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Rewriting Systems and If-Then Rules in Fuzzy Setting

Dissertation Thesis

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Shared responsibility statement

I hereby declare that the thesis is my original work. Some parts of this thesis are outcomes of joint scientific work with Radim Bělohlávek and Vilém Vychodil (Sections 3.1 and 3.4); and Vilém Vychodil (Sections 3.2 and 3.3, Chapter 4). All authors have even share in the results and findings contained in the respective parts.

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Abstract

This thesis consists two main parts which are dealing with quite different fuzzy relational systems. In the first part, the foundations for fuzzy rewriting systems are presented. The fuzzy counterparts of the notions of reducibility, convergence, divergence, convertibility, confluence, Church-Rosser property, termination, well-foundedness, inductive property, and normal form are introduced for a given fuzzy relation which serves as a reduction relation in this approach. Moreover, the ordinary notions are left as a particular case when the underlying structure of truth degrees is the two-valued Boolean algebra. Some essential properties of these generalized notions are also investigated in the first part of this thesis.

There are also considered two ways of possible deeper generalization of the notions mentioned above which can be seen as foundations for rewriting on similarity and metric spaces. A given fuzzy equivalence or a generalized pseudometric serves as an additional knowledge representing the indistinguishability of elements which can be specified by an expert in a particular domain. Basic similarity issues as well as research on derived fuzzy reductions are also described in the first part.

The second part of this thesis describes a link between two types of logic systems for reasoning with graded if-then rules: the system of fuzzy logic programming (FLP) in sense of Vojtáš and the system of fuzzy attribute logic (FAL) in sense of Bělohlávek and Vychodil. The main result in this part is that each finite theory consisting of formulas of FAL can be represented by a definite program so that the semantic entailment in FAL can be characterized by correct answers for the program and a query. Conversely, for each definite program there is a collection of formulas of FAL so that the correct answers can be represented by the entailment in FAL. Using this link, someone can transport results from FAL to FLP and *vice versa*. Research focused on reducing entailment in FLP to reasoning with Boolean attribute implications as well as investigation of properties of least models are also included.

Contents

Chapter 1

Introduction

In our everyday lives, we are being encountered with various kinds of information. Some pieces of information are precisely formulated and one usually can immediately say if a given statement or observation is true or not. For example, the sentence "Washington is the capital of the USA." is true without any doubts. On the other hand, there are plenty of equally simple propositions which can not be easily tagged as true or false. Sentences like "It will be sunny tomorrow." or "John is a tall person." show us fundamental limitations of the classical bivalent logic [43]. Usually, we can deal with them using some sort of intuition but there are situations (e.g. in scientific exploration) when precise dealing with somehow uncertain statements or data is needed. For this reason, a large scale of various mathematical theories for dealing with different types of uncertainty was developed recently and is still being intensively investigated. The most important and well-known examples are the probability theory [32], the rough sets theory [49], the theory of quantum mechanics [22, 45], and plenty of multi-valued logics [3, 27, 29].

One of the most famous theory for dealing with vague statements is the fuzzy logic [29] which is based on the fuzzy set theory [55, 56]. Let us return now to the example sentence about John. What is the problem which the classical logic is not able to solve satisfactorily? Even if we know John well or have measured exactly his height, we sometimes can't determine whether the sentence should be true or not. Clearly, if John's height is for example 190 cm he will be considered to be a tall person, and the sentence should be true of course. Contrarily, a person who is 150 cm tall is usually not considered to be tall at all, i.e. the corresponding statement is simply false. However, what decision should we make if John's height is not so extremal, e.g. 165 cm. While using the classical bivalent logic, we always need to have a strict border between elements (e.g. persons) which have some property (e.g. being tall) and the elements which don't have it. In our example, we can determine that anyone who has 170 cm or more is a tall person and anyone smaller than 170 cm is not tall. This approach works theoretically well, but from the practical point of view, it is quite unpleasant to possibly handle with two people who have almost the same height (e.g. 170 cm and 169.9 cm) and one of them is considered fully to be tall whereas the other is not tall at all.

