex eigenvalues of magnitude always equal to one, the phase space of a quantum rigid rotor is a set of equidistant rings (each ring is one unit away from the other) on a surface of a cylinder with radius one.

The Wigner function is a uniquely defined concept for the system of the linear harmonic oscillator. For the quantum rotor we have found that, in developing the Wigner function, one can adopt different phase conventions leading to different Wigner functions with generally different properties. In order to obtain a Wigner function with the maximum amount of required properties, such as reality, providing the correct marginals, normalization, etc., the phase convention has to satisfy certain conditions. We have found these conditions. We reviewed some phase conventions which already appeared in the literature and discussed their strengths and weaknesses. Surprisingly, we were able to solve the set of equations given by the conditions and derived a new Wigner function satisfying all of the required conditions.

Additionally, we found that by sacrificing the reality property of the Wigner function, we can arrive at a particularly simple phase space quasiprobability distribution, which has very simple expression and allows for easy computations.

The newly introduced Wigner function for the quantum rotor can be applied analogously as in the case of the linear harmonic oscillator. This includes, for instance, analysis of the performance of quantum information protocols with mixed states or development of the inseparability criteria in phase space.

Finding of a Wigner function of a quantum rotor, which is equipped with the largest number of the required properties, can be also viewed as the first step towards development of the complete phase space formalism for this system. When undertaking this endeavour a number of questions have to be addressed encompassing, e.g., the question what would be the analogy of the normal, symmetrical and anti-normal orderings for the observables of the angular momentum and angular position, how would thermal state and the Gaussian state look like.

Our results also unveil that the simplest quasiprobability distribution is obtained if one resigns one of the basic properties of the Wigner function, which is its reality. This rises a question as to whether one really has to be guided by the properties of the Wigner function of the linear harmonic oscillator when dealing with other quantum systems.

We believe that our results will stimulate further research aiming to harness quantum properties of systems beyond position and momentum of a linear harmonic oscillator.

Appendix A

Modified Bessel function

The modified Bessel function of an integer order n and complex argument z can be expressed as the following integral [50]

$$I_n(z) = \int_{\alpha_0}^{\alpha_0+2\pi} \frac{d\phi}{2\pi} e^{z\cos\phi + in\phi}, \tag{A.1}$$

where α_0 is an arbitrary angle. Hence, one can easily find that

$$I_n(z) = I_{-n}(z) , \quad I_n(-z) = (-1)^n I_n(z) , \quad I_n(0) = \delta_{n,0} , \tag{A.2}$$

and $I_n(z)$ is real for real z, where in the derivation of the last equality we used a result from complex analysis

$$\int_{\alpha_0}^{\alpha_0+2\pi} \frac{d\phi}{2\pi} e^{i(m-n)\phi} = \delta_{m,n} , \tag{A.3}$$

where α_0 is an arbitrary angle. Besides, the modified Bessel function fulfils the recurrence relations [50]

$$I_{n-1}(z) - I_{n+1}(z) = \frac{2n}{z} I_n(z) , \tag{A.4}$$

and

$$I_{n-1}(z) + I_{n+1}(z) = 2\frac{\mathrm{d}}{\mathrm{d}z} I_n(z) . \tag{A.5}$$

Our calculations with the modified Bessel functions are greatly simplified by the addition theorem [50]

$$\sum_{m\in\mathbb{Z}} (-1)^m I_{\nu+m}(Z)\, I_m(z)\, e^{im\phi} = e^{i\nu\psi} I_\nu(\omega) , \tag{A.6}$$

where $\nu \in \mathbb{Z}$ and

$$\begin{aligned} \omega &= \sqrt{Z^2 + z^2 - 2Zz\cos\phi}\,, \\ \omega\cos\psi &= Z - z\cos\phi\,, \\ \omega\sin\psi &= z\sin\phi\,. \end{aligned} \tag{A.7}$$

In particular, the addition theorem yields

$$\sum_{m\in\mathbb{Z}} I_m(\kappa)\, I_{m+\nu}(\kappa)\, e^{im\phi} = e^{-i\nu\frac{\phi}{2}} I_\nu\left[2\kappa\cos\left(\frac{\phi}{2}\right)\right] , \tag{A.8}$$

which for the special case $\nu = \phi = 0$ gives

$$\sum_{m\in\mathbb{Z}} I_m^2(\kappa) = I_0\left(2\kappa\right)\ . \tag{A.9}$$

From the definition of the modified Bessel function (A.1), using the Dirac comb (1.14), one can swiftly derive the following generating function

$$\sum_{m\in\mathbb{Z}} I_m\left(z\right) e^{im\phi} = e^{z\cos\phi}\,. \tag{A.10}$$

Let us also note that R.P.Soni [51] established the following inequality

$$0 < I_{\nu+1}\left(x\right) < I_\nu\left(x\right)\ ,\quad \nu > -\frac{1}{2}\,,\quad x > 0\,. \tag{A.11}$$

Since we work with modified Bessel functions of integer order we get

$$\begin{aligned} 0 < I_{n+1}\left(x\right) < I_n\left(x\right)\ , &\quad n\in\mathbb{Z}_0^+\,,\quad x>0\,,\\ 0 < I_{n-1}\left(x\right) < I_n\left(x\right)\ , &\quad n\in\mathbb{Z}_0^-\,,\quad x>0\,, \end{aligned} \tag{A.12}$$

and most importantly

$$I_n\left(x\right) > 0\,,\quad n\in\mathbb{Z}\,,\quad x>0\,. \tag{A.13}$$

Appendix B

Computation of certain mathematical objects

B.1 Independence of $W^{|n\rangle}(m,\beta)$ and $W^{|\phi\rangle}(m,\beta)$ on the phase choice

From the definition of the Wigner function as the mean

$$W_j^{\hat{\rho}}(n,\alpha) = \frac{1}{2\pi}\,\mathrm{Tr}\left[\hat{\rho}\hat{\mathcal{W}}_j(n,\alpha)\right]\,, \tag{3.25}$$

of the operator

$$\hat{\mathcal{W}}_j(n,\alpha) = \sum_{l\in\mathbb{Z}}\int_0^{2\pi}\frac{d\phi}{2\pi}e^{i(n\phi-\alpha l)}\hat{D}_j(l,\phi)\,, \tag{3.26}$$

we see, that

$$W_j^{|n\rangle}(m,\beta) = \frac{1}{2\pi}\,\mathrm{Tr}\left[|n\rangle\langle n|\,\hat{\mathcal{W}}_j(m,\beta)\right] = \frac{1}{2\pi}\,\langle n|\hat{\mathcal{W}}_j(m,\beta)|n\rangle =$$
$$= \frac{1}{2\pi}\,\langle n|\sum_{l\in\mathbb{Z}}\int_0^{2\pi}\frac{d\phi}{2\pi}e^{i(m\phi-\beta l)}\hat{D}_j(l,\phi)|n\rangle = \frac{1}{2\pi}\sum_{l\in\mathbb{Z}}\int_0^{2\pi}\frac{d\phi}{2\pi}e^{i(m\phi-\beta l)}\,\langle n|\hat{D}_j(l,\phi)|n\rangle\ .$$