Fuzzy logic (in the broad sense) [5, 13, 29] solves this discontinuity in a quite simple way. Statements in fuzzy logic are allowed to be not only fully true (usually denoted by truth value 1) or fully false (denoted by 0), but they can be evaluated also to intermediate truth degrees from a given structure of truth values. These structures have to be partially ordered with 0 being the least and 1 the greatest element. Sometimes, additional properties of the structures of truth degrees are required. In various theories and applications based on the fuzzy logic, the most common structure of truth values is the real unit interval. Return to our example last time. Assuming that John's height is exactly 165 cm, we can assigned the degree 0.3 to the sentence "John is a tall person." whose meaning will be that it is almost not true that John is tall. Due to its ability to deal sensible with vague propositions and data, fuzzy logic has plenty of applications in industry automation, control and analytical systems.

This thesis summarizes our results in the field of three different systems based on the fuzzy logic. In particular, Chapter 3 of this thesis introduces the crucial notions for theoretical foundations of fuzzy rewriting systems. Our motivation is the fact that the phenomenon of substitutability may not be bivalent. It is a common practice of everyday life to substitute y for x whenever x is too complex or expensive to handle and y does the job of x sufficiently well. For example, instead of working with a whole article, one may work with its summary only which is sufficiently informative; or instead of using an expensive option poll based on survey of a sample of $10,000$ persons, one may use a cheaper option poll based on a few hundreds of persons which will give sufficiently similar result. The common characteristic of these examples is that one works with a substitutability relation which is a fuzzy relation rather than a bivalent one. In Chapter 3, we have focused on the two most important properties of a reduction relation – confluence and termination, developed their fuzzy counterparts and studied their properties. We have also introduced some notions related to confluence which will respect a given fuzzy reduction as well as a given indistinguishability fuzzy relation (formalized by fuzzy equivalence).
Confluence and termination are properties of binary relations related to the idea of performing substitutions specified by the respective binary relation. The notions have been introduced in the theory of abstract rewriting systems [2, 47, 54] which deal with the idea of substituting elements by other elements which are indistinguishable from the original ones from a certain point of view. Particularly, in a term rewriting system, complex terms are substituted by simpler ones that have the same meaning according to a predefined semantics. As an example, $x+x+x$ can be replaced by an equivalent but simpler term $3\times x$ considering the usual interpretation of $+$ and $\times$ as arithmetic operations. The fact that elements can be equally substituted by other elements is formalized by a binary relation $\rightsquigarrow$ on a universe set X of all elements, $x \rightsquigarrow y$ is interpreted so that one may substitute y for x. An element $x \in X$ is called reducible if $x \rightsquigarrow y$ for some $y \in Y$; otherwise, x is called irreducible. By a reduction we mean any sequence $x_1, x_2, \ldots, x_n$ of elements from X such that $x_1 \rightsquigarrow x_2 \rightsquigarrow \cdots \rightsquigarrow x_n$. In that case, we say that x_1 is reducible to x_n. A reduction is called terminating if x_n is irreducible. Relation $\rightsquigarrow$ is called terminating if it has only terminating reductions. Furthermore, relation $\rightsquigarrow$ is called confluent whenever x is reducible to both y and y' then there is some z such that both y and y' are reducible to z. Therefore, if $\rightsquigarrow$ is confluent we always end up with the same result if we apply substitutions in different order until no more substitutions can be applied. So, there is a synergy between termination and confluence—a relation which is both terminating and confluent has normal forms, i.e. each element is reducible to a unique irreducible element.
There were also many applications of abstract rewriting systems evolved. For instance, term rewriting systems can be used as a theoretical background for functional programming, various logical deductive systems can be formalized by rewriting systems, simplification of mathematical expressions can be seen as a reduction, rewriting plays an important role in the theory of formal grammars, etc. A good overview of rewriting systems and their applications can be found in [2, 47].

Chapter 4 of the thesis summarizes the results of our research focused on establishing a link between two logic systems that have been proposed and studied independently in the past. Namely, we have investigated the relationship between fuzzy logic programming (shortly FLP) in sense of [53] and fuzzy attribute logic (FAL) presented in [12]. Note that there are also more general approaches to FLP, e.g. [21]. We have chosen [53] for its simple and straightforward form. Although the systems are technically different and were developed to serve different purposes, they share common features: (i) they are based on residuated structures of truth degrees, (ii) use truth-functional interpretation of logical connectives, (iii) both the systems can be used to describe if-then dependencies in problem domains when one requires a formal treatment of inexact matches, (iv) models of theories form particular closure systems and semantic entailment can be expressed by means of least models. Both the systems play an important role in computer science and artificial intelligence as they can be used for approximate knowledge representation and inference, description of dependencies found in data, representing approximate constraints in relational similarity-based databases, etc. It is therefore appealing to look closer at their mutual relationship. Furthermore, a possible link between the two systems can bring forth new results.

Fuzzy attribute logic [12] was developed primarily for the purpose of describing if-then dependencies that hold in object-attribute relational data where objects are allowed to have attributes to degrees. The formulas of FAL, so-called fuzzy attribute implications (FAIs) can be seen as implications $A \Rightarrow B$ between two graded sets of attributes, saying that if an object has all the attributes from A, then it has all the attributes from B. The fact that A and B are fuzzy sets allows us to express graded dependencies between attributes. As an example

$$\left\{{}^{0.7}/\mathit{lowAge}, {}^{0.9}/\mathit{lowMileage}\right\} \Rightarrow \left\{{}^{0.6}/\mathit{highFuelEconomy}, {}^{0.9}/\mathit{highPrice}\right\} \tag{1.1}$$

is an attribute implication saying that cars with low age (at least to degree 0.7) and low mileage (at least to 0.9) have also high fuel economy (at least to 0.6) and high price (at least to 0.9). Formulas of this form can be prescribed by an expert or inferred from object-attribute relational data [28]. FAIs have also an alternative interpretation as similarity-based functional dependencies [11] in relational databases [19, 41].
The main results on FAL include syntactico-semantically complete axiomatization with ordinary-style and graded-style (Pavelka style, see [48]) notions of provability and results on descriptions of nonredundant bases of FAIs describing dependencies present in object-attribute data and ranked data tables over domains with similarities [11, 12, 16].

Fuzzy logic programming [21, 42, 53] is a generalization of the ordinary logic programming [40] in which logic programs consist of facts and complex rules containing a head (an atomic predicate formula) and a tail (a formula composed from atomic predicate formulas using connectives and aggregations interpreted by monotone truth functions) connected by a residuated implication. In addition, each formula in a program is assumed to be valid to a degree, i.e., programs are theories in sense of Pavelka's abstract fuzzy logic [29, 48]. As a consequence, fuzzy logic programs are capable of expressing graded dependencies between facts.
As an example, we can consider the following rule:

$$\mathit{suitable}(\mathbb{X}) \overset{0.8}{\Leftarrow} \mathbb{wa}\big(\mathit{near}(\mathbb{X}, \mathsf{stadium}) \mathbin{\&} \mathit{near}(\mathbb{X}, \mathsf{center}), \mathit{quality}(\mathbb{X}), \mathit{cost}(\mathbb{X})\big), \tag{1.2}$$

which expresses how much a hotel (variable $\mathbb{X}$) is suitable for a sport fan. This rule describes the degree of hotel suitability (atomic formula $\mathit{suitable}(\mathbb{X})$) as weighted average

(aggregator $\mathtt{wa}$) of degrees of being conveniently located, having high quality ($quality(\mathbb{X})$) and having low prices ($cost(\mathbb{X})$). The convenience of hotel location is specified here as a conjunction ($\wedge$) of being near to the stadium ($near(\mathbb{X}, \mathsf{stadium})$) and being near to the city center ($near(\mathbb{X}, \mathsf{center})$). The rule is valid to degree 0.8, that can be understood so that we put almost full emphasis on the rule.
The basic result of FLP is the completeness which puts in correspondence the declarative and procedural semantics of logic programs [53, Theorem 1 and Theorem 3] represented by correct answers and computed answers.

Chapter 2

Preliminaries

In this chapter, I will briefly recall basic notions from the fuzzy set theory, similarity spaces, generalized pseudometric spaces, fuzzy attribute logic and fuzzy logic programming. I will also mention some properties of these notions which are necessary for understanding the results in Chapters 3 and 4.