Now, noticing that

$$\hat{D}_j(l,\phi)\,|n\rangle = e^{i\delta_j(l,\phi)}\hat{E}^{-l}e^{-i\hat{L}\phi}\,|n\rangle = e^{i[\delta_j(l,\phi)-n\phi]}\,|n+l\rangle\ , \tag{B.1}$$

so the expectation value is

$$\langle n|\hat{D}_j(l,\phi)|n\rangle = e^{i[\delta_j(l,\phi)-n\phi]}\delta_{l,0} = e^{-in\phi}\delta_{l,0}\,, \tag{B.2}$$

where we assume, that the condition for marginal distributions

$$\delta_j(0,\phi) = 2n\pi\,,\quad n\in\mathbb{Z}\,, \tag{3.53}$$

holds. From this it follows that

$$W^{|n\rangle}(m,\beta) = \frac{1}{2\pi}\sum_{l\in\mathbb{Z}}\int_0^{2\pi}\frac{d\phi}{2\pi}e^{i(m\phi-\beta l)}e^{-in\phi}\delta_{l,0} = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\phi}{2\pi}e^{i(m-n)\phi} = \frac{1}{2\pi}\delta_{m,n}\,. \tag{4.1}$$

Similarly

$$W_j^{|\phi\rangle}(m,\beta) = \frac{1}{2\pi}\,\mathrm{Tr}\left[|\phi\rangle\langle\phi|\,\hat{\mathcal{W}}_j(m,\beta)\right] = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_0^{2\pi}\frac{d\alpha}{2\pi}e^{i(m\alpha-\beta n)}\,\langle\phi|\hat{D}_j(n,\alpha)|\phi\rangle\ , \tag{B.3}$$

where the expectation value

$$\begin{aligned}\langle\phi|\hat{D}_j(n,\alpha)|\phi\rangle &= \langle\phi|e^{i\delta_j(n,\alpha)}\hat{E}^{-n}e^{-i\hat{L}\alpha}|\phi\rangle = \langle\phi|\,e^{i[\delta_j(n,\alpha)+n(\phi+\alpha)]}\,|\phi+\alpha\rangle = \\ &= e^{i[\delta_j(n,\alpha)+n(\phi+\alpha)]}\delta_{2\pi}(\alpha) = e^{in\phi}\delta_{2\pi}(\alpha)\,,\end{aligned}$$

where we assume, that the condition for marginal distributions

$$\delta_j(l,0) = 2r\pi\,,\quad r\in\mathbb{Z}\,, \tag{3.54}$$

holds. Finally

$$\begin{aligned}W^{|\phi\rangle}(m,\beta) &= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_0^{2\pi}\frac{d\alpha}{2\pi}e^{i(m\alpha-\beta n)}e^{in\phi}\delta_{2\pi}(\alpha) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\frac{1}{2\pi}e^{in(\phi-\beta)} = \\ &= \frac{1}{2\pi}\delta_{2\pi}(\beta-\phi)\,.\end{aligned} \tag{4.2}$$

B.2 Computation of the Wigner function $W_1^{|n,\alpha\rangle}(m,\beta)$

We start with the decomposition of the Wigner function $W_1^{|n,\alpha\rangle}(m,\beta)$

$$W_1^{|n,\alpha\rangle}(m,\beta) = W_+^{|n,\alpha\rangle}(m,\beta) + W_-^{|n,\alpha\rangle}(m,\beta)\,, \tag{4.12}$$

which comes from the form of the Wigner operator

$$\hat{\mathcal{W}}_1(n,\alpha) = \sum_{p\in\mathbb{Z}}e^{-i2p\alpha}\,|n+p\rangle\langle n-p| + \frac{1}{\pi}\sum_{p,q\in\mathbb{Z}}\frac{(-1)^{n-q}}{q-n+\frac{1}{2}}e^{-i(2p+1)\alpha}\,|p+q+1\rangle\langle q-p|\ . \tag{4.9}$$

From here, using Eq. (3.25), we get

$$\begin{aligned}W_+^{|n,\alpha\rangle}(m,\beta) &= \frac{1}{2\pi}\,\langle n,\alpha|\left(\sum_{p\in\mathbb{Z}}e^{-i2p\beta}\,|m+p\rangle\langle m-p|\right)|n,\alpha\rangle = \\ &= \frac{1}{2\pi I_0(2\kappa)}\sum_{p\in\mathbb{Z}}e^{-i2p\beta}e^{-i[n-(m+p)]\alpha}I_{n-(m+p)}(\kappa)\,e^{i[n-(m-p)]\alpha}I_{n-(m-p)}(\kappa)\ ,\end{aligned}$$

where we used the overlap (4.18). After some cancellations and using the symmetry (A.2) we get

$$\begin{aligned}W_+^{|n,\alpha\rangle}(m,\beta) &= \frac{1}{2\pi I_0(2\kappa)}\sum_{p\in\mathbb{Z}}e^{-i2p(\alpha-\beta)}I_{p+(m-n)}(\kappa)\,I_{p-(m-n)}(\kappa) = \\ &= \frac{e^{-i2(m-n)(\alpha-\beta)}}{2\pi I_0(2\kappa)}\sum_{l\in\mathbb{Z}}I_l(\kappa)\,I_{l-2(m-n)}(\kappa)\,e^{i2l(\alpha-\beta)} = \\ &= \frac{I_{2(m-n)}\left[2\kappa\cos(\alpha-\beta)\right]}{2\pi I_0(2\kappa)}\,,\end{aligned} \tag{4.19}$$

where to get the last equality we used the addition theorem (A.8).

To get the second part of Eq. (4.12) we again use Eq. (3.25) and overlap (4.18) to find that

$$W_-^{|n,\alpha\rangle}(m,\beta) = \frac{1}{2\pi^2 I_0(2\kappa)} \sum_{p,q\in\mathbb{Z}} \frac{(-1)^{m-q} e^{i(2p+1)(\beta-\alpha)}}{q-m+\frac{1}{2}} I_{n-(q-p)}(\kappa)\, I_{n-(p+q+1)}(\kappa) =$$
$$= \sum_{q\in\mathbb{Z}} \frac{(-1)^{m-q} e^{i(\beta-\alpha)}}{2\pi^2 I_0(2\kappa)(q-m+\frac{1}{2})} \sum_{p\in\mathbb{Z}} I_{n-(q-p)}(\kappa)\, I_{n-(p+q+1)}(\kappa)\, e^{i2p(\beta-\alpha)} =$$
$$= \sum_{q\in\mathbb{Z}} \frac{(-1)^{m-q} e^{i[2(q-n)+1](\beta-\alpha)}}{2\pi^2 I_0(2\kappa)(q-m+\frac{1}{2})} \sum_{l\in\mathbb{Z}} I_l(\kappa)\, I_{l+2(q-n)+1}(\kappa)\, e^{i2l(\beta-\alpha)},$$

using the addition theorem (A.8) and after some simple cancellations we arrive at

$$W_-^{|n,\alpha\rangle}(m,\beta) = \frac{1}{2\pi^2 I_0(2\kappa)} \sum_{q\in\mathbb{Z}} \frac{(-1)^{m-q}}{q-m+\frac{1}{2}} I_{2(q-n)+1}\left[2\kappa\cos(\beta-\alpha)\right]. \tag{4.20}$$

B.3 Computation of the kernel $k_2(n,\alpha)$

To achieve the expression for the kernel $k_2(n,\alpha)$ we first inspect the overlap

$$o_2(l,\phi) = e^{i\gamma_2(l,\phi)} \frac{I_l\left[2\kappa\cos\left(\frac{\phi}{2}\right)\right]}{I_0(2\kappa)} = \frac{e^{i\left(\frac{l}{2}-\left\lfloor\frac{l}{2}\right\rfloor\right)\phi} e^{i\frac{l}{2}\phi}}{I_0(2\kappa)} \sum_{m\in\mathbb{Z}} I_m(\kappa)\, I_{m+l}(\kappa)\, e^{im\phi},$$

where to get the last equality the addition theorem (A.8) has been used. Next, we collect terms and get

$$o_2(l,\phi) = \frac{1}{I_0(2\kappa)} \sum_{m\in\mathbb{Z}} e^{i\left(l-\left\lfloor\frac{l}{2}\right\rfloor+m\right)\phi} I_m(\kappa)\, I_{m+l}(\kappa),$$

and using the identity (4.25) we get

$$o_2(l,\phi) = \sum_{m\in\mathbb{Z}} \frac{e^{i\left(m-\left\lfloor-\frac{l}{2}\right\rfloor\right)\phi} I_m(\kappa)\, I_{m+l}(\kappa)}{I_0(2\kappa)}.$$

Now from the definition of the kernel (3.31) it follows

$$k_2(n,\alpha) = \sum_{l,m\in\mathbb{Z}} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i(n\phi-l\alpha)} \frac{e^{i\left(m-\left\lfloor-\frac{l}{2}\right\rfloor\right)\phi} I_m(\kappa)\, I_{m+l}(\kappa)}{I_0(2\kappa)}$$
$$= \sum_{l,m\in\mathbb{Z}} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-il\alpha} \frac{e^{i\left(m-\left\lfloor-\frac{l}{2}\right\rfloor+n\right)\phi} I_m(\kappa)\, I_{m+l}(\kappa)}{I_0(2\kappa)}.$$

Due to the definition of the floor function (4.24) we need to split the sum into an odd part and an even part, i.e.

$$k_2(n,\alpha) = \sum_{m,j\in\mathbb{Z}} \int_0^{2\pi} \frac{d\phi}{2\pi I_0(2\kappa)} I_m(\kappa)\, I_{m+2j}(\kappa)\, e^{i(n+m+j)\phi} e^{-i2j\alpha} +$$
$$+ \sum_{m,j\in\mathbb{Z}} \int_0^{2\pi} \frac{d\phi}{2\pi I_0(2\kappa)} I_m(\kappa)\, I_{m+2j-1}(\kappa)\, e^{i(n+m+j)\phi} e^{-i(2j-1)\alpha},$$

using the Kronecker delta (A.3) we get

$$k_2(n,\alpha) = \sum_{m\in\mathbb{Z}} \frac{I_m(\kappa)}{I_0(2\kappa)} \left[I_{m+2(-n-m)}(\kappa)\, e^{-i2(-n-m)\alpha} + I_{m+2(-n-m)-1}(\kappa)\, e^{-i[2(-n-m)-1]\alpha} \right] =$$

$$= \sum_{m\in\mathbb{Z}} \frac{I_m(\kappa)}{I_0(2\kappa)} \left[I_{-m-2n}(\kappa)\, e^{i2n\alpha} e^{i2m\alpha} + I_{-m-2n-1}(\kappa)\, e^{i2m\alpha} e^{i(2n+1)\alpha} \right] =$$

$$= \frac{1}{I_0(2\kappa)} \left[\left(\sum_{m\in\mathbb{Z}} I_m(\kappa)\, I_{m+2n}(\kappa)\, e^{i2m\alpha} \right) e^{i2n\alpha} + \left(\sum_{m\in\mathbb{Z}} I_m(\kappa)\, I_{m+2n+1}(\kappa)\, e^{i2m\alpha} \right) e^{i(2n+1)\alpha} \right],$$

where, to get the last equality, we used the symmetry property (A.2). After applying the addition theorem (A.8) for the sums over the index m, we arrive at

$$k_2(n,\alpha) = \frac{1}{I_0(2\kappa)} [I_{2n}(2\kappa\cos\alpha) + I_{2n+1}(2\kappa\cos\alpha)] \tag{4.26}$$

B.4 Computation of the Wigner operator $\hat{\mathcal{W}}_2(0,0)$

The operator $\hat{\mathcal{W}}_2(0,0)$ is given by Eq. (3.27)

$$\hat{\mathcal{W}}_2(0,0) = \sum_{l\in\mathbb{Z}} \int_0^{2\pi} \frac{d\phi}{2\pi} \hat{D}_2(l,\phi) = \sum_{l\in\mathbb{Z}} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i\delta_2(l,\phi)} \hat{E}^{-l} e^{-i\hat{L}\phi} =$$

$$= \sum_{l\in\mathbb{Z}} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i\lfloor -\frac{l}{2} \rfloor \phi} \sum_{k\in\mathbb{Z}} e^{-ik\phi} |k+l\rangle\langle k|,$$

applying the rules of the floor function (4.24) we need to split the sum into an odd part and an even part, i.e.