2.1 Structures of Truth Degrees

In the following chapters, I will use residuated structures based on complete lattices. Recall that a *complete lattice* is an algebra $\mathbf{L} = \langle L, \wedge, \vee, 0, 1\rangle$ with L representing a set of degrees (bounded by 0 and 1). The corresponding lattice order $\leq$ is induced so that $a \leq b$ iff $a = a \wedge b$ (or equivalently, $a \vee b = b$). As usual, 0 and 1 are interpreted as degrees representing the full falsity and full truth. In order to express truth functions of general logical connectives, I assume that $\mathbf{L}$ is equipped by a collection of pairs in the form $\langle \otimes, \rightarrow\rangle$ such that $\langle L, \otimes, 1\rangle$ is a commutative monoid, and $\otimes$ and $\rightarrow$ satisfy the adjointness property:

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \rightarrow c \tag{2.1}$$

for any $a, b, c \in L$. As usual, $\otimes$ (called a multiplication) and $\rightarrow$ (called a residuum) serve as truth functions of binary logical connectives "fuzzy conjunction" and "fuzzy implication". The mutual relationship of $\otimes$ and $\rightarrow$ posed by (2.1) has been derived from a graded counterpart to the classic deduction rule modus ponens. This seminal observation due to J. Goguen [26] was later elaborated by Pavelka [48] in his general logic with graded semantic and syntactic entailments, see also an important monograph [25] devoted to this particular branch of multiple-valued logics. If $\otimes$ and $\rightarrow$ satisfy (2.1), then $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ is called a *complete residuated lattice*. A thorough information about the role of residuated lattices in fuzzy logic can be obtained from monographs [27, 29].

Note that the two-element Boolean algebra is a particular case of a complete residuated lattice where $L = \{0, 1\}$, $\wedge$ and $\vee$ being the minimum and maximum, respectively, $\otimes = \wedge$, and $\rightarrow$ being truth function of the two-element implication. In the sequel, the two-element Boolean algebra will be denoted by $\mathbf{2}$. The class of complete residuated lattices includes also structures of truth degrees on the real unit interval with $\otimes$ and $\rightarrow$ being a left-continuous t-norm and its corresponding residuum, respectively. Three most important pairs of adjoint operations on the unit interval are: Łukasiewicz: $a \otimes b = \max(0, a + b - 1)$, $a \rightarrow b = \min(1, 1 - a + b)$; Gödel: $a \otimes b = a \wedge b$, $a \rightarrow b = b$ for $a > b$ and $a \rightarrow b = 1$ for $a \leq b$; Goguen: $a \otimes b = a \cdot b$, $a \rightarrow b = \frac{b}{a}$ for $a > b$ and $a \rightarrow b = 1$ for $a \leq b$.

In this thesis, I will also use some properties of a complete residuated lattice $\mathbf{L}$ as well as properties of a particular truth degree $a \in L$. Recall that $a \in L$ is called *idempotent* if $a \otimes a = a$. Furthermore, $0 \neq a \in L$ is called a *zero-divisor* of $\otimes$ if there is $0 \neq b \in L$ such that $a \otimes b = 0$. A t-norm $\otimes$ is called *Archimedean* if 0 and 1 are its only idempotents. A complete residuated lattice $\mathbf{L}$ is called a *chain* if it is linearly ordered. If for any $a_i \in L$ $(i \in I)$ there is a finite subset $I' \subseteq I$ such that $\bigvee_{i \in I} a_i = \bigvee_{i \in I'} a_i$ then $\mathbf{L}$ is called a *Noetherian residuated lattice* [18]. In the sequel, I am going to use the following laws which directly follow as properties of complete residuated lattices:

$$a \otimes (a \to b) \leq b, \tag{2.2}$$
$$(a \otimes b) \to c = a \to (b \to c), \tag{2.3}$$
$$a \to (b \to c) = b \to (a \to c), \tag{2.4}$$
$$a \otimes (b \to c) \leq b \to (a \otimes c), \tag{2.5}$$
$$(a \to b) \otimes (c \to d) \leq (a \otimes c) \to (b \otimes d), \tag{2.6}$$
$$\textstyle\bigotimes_{i \in I}(a_i \to b_i) \leq \bigotimes_{i \in I} a_i \to \bigotimes_{i \in I} b_i, \tag{2.7}$$
$$a \otimes \textstyle\bigvee_{i \in I} b_i = \bigvee_{i \in I}(a \otimes b_i), \tag{2.8}$$
$$a \otimes \textstyle\bigwedge_{i \in I} b_i \leq \bigwedge_{i \in I}(a \otimes b_i), \tag{2.9}$$
$$a \to \textstyle\bigwedge_{i \in I} b_i = \bigwedge_{i \in I}(a \to b_i), \tag{2.10}$$
$$\textstyle\bigvee_{i \in I} a_i \to b = \bigwedge_{i \in I}(a_i \to b), \tag{2.11}$$
$$\textstyle\bigvee_{i \in I}(a_i \to b) \leq \bigwedge_{i \in I} a_i \to b, \tag{2.12}$$
$$\textstyle\bigvee_{i \in I}(a \to b_i) \leq a \to \bigvee_{i \in I} b_i, \tag{2.13}$$
$$\textstyle\bigwedge_{i \in I}(a_i \to b_i) \leq \bigvee_{i \in I} a_i \to \bigvee_{i \in I} b_i, \tag{2.14}$$
$$(a \leftrightarrow b) \otimes (c \leftrightarrow d) \leq (a \to c) \leftrightarrow (b \to d), \tag{2.15}$$
$$\textstyle\bigwedge_{i \in I}(a_i \leftrightarrow b_i) \leq \bigwedge_{i \in I} a_i \leftrightarrow \bigwedge_{i \in I} b_i, \tag{2.16}$$
$$\textstyle\bigwedge_{i \in I}(a_i \leftrightarrow b_i) \leq \bigvee_{i \in I} a_i \leftrightarrow \bigvee_{i \in I} b_i. \tag{2.17}$$

Further properties of complete residuated lattices can be found in [5, 23, 27, 29].

2.2 Fuzzy Sets and Relations

I will now recall basic notions of fuzzy sets and fuzzy relations. Let $\mathbf{L}$ be a complete residuated lattice. An $\mathbf{L}$-*set* in a universe set X is any map $A\colon X \to L$, $A(x) \in L$ being interpreted as the truth value of "x belongs to A". The set of all $\mathbf{L}$-sets in X will be denoted by L^X. Analogously, an n-ary $\mathbf{L}$-*relation* on a universe set X is an $\mathbf{L}$-set in the universe set X^n. For instance, a binary $\mathbf{L}$-relation R on X is a map $R\colon X \times X \to L$. Binary $\mathbf{L}$-relations will be denoted by capital letters $R, R', \ldots$ or symbols $\rightharpoonup, \leftharpoondown, \ldots$ in which case I will write for example $u \rightharpoonup v$ instead of $\rightharpoonup(u, v)$.
For an $\mathbf{L}$-set A in X its *strong* 0-*cut* is defined by $A^+ \subseteq X$ by $A^+ = \{x \in X \mid A(x) > 0\}$. For binary $\mathbf{L}$-relations R_1, R_2 on X the $\circ$-*composition* of R_1 and R_2 is a binary $\mathbf{L}$-relation $R_1 \circ R_2$ on X defined by

$$(R_1 \circ R_2)(x, y) = \textstyle\bigvee_{z \in X}\big(R_1(x, z) \otimes R_2(z, y)\big). \tag{2.18}$$

For $\mathbf{L}$-sets A and B in X one can define degrees $S(A, B) \in L$ and $A \approx B \in L$ as follows:

$$S(A, B) = \textstyle\bigwedge_{x \in X}(A(x) \to B(x)), \tag{2.19}$$
$$E(A, B) = \textstyle\bigwedge_{x \in X}(A(x) \leftrightarrow B(x)). \tag{2.20}$$