$$\hat{\mathcal{W}}_2(0,0) = \sum_{j\in\mathbb{Z}} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-ij\phi} \sum_{k\in\mathbb{Z}} e^{-ik\phi} |k+2j\rangle\langle k| + \sum_{j\in\mathbb{Z}} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-ij\phi} \sum_{k\in\mathbb{Z}} e^{-ik\phi} |k+2j-1\rangle\langle k| =$$

$$= \sum_{j,k\in\mathbb{Z}} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-i(j+k)\phi} |k+2j\rangle\langle k| + \sum_{j,k\in\mathbb{Z}} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-i(j+k)\phi} |k+2j-1\rangle\langle k| =$$

$$= \sum_{k\in\mathbb{Z}} |k-2k\rangle\langle k| + \sum_{k\in\mathbb{Z}} |k-2k-1\rangle\langle k| = \sum_{k\in\mathbb{Z}} |-k\rangle\langle k| + \sum_{k\in\mathbb{Z}} |-k-1\rangle\langle k| =$$

$$= \hat{P} + \hat{E} \sum_{k\in\mathbb{Z}} |-k\rangle\langle k| = \hat{P} + \hat{E}\hat{P} = (\mathbb{1} + \hat{E})\hat{P}. \tag{4.28}$$

B.5 Computation of the kernel $\widetilde{k}_2(n,\alpha)$

First we inspect the overlap

$$\widetilde{o}_2(l,\phi) = e^{i\widetilde{\gamma}_2(l,\phi)} \frac{I_l\left[2\kappa\cos\left(\frac{\phi}{2}\right)\right]}{I_0(2\kappa)} = \frac{e^{i\left(\lfloor \frac{l}{2} \rfloor - \frac{l}{2}\right)\phi} e^{i\frac{l}{2}\phi}}{I_0(2\kappa)} \sum_{m\in\mathbb{Z}} I_m(\kappa)\, I_{m+l}(\kappa)\, e^{im\phi},$$

where to get the last equality the addition theorem (A.8) has been used. Next, we collect terms and get

$$o_2(l,\phi) = \frac{1}{I_0(2\kappa)} \sum_{m\in\mathbb{Z}} e^{i\left(\lfloor \frac{l}{2} \rfloor + m\right)\phi} I_m(\kappa)\, I_{m+l}(\kappa).$$

From the definition of the kernel (3.31) it follows

$$\widetilde{k}_2(n,\alpha) = \sum_{l,m\in\mathbb{Z}} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i(n\phi - l\alpha)} \frac{e^{i\left(\left\lfloor\frac{l}{2}\right\rfloor + m\right)\phi} I_m(\kappa)\, I_{m+l}(\kappa)}{I_0(2\kappa)}$$

$$= \sum_{l,m\in\mathbb{Z}} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-il\alpha} \frac{e^{i\left(\left\lfloor\frac{l}{2}\right\rfloor + m + n\right)\phi} I_m(\kappa)\, I_{m+l}(\kappa)}{I_0(2\kappa)} .$$

Because of the definition of the floor function (4.24) we need to split the sum into an odd part and an even part, i.e.

$$\widetilde{k}_2(n,\alpha) = \sum_{m,j\in\mathbb{Z}} \int_0^{2\pi} \frac{d\phi}{2\pi I_0(2\kappa)} I_m(\kappa)\, I_{m+2j}(\kappa)\, e^{i(n+m+j)\phi} e^{-i2j\alpha} +$$

$$+ \sum_{m,j\in\mathbb{Z}} \int_0^{2\pi} \frac{d\phi}{2\pi I_0(2\kappa)} I_m(\kappa)\, I_{m+2j+1}(\kappa)\, e^{i(n+m+j)\phi} e^{-i(2j+1)\alpha} ,$$

and using the Kronecker delta (A.3) we get

$$\widetilde{k}_2(n,\alpha) = \sum_{m\in\mathbb{Z}} \frac{I_m(\kappa)}{I_0(2\kappa)} \left[I_{m+2(-n-m)}(\kappa)\, e^{-i2(-n-m)\alpha} + I_{m+2(-n-m)+1}(\kappa)\, e^{-i[2(-n-m)+1]\alpha} \right] =$$

$$= \sum_{m\in\mathbb{Z}} \frac{I_m(\kappa)}{I_0(2\kappa)} \left[I_{-m-2n}(\kappa)\, e^{i2n\alpha} e^{i2m\alpha} + I_{-m-2n+1}(\kappa)\, e^{i2m\alpha} e^{i(2n-1)\alpha} \right] =$$

$$= \frac{1}{I_0(2\kappa)} \left[\left(\sum_{m\in\mathbb{Z}} I_m(\kappa)\, I_{m+2n}(\kappa)\, e^{i2m\alpha} \right) e^{i2n\alpha} + \left(\sum_{m\in\mathbb{Z}} I_m(\kappa)\, I_{m+2n-1}(\kappa)\, e^{i2m\alpha} \right) e^{i(2n-1)\alpha} \right] ,$$

where, to get the last equality, we used the symmetry property (A.2). After applying the addition theorem (A.8) for the sums over the index m, we arrive at

$$k_2(n,\alpha) = \frac{1}{I_0(2\kappa)} \left[I_{2n}(2\kappa\cos\alpha) + I_{2n-1}(2\kappa\cos\alpha) \right] . \tag{4.32}$$

B.6 Computation of the Wigner operator $\hat{\widetilde{\mathcal{W}}}_2(n,\alpha)$

The operator $\hat{\widetilde{\mathcal{W}}}_2(0,0)$ is given by Eq. (3.27)

$$\hat{\widetilde{\mathcal{W}}}_2(0,0) = \sum_{l\in\mathbb{Z}} \int_0^{2\pi} \frac{d\phi}{2\pi} \hat{\widetilde{D}}_2(l,\phi) = \sum_{l\in\mathbb{Z}} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i\tilde{\delta}_2(l,\phi)} \hat{E}^{-l} e^{-i\hat{L}\phi} =$$

$$= \sum_{l\in\mathbb{Z}} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-i\left\lfloor\frac{l}{2}\right\rfloor\phi} \sum_{k\in\mathbb{Z}} e^{-ik\phi} |k+l\rangle\langle k| ,$$

applying the rules of the floor function (4.24) we need to split the sum into an odd part and an even part, i.e.

$$\hat{\widetilde{\mathcal{W}}}_2(0,0) = \sum_{j\in\mathbb{Z}} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-ij\phi} \sum_{k\in\mathbb{Z}} e^{-ik\phi} |k+2j\rangle\langle k| + \sum_{j\in\mathbb{Z}} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-ij\phi} \sum_{k\in\mathbb{Z}} e^{-ik\phi} |k+2j+1\rangle\langle k| =$$

$$= \sum_{j,k\in\mathbb{Z}} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-i(j+k)\phi} |k+2j\rangle\langle k| + \sum_{j,k\in\mathbb{Z}} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-i(j+k)\phi} |k+2j+1\rangle\langle k| ,$$

and using the Kronecker delta (A.3) we get

$$\hat{\widetilde{\mathcal{W}}}_2(0,0) = \sum_{k\in\mathbb{Z}} |k-2k\rangle\langle k| + \sum_{k\in\mathbb{Z}} |k-2k+1\rangle\langle k| = \sum_{k\in\mathbb{Z}} |-k\rangle\langle k| + \sum_{k\in\mathbb{Z}} |-k+1\rangle\langle k| =$$
$$= \hat{P} + \hat{E}^\dagger \sum_{k\in\mathbb{Z}} |-k\rangle\langle k| = \hat{P} + \hat{E}^\dagger \hat{P} = (\mathbb{1} + \hat{E}^\dagger)\hat{P} \,. \tag{4.34}$$

B.7 Relation between $\widetilde{W}_2^{|n,\alpha\rangle}(m,\beta)$ and $W_2^{|n,\alpha\rangle}(m,\beta)$

We start by stating that both Wigner operators

$$\hat{\mathcal{W}}_2(0,0) = (\mathbb{1} + \hat{E})\hat{P} \,,$$
$$\hat{\widetilde{\mathcal{W}}}_2(0,0) = (\mathbb{1} + \hat{E}^\dagger)\hat{P} \,,$$

are Hermitian, since for both choices of $\delta_2(l,\phi)$ and $\widetilde{\delta}_2(l,\phi)$ the reality condition (3.51) is satisfied. That, in turn, tells us that the operators

$$\hat{\mathcal{W}}_2(n,\alpha) = \hat{D}_2(n,\alpha)\hat{\mathcal{W}}_2(0,0)\hat{D}_2^\dagger(n,\alpha) = \hat{E}^{-n}e^{-i\hat{L}\alpha}\hat{\mathcal{W}}_2(0,0)\left[\hat{E}^{-n}e^{-i\hat{L}\alpha}\right]^\dagger =$$
$$= \hat{D}(n,\alpha)\hat{\mathcal{W}}_2(0,0)\hat{D}^\dagger(n,\alpha) \,,$$

$$\hat{\widetilde{\mathcal{W}}}_2(n,\alpha) = \hat{\widetilde{D}}_2(n,\alpha)\hat{\widetilde{\mathcal{W}}}_2(0,0)\hat{\widetilde{D}}_2^\dagger(n,\alpha) = \hat{E}^{-n}e^{-i\hat{L}\alpha}\hat{\widetilde{\mathcal{W}}}_2(0,0)\left[\hat{E}^{-n}e^{-i\hat{L}\alpha}\right]^\dagger =$$
$$= \hat{D}(n,\alpha)\hat{\widetilde{\mathcal{W}}}_2(0,0)\hat{D}^\dagger(n,\alpha) \,,$$

are also Hermitian. Note that for the purposes of this section we defined the operator $\hat{D}(n,\alpha)$ as

$$\hat{D}(n,\alpha) = \hat{E}^{-n}e^{-i\hat{L}\alpha} \,. \tag{B.4}$$

This allows us to write the following

$$\hat{\widetilde{\mathcal{W}}}_2(n,\alpha) = \hat{\widetilde{D}}(n,\alpha)\hat{\widetilde{\mathcal{W}}}_2(0,0)\hat{\widetilde{D}}^\dagger(n,\alpha) = \hat{\widetilde{D}}(n,\alpha)\hat{\widetilde{\mathcal{W}}}_2^\dagger(0,0)\hat{\widetilde{D}}^\dagger(n,\alpha) =$$
$$= \hat{\widetilde{D}}(n,\alpha)\hat{P}(\mathbb{1} + \hat{E})\hat{\widetilde{D}}^\dagger(n,\alpha) = \hat{P}\hat{P}\hat{\widetilde{D}}(n,\alpha)\hat{P}(\mathbb{1} + \hat{E})\hat{P}\hat{P}\hat{\widetilde{D}}^\dagger(n,\alpha)\hat{P}\hat{P} =$$
$$= \hat{P}\left\{\left[\hat{P}\hat{\widetilde{D}}(n,\alpha)\hat{P}\right]\hat{\mathcal{W}}_2(0,0)\left[\hat{P}\hat{\widetilde{D}}(n,\alpha)\hat{P}\right]^\dagger\right\}\hat{P} \,,$$

where we have used the property of the parity operator $\hat{P}$ (3.61). The action of the parity operator on the displacement operator $\hat{\widetilde{D}}(n,\alpha)$ is

$$\hat{P}\hat{\widetilde{D}}(n,\alpha)\hat{P} = \hat{P}\hat{E}^{-n}\hat{P}\hat{P}e^{-i\hat{L}\alpha}\hat{P} = \hat{E}^{n}e^{i\hat{L}\alpha} = \hat{E}^{-(-n)}e^{-i\hat{L}(-\alpha)} = \hat{D}(-n,-\alpha) \,,$$

where we have used the actions of the parity operator on the operators $\hat{E}$ and $\hat{L}$ (3.62). Thus we get

$$\hat{\widetilde{\mathcal{W}}}_2(n,\alpha) = \hat{P}\hat{D}(-n,-\alpha)\hat{\mathcal{W}}_2(0,0)\hat{D}^\dagger(-n,-\alpha)\hat{P} = \hat{P}\hat{\mathcal{W}}_2(-n,-\alpha)\hat{P} \,. \tag{B.5}$$

Looking at the definition of the Wigner function

$$W_j^{\hat{\rho}}(n,\alpha) = \frac{1}{2\pi}\,\mathrm{Tr}\left[\hat{\rho}\hat{\mathcal{W}}_j(n,\alpha)\right] \,, \tag{3.25}$$

we see that

$$\widetilde{W}_2^{\hat{\rho}}(m,\beta) = \frac{1}{2\pi}\,\mathrm{Tr}\left[\hat{\rho}\hat{\widetilde{\mathcal{W}}}_2(m,\beta)\right] = \frac{1}{2\pi}\,\mathrm{Tr}\left[\hat{\rho}\hat{P}\hat{\mathcal{W}}_2(-m,-\beta)\hat{P}\right] = \frac{1}{2\pi}\,\mathrm{Tr}\left[\hat{P}\hat{\rho}\hat{P}\hat{\mathcal{W}}_2(-m,-\beta)\right] =$$
$$= W_2^{\hat{P}\hat{\rho}\hat{P}}(-m,-\beta)\,. \tag{B.6}$$