$S(A,B)$ is called a *degree of subsethood* of A in B; $E(A,B)$ is called a *degree of equality* of A and B. Note, that $a \leftrightarrow b$ in (2.20) is an abbreviation for $(a \to b) \wedge (b \to a)$. Clearly, $E(A,B) = S(A,B) \wedge S(B,A)$. Furthermore, I will write $A \subseteq B$ if $S(A,B) = 1$, i.e. if $A(x) \leq B(x)$ is true for each $x \in X$.
For an **L**-set A in X and an element $a \in L$, I define an *a-multiple* $a \otimes A$ of A and *a-shift* $a \to A$ of A as **L**-sets given by

$$(a \otimes A)(x) = a \otimes A(x) \tag{2.21}$$

$$(a \to A)(x) = a \to A(x) \tag{2.22}$$

for all $x \in X$. In words, the degree to which x belongs to $a \otimes A$ is equal to the degree to which x is in A multiplied by a, analogously for $a \to A$. Replacing an **L**-set A by an **L**-relation R, we can define a-multiples and a-shift as derived **L**-relations in the same ways. Further details on fuzzy structures and their properties can be found in [5, 13, 27].

2.3 Similarity Spaces

A binary **L**-relation $\approx: X \times X \to L$ is called an **L**-*equivalence* (**L**-similarity or shortly a similarity) if it is (i) reflexive, i.e., $x \approx x = 1$ for each $x \in X$, (ii) symmetric, i.e., $x \approx y = y \approx x$ for each $x, y \in X$, and (iii) $\otimes$-transitive, i.e., $x \approx y \otimes y \approx z \leq x \approx z$ for each $x, y, z \in X$. A set X with an **L**-similarity $\approx: X \times X \to L$, denoted $\langle X, \approx \rangle$, is called a *similarity space*, see [5]. A binary **L**-relation R on X is called *compatible with* $\approx$ (or $\approx$-*extensional*) if, for each $x_1, x_2, y_1, y_2 \in X$:

$$x_1 \approx x_2 \otimes y_1 \approx y_2 \otimes R(x_1, y_1) \leq R(x_2, y_2). \tag{2.23}$$

Recall that systems of all reflexive, symmetric, $\otimes$-transitive, and $\approx$-extensional **L**-relations in X are closed under arbitrary intersections, i.e., they form closure systems. For each **L**-relation, I will denote by $\rightleftharpoons$ the symmetric closure of $\rightharpoonup$ which can obviously be expressed as $\rightleftharpoons \, = \, \rightharpoonup \cup \leftharpoondown \, = \, \rightharpoonup \cup \rightharpoonup^{-1}$. Analogously, I will denote by $\rightharpoonup^*$ the reflexive and $\otimes$-transitive closure of $\rightharpoonup$ which can be expressed by $\rightharpoonup^* = \bigcup_{n=0}^{\infty} \rightharpoonup^n$, where $\rightharpoonup^0$ is the identity **L**-relation (i.e., $x \rightharpoonup^0 x = 1$ and $x \rightharpoonup^0 y = 0$ for all $x \neq y$) and $\rightharpoonup^n \, = \, \rightharpoonup^{n-1} \circ \rightharpoonup$. In case of $\rightharpoonup$ defined on $\langle X, \approx \rangle$, the $\approx$-extensional closure of $\rightharpoonup$, i.e., the least **L**-relation containing $\rightharpoonup$ which is compatible with $\approx$, is equal to $\approx \circ \rightharpoonup \circ \approx$, see [6, 13, 33].

2.4 Generalized Pseudometric Spaces

Now, I will recall basic notions related to generalized pseudometric spaces and to the link between pseudometrics and similarities, see [5, 4] for more details. A *generalized pseudometric space* is a pair $\langle X, \delta \rangle$, where X is a nonempty set and $\delta: X \times X \to [0, +\infty]$ is a mapping (so-called *generalized pseudometric*) satisfying the following properties: (i) $\delta(x,x) = 0$, (ii) $\delta(x,y) = \delta(y,x)$, (iii) $\delta(x,z) \leq \delta(x,y) + \delta(y,z)$.
If $L = [0,1]$ and $\otimes$ is a continuous Archimedean t-norm on L, then one can represent $\otimes$ by a continuous additive generator of $\otimes$ and its pseudoinverse. Recall that a *continuous additive generator* f is a strictly decreasing continuous mapping $f: [0,1] \to [0, +\infty]$ with $f(1) = 0$ such that $a \otimes b = f^{(-1)}(f(a) + f(b))$ for all $a, b \in L$, where $f^{(-1)}$ denotes the pseudoinverse of f defined by $f^{(-1)}(x) = f^{-1}(x)$ if $x \leq f(0)$ and $f^{(-1)}(x) = 0$ otherwise.
Let **L** be a complete residuated lattice on the real unit interval and its conjunction $\otimes$ be a continuous Archimedean t-norm with an additive generator f. For given similarity $\approx$