Inspecting what is the action of the parity operator on the von Mises state (1.28), we get

$$\hat{P}\,|n,\alpha\rangle = \frac{1}{\sqrt{I_0(2\kappa)}}\sum_{l\in\mathbb{Z}} e^{i(n-l)\alpha} I_{n-l}(\kappa)\,\hat{P}\,|l\rangle = \frac{1}{\sqrt{I_0(2\kappa)}}\sum_{l\in\mathbb{Z}} e^{i(n-l)\alpha} I_{n-l}(\kappa)\,|-l\rangle =$$
$$= \frac{1}{\sqrt{I_0(2\kappa)}}\sum_{k\in\mathbb{Z}} e^{i(n+k)\alpha} I_{n+k}(\kappa)\,|k\rangle = \frac{1}{\sqrt{I_0(2\kappa)}}\sum_{k\in\mathbb{Z}} e^{i(-n-k)(-\alpha)} I_{-n-k}(\kappa)\,|k\rangle\ ,$$

where in the last equality we used the symmetry property (A.2), thus

$$\hat{P}\,|n,\alpha\rangle = |-n,-\alpha\rangle\ , \tag{B.7}$$

and finally, using Eq. (B.6) with Eq. (B.7), we arrive at

$$\widetilde{W}_2^{|n,\alpha\rangle}(m,\beta) = W_2^{|-n,-\alpha\rangle}(-m,-\beta)\,. \tag{4.36}$$

B.8 Computation of the kernel $k_3(n,\alpha)$

To find the kernel $k_3(n,\alpha)$ we need to preform the Fourier transform of the overlap $o_3(l,\phi)$

$$k_3(n,\alpha) = (\mathcal{F}o_1(l,\phi))(n,\alpha) = \sum_{l\in\mathbb{Z}}\int_0^{2\pi}\frac{d\phi}{2\pi}e^{i(n\phi-\alpha l)}o_1(l,\phi) =$$
$$= \sum_{l\in\mathbb{Z}}\int_0^{2\pi}\frac{d\phi}{2\pi}e^{i(n\phi-\alpha l)}e^{-il\frac{\phi}{2}}\frac{I_l\left[2\kappa\cos\left(\frac{\phi}{2}\right)\right]}{I_0(2\kappa)} = \sum_{l\in\mathbb{Z}}\int_0^{2\pi}\frac{d\phi}{2\pi I_0(2\kappa)}e^{i(n\phi-\alpha l)}\sum_{m\in\mathbb{Z}}I_m(\kappa)\,I_{m+l}(\kappa)\,e^{im\phi}\,,$$

where to get the last equality the addition theorem (A.8) has been used. Further, using the Kronecker delta (A.3) we get

$$k_3(n,\alpha) = \frac{1}{I_0(2\kappa)}\sum_{l,m\in\mathbb{Z}} I_m(\kappa)\,I_{m+l}(\kappa)\,e^{-il\alpha}\delta_{m,-n} = \frac{I_n(\kappa)}{I_0(2\kappa)}\sum_{l\in\mathbb{Z}} I_{l-n}(\kappa)\,e^{-il\alpha}\,,$$

where the symmetry in orders of the modified Bessel function (A.2) has been used,

$$k_3(n,\alpha) = \frac{e^{-in\alpha}}{I_0(2\kappa)}I_n(\kappa)\sum_{m\in\mathbb{Z}} I_m(\kappa)\,e^{-im\alpha} = \frac{I_n(\kappa)}{I_0(2\kappa)}e^{\kappa\cos\alpha-in\alpha}\,, \tag{4.37}$$

where the generating function (A.10) has been used.

B.9 Computation of the Wigner operator $\hat{\mathcal{W}}_3(n,\alpha)$

From the definition of the Wigner operator (3.26) we see that

$$\hat{\mathcal{W}}_3(n,\alpha) = (\mathcal{F}\hat{D}_3)(n,\alpha) = \sum_{l\in\mathbb{Z}} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i(n\phi-\alpha l)} \hat{D}_3(l,\phi) =$$
$$= \sum_{l\in\mathbb{Z}} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i(n\phi-\alpha l)} \hat{E}^{-l} e^{-i\hat{L}\phi} = \sum_{l\in\mathbb{Z}} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i(n\phi-\alpha l)} \sum_{k\in\mathbb{Z}} e^{-ik\phi} |k+l\rangle\langle k|$$

using the Kronecker delta (A.3) we get

$$\hat{\mathcal{W}}_3(n,\alpha) = \sum_{l,k\in\mathbb{Z}} e^{-il\alpha} |k+l\rangle\langle k| \, \delta_{n,k} = \sum_{l\in\mathbb{Z}} e^{-il\alpha} |n+l\rangle\langle n| =$$
$$= e^{in\alpha} \sum_{m\in\mathbb{Z}} e^{-im\alpha} |m\rangle\langle n| = e^{in\alpha}\sqrt{2\pi}\, |\alpha\rangle\langle n| \ ,$$

where to get the last equality we used the expression for the $|\phi\rangle$ state in the $\{|n\rangle\}_{n\in\mathbb{Z}}$ basis

$$|\phi\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} e^{-in\phi} |n\rangle \ . \tag{1.12}$$

Finally, noticing the overlap

$$\langle\phi|k\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} e^{in\phi} \langle n|k\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} e^{in\phi} \delta_{n,k} = \frac{1}{\sqrt{2\pi}} e^{ik\phi} \, ,$$

we arrive at a very simple Wigner operator

$$\hat{\mathcal{W}}_3(n,\alpha) = 2\pi \, |\alpha\rangle \, \langle\alpha|n\rangle \, \langle n| \ . \tag{4.39}$$

Appendix C

Analysis of the Wigner functions

C.1 Behavior as functions of m and β

Here we will analyze the Wigner functions $W_2^{|0,0\rangle}(m,\beta)$ and $\widetilde{W}_2^{|0,0\rangle}(m,\beta)$. We will start with $W_2^{|0,0\rangle}(m,\beta)$, since the transition to $\widetilde{W}_2^{|0,0\rangle}(m,\beta)$ is easy due to the fact that

$$\widetilde{W}_2^{|0,0\rangle}(m,\beta) = W_2^{|0,0\rangle}(-m,\beta)\,, \quad \forall m \in \mathbb{Z} \quad \text{and} \quad \beta \in \mathbb{R}\,. \tag{4.44}$$

Let us write down $W_2^{|0,0\rangle}(m,\beta)$ explicitly

$$W_2^{|0,0\rangle}(m,\beta) = \frac{1}{2\pi I_0\left(2\kappa\right)}\left[I_{2m}\left(2\kappa\cos\beta\right) + I_{2m-1}\left(2\kappa\cos\beta\right)\right], \tag{4.43}$$

from which it is obvious that $W_2^{|0,0\rangle}(m,\beta)$ is 2π-periodic and even as a function of β. Because the function is even we can further restrict the analysis to the interval $[0,\pi]$. Note that for the case of the spread parameter $\kappa = 0$, using $I_n\left(0\right) = \delta_{n,0}$, we get

$$W_2^{|0,0\rangle}(m,\beta) = \frac{1}{2\pi}\delta_{m,0}\,, \tag{4.45}$$

which is the Wigner function of the angular momentum eigenstate, Eq. (4.1) with $n = 0$. Bellow we will analyze the case when $\kappa > 0$.

Computing the first derivative and setting it equal to zero we get

$$\frac{\mathrm{d}W_2^{|0,0\rangle}(m,\beta)}{\mathrm{d}\beta} = -\frac{\sin\beta}{2\pi I_0\left(2\kappa\right)}\left[\frac{\mathrm{d}I_{2m}\left(2\kappa\cos\beta\right)}{\mathrm{d}\cos\beta} + \frac{\mathrm{d}I_{2m-1}\left(2\kappa\cos\beta\right)}{\mathrm{d}\cos\beta}\right] = 0\,, \tag{C.1}$$

which is satisfied if $\beta = 0$, $\beta = \pi$ or

$$\frac{\mathrm{d}I_{2m}\left(2\kappa\cos\beta\right)}{\mathrm{d}\cos\beta} = \frac{\mathrm{d}I_{2m-1}\left(-2\kappa\cos\beta\right)}{\mathrm{d}\cos\beta}\,, \tag{C.2}$$

where we used the fact that $I_n\left(-z\right) = (-1)^n I_n\left(z\right)$ (see Appemdix A for the properties of the modified Bessel function). This equation boils down to

$$e^{2\kappa\cos\beta\cos\phi + i2m\phi} = e^{-2\kappa\cos\beta\cos\phi + i(2m+1)\phi}\,, \tag{C.3}$$

which is satisfied if $\beta = \frac{\pi}{2}$. So the points of potential exetrema are at $\beta = 0$, $\beta = \frac{\pi}{2}$ and $\beta = \pi$.

Next we will investigate these points. Computing the second derivative and setting $\beta = 0$ we get

$$\left.\frac{\mathrm{d}^2 W_2^{|0,0\rangle}(m,\beta)}{\mathrm{d}\beta^2}\right|_{\beta=0} = -\frac{2\kappa\cos\beta}{2\pi I_0(2\kappa)}\left[\frac{\mathrm{d}I_{2m}(2\kappa\cos\beta)}{\mathrm{d}2\kappa\cos\beta} + \frac{\mathrm{d}I_{2m-1}(2\kappa\cos\beta)}{\mathrm{d}2\kappa\cos\beta}\right]\Bigg|_{\beta=0} - \\ -\frac{\sin\beta}{2\pi I_0(2\kappa)}\frac{\mathrm{d}}{\mathrm{d}\beta}\left[\frac{\mathrm{d}I_{2m}(2\kappa\cos\beta)}{\mathrm{d}\cos\beta} + \frac{\mathrm{d}I_{2m-1}(2\kappa\cos\beta)}{\mathrm{d}\cos\beta}\right]\Bigg|_{\beta=0},$$

noticing that the second term is equal to zero and using the derivative of the modified Bessel function Eq. (A.5), we arrive at

$$\left.\frac{\mathrm{d}^2 W_2^{|0,0\rangle}(m,\beta)}{\mathrm{d}\beta^2}\right|_{\beta=0} = -\frac{\kappa\cos\beta}{2\pi I_0(2\kappa)}\left[I_{2m-1}(2\kappa\cos\beta) + I_{2m+1}(2\kappa\cos\beta)\right]\Big|_{\beta=0} - \\ -\frac{\kappa\cos\beta}{2\pi I_0(2\kappa)}\left[I_{2m-2}(2\kappa\cos\beta) + I_{2m}(2\kappa\cos\beta)\right]\Big|_{\beta=0} = \\ = -\frac{\kappa}{2\pi I_0(2\kappa)}\left[I_{2m-1}(2\kappa) + I_{2m+1}(2\kappa) + I_{2m-2}(2\kappa) + I_{2m}(2\kappa)\right],$$

since Eq. (A.13) holds, the above expression is negative irrespective of the order and therefore $W_2^{|0,0\rangle}(m,\beta)$ has a local maximum at $\beta = 0$ for every $m \in \mathbb{Z}$.
For $\beta = \frac{\pi}{2}$ we get

$$W_2^{|0,0\rangle}\left(m,\frac{\pi}{2}\right) = \frac{1}{2\pi I_0(2\kappa)}\delta_{m,0}, \tag{4.46}$$

and except for $W_2^{|0,0\rangle}(0,\beta)$, all the Wigner functions $W_2^{|0,0\rangle}(m,\beta)$ are equal to zero at $\beta = \frac{\pi}{2}$.
For $\beta = \pi$ the Wigner function takes the following form

$$W_2^{|0,0\rangle}(m,\pi) = \frac{1}{2\pi I_0(2\kappa)}\left[I_{2m}(2\kappa) - I_{2m-1}(2\kappa)\right], \tag{C.4}$$

where we used (A.2). Further

$$\frac{1}{2\pi I_0(2\kappa)}\left[I_{2m}(2\kappa) - I_{2m-1}(2\kappa)\right]\begin{cases} < 0, & \text{if} \quad m \in \mathbb{Z}^+, \\ > 0, & \text{if} \quad m \in \mathbb{Z}_0^-, \end{cases} \tag{4.47}$$

where we used (A.12). Here we can see that for positive integers m the corresponding Wigner functions $W_2^{|0,0\rangle}(m,\pi)$ take on negative values, and for non-positive integers m the corresponding Wigner functions $W_2^{|0,0\rangle}(m,\beta)$ are non-negative on the whole interval $(-\pi,\pi]$.
Lastly notice that

$$W_2^{|0,0\rangle}(m,0) > \left|W_2^{|0,0\rangle}(m,\pi)\right|, \tag{C.5}$$

is the same as saying

$$I_{2m+1}(2\kappa) > 0 \quad \forall m \in \mathbb{Z}^-, \quad I_{2m}(2\kappa) > 0 \quad \forall m \in \mathbb{Z}_0^+, \tag{C.6}$$

which is obviously true because of Eq. (A.13). So $W_2^{|0,0\rangle}(m,\beta)$ has a global maximum at $\beta = 0$ for any $m \in \mathbb{Z}$.