on X, a mapping $\delta_{\approx}\colon X \times X \to [0,+\infty]$ defined by $\delta_{\approx}(x,y) = f(x \approx y)$ is a generalized pseudometric. Conversely, let δ be a generalized pseudometric on X. Then $\approx_{\delta}\colon X \times X \to [0,1]$ defined by $(x \approx_{\delta} y) = f^{(-1)}(\delta(x,y))$ is a similarity on X.

2.5 Fuzzy Attribute Logic

Fuzzy attribute logic (shortly FAL) was developed for the purpose of describing if-then dependencies that hold in object-attribute relational data where objects are allowed to have attributes to degrees, see [12] for details. Assume here that $\mathbf{L}$ is a complete residuated lattice and let Y be a nonempty set of *attributes*. A *fuzzy attribute implication* (shortly FAI) is an expression $A \Rightarrow B$, where $A, B \in L^Y$. Note that FAIs serve as formulas in FAL. For example, (1.1) can be seen as a FAI with $A \in L^Y$ being an $\mathbf{L}$-set in $Y = \{\mathit{lowAge}, \mathit{lowMileage}, \mathit{highFuelEconomy}, \mathit{highPrice}, \ldots\}$ such that $A(\mathit{lowAge}) = 0.7$, $A(\mathit{lowMileage}) = 0.9$ and $A(\cdots) = 0$ otherwise, analogously for B. The intended meaning of $A \Rightarrow B$ is: "if it is (very) true that an object has all attributes from A, then it has also all attributes from B". Formally, for an $\mathbf{L}$-set $M \in L^Y$ of attributes, a degree $||A \Rightarrow B||_M \in L$ to which $A \Rightarrow B$ is true in M is defines by

$$||A \Rightarrow B||_M = S(A,M)^* \to S(B,M), \tag{2.24}$$

where $S(\cdots)$ denote subsethood degrees (2.19), $\to$ is the residuum from $\mathbf{L}$ and * is an additional unary operation on L satisfying the following conditions: (i) $1^* = 1$, (ii) $a^* \leq a$, (iii) $(a \to b)^* \leq a^* \to b^*$, and (iv) $a^{**} = a^*$ for all $a, b \in L$. An operation * satisfying (i)–(iv) shall be called an *idempotent truth-stressing hedge* (shortly hedge). The requirements (i)–(iv) have appeared as parameters of interpretations of if-then rules in fuzzy Horn logic [14, 15] and have been later used in FAL and formal concept analysis [24] with linguistic hedges [17]. Similar conditions (without the idempotency and with an additional axiom of linearity) appear in [30] where hedges serve as truth functions of logical connectives "very true". In (2.24), * is used as a parameter of the interpretation of $A \Rightarrow B$ in a similar sense as in [14, 15]. Namely, if * is set to identity, then $||A \Rightarrow B||_M = 1$ means that $S(A,M) \leq S(B,M)$, i.e. B is contained in M at least to the degree to which A is contained in M. On the other hand, if * is defined as a globalization [51]:

$$a^* = \begin{cases} 1, & \text{if } a = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.25}$$

then $||A \Rightarrow B||_M = 1$ means that if A is fully contained in M, then B is fully contained in M. Thus, two different important ways to interpret $||\cdots||_M = 1$ are obtained from the general definition (2.24) by different choices of * and can be approached in a single theory instead of having two separatre theories dealing with both the possibly interesting interpretations independently.

Two types of entailment of FAIs are usually considered: (i) semantic entailment based on satisfaction of FAIs in systems of models, and (ii) syntactic entailment based on the notion of provability