Because of Eq. (C.5) it suffices to investigate the behavior of the peaks $W_2^{|0,0\rangle}(m,0)$ to get an intuition on the behavior of the Wigner functions $W_2^{|0,0\rangle}(m,\beta)$ as functions of m. Using Eq. (A.12) we find the following inequalities:

$$W_2^{|0,0\rangle}(0,0) > W_2^{|0,0\rangle}(m,0) > 0\,, \quad \forall m \in \mathbb{Z} - \{0\}\,, \tag{C.7}$$

and $W_2^{|0,0\rangle}(m,0)$ has a global maximum at $m=0$.

$$W_2^{|0,0\rangle}(m,0) > W_2^{|0,0\rangle}(m+1,0) > 0\,, \quad \forall m \in \mathbb{Z}^+\,, \tag{C.8}$$

and the Wigner functions are descending for $m \in \mathbb{Z}^+$.

$$W_2^{|0,0\rangle}(m+1,0) > W_2^{|0,0\rangle}(m,0) > 0\,, \quad \forall m \in \mathbb{Z}^-\,, \tag{C.9}$$

and the Wigner functions are ascending for $m \in \mathbb{Z}^-$.

Due to Eq. (4.44) the only difference in the analysis of $\widetilde{W}_2^{|0,0\rangle}(m,\beta)$ is in the following inequalities

$$\widetilde{W}_2^{|0,0\rangle}(m,\pi) = \frac{1}{2\pi I_0(2\kappa)}\left[I_{2m}(2\kappa) - I_{2m+1}(2\kappa)\right] \begin{cases} >0\,, & \text{if} \quad m \in \mathbb{Z}_0^+\,, \\ <0\,, & \text{if} \quad m \in \mathbb{Z}^-\,, \end{cases} \tag{4.48}$$

i.e. the Wigner functions $\widetilde{W}_2^{|0,0\rangle}(m,\beta)$ of negative integers m take on negative values at $\beta=\pi$ and the Wigner functions $\widetilde{W}_2^{|0,0\rangle}(m,\beta)$ of non-negative integers m are non-negative on the whole interval $\beta \in (-\pi,\pi]$.

C.2 Descending behavior

Here we estimate the speed with which $W_2^{|0,0\rangle}(m,\beta)$ decreases as a function of m. Due to (C.5) it suffices to analyze the decreasing of $W_2^{|0,0\rangle}(m,0)$. From (A.12) we can estimate the Wigner functions $W_2^{|0,0\rangle}(m,0)$ from bellow and above in the following way

$$0 < 2\mathcal{N} I_{2m-1}(2\kappa) < W_2^{|0,0\rangle}(m,0) = \mathcal{N}[I_{2m}(2\kappa) + I_{2m-1}(2\kappa)] < 2\mathcal{N} I_{2m}(2\kappa)\ , \tag{C.10}$$

for all $m \in \mathbb{Z}_0^-$, and

$$0 < 2\mathcal{N} I_{2m}(2\kappa) < W_2^{|0,0\rangle}(m,0) = \mathcal{N}[I_{2m}(2\kappa) + I_{2m-1}(2\kappa)] < 2\mathcal{N} I_{2m-1}(2\kappa)\ , \tag{C.11}$$

for all $m \in \mathbb{Z}^+$, where $\mathcal{N} = 1/[2\pi I_0(2\kappa)]$. Then

$$\frac{W_2^{|0,0\rangle}(m,0)}{W_2^{|0,0\rangle}(m+1,0)} < \frac{I_{2m}(2\kappa)}{I_{2m+1}(2\kappa)} < 1\,, \quad \forall m \in \mathbb{Z}^-\,, \tag{C.12}$$

and

$$\frac{W_2^{|0,0\rangle}(m+1,0)}{W_2^{|0,0\rangle}(m,0)} < \frac{I_{2m+1}(2\kappa)}{I_{2m}(2\kappa)} < 1\,, \quad \forall m \in \mathbb{Z}_0^+\,. \tag{C.13}$$

Due to the ascending (descending) behavior of the modified Bessel functions for negative (non-negative) integer orders (A.12) we can estimate the quotients by the first terms in the set as

$$\frac{W_2^{|0,0\rangle}(m,0)}{W_2^{|0,0\rangle}(m+1,0)} < \frac{I_2(2\kappa)}{I_1(2\kappa)} < 1\,, \quad \forall m \in \mathbb{Z}^-\,, \tag{C.14}$$

and

$$\frac{W_2^{|0,0\rangle}(m+1,0)}{W_2^{|0,0\rangle}(m,0)} < \frac{I_1(2\kappa)}{I_0(2\kappa)} < 1\,, \quad \forall m \in \mathbb{Z}_0^+\,. \tag{C.15}$$

From here it is easy to show that

$$0 < \frac{W_2^{|0,0\rangle}(-m,0)}{W_2^{|0,0\rangle}(0,0)} \leq \left[\frac{I_2(2\kappa)}{I_1(2\kappa)}\right]^m, \; 0 < \frac{W_2^{|0,0\rangle}(m,0)}{W_2^{|0,0\rangle}(0,0)} \leq \left[\frac{I_1(2\kappa)}{I_0(2\kappa)}\right]^m, \quad \forall m \in \mathbb{Z}_0^+\,, \tag{4.49}$$

where equality occurs if $m = 0$. So for a given $\kappa > 0$ the Wigner functions $W_2^{|0,0\rangle}(m,\beta)$, as functions of m, decrease at least as fast as a geometric series with the ratio $I_2(2\kappa)/I_1(2\kappa)$ for the Wigner function of negative integers m and $I_1(2\kappa)/I_0(2\kappa)$ for the Wigner function of positive integers m, as can be seen in Figure 4.2 and Figure 4.3. Further, Eq (4.44) tells us that

$$0 < \frac{\widetilde{W}_2^{|0,0\rangle}(m,0)}{\widetilde{W}_2^{|0,0\rangle}(0,0)} \leq \left[\frac{I_2(2\kappa)}{I_1(2\kappa)}\right]^m, \; 0 < \frac{\widetilde{W}_2^{|0,0\rangle}(-m,0)}{\widetilde{W}_2^{|0,0\rangle}(0,0)} \leq \left[\frac{I_1(2\kappa)}{I_0(2\kappa)}\right]^m, \quad \forall m \in \mathbb{Z}_0^+\,, \tag{4.50}$$

where equality occurs if $m = 0$ and in the case of $\widetilde{W}_2^{|0,0\rangle}(m,\beta)$ the ratios have been switched.

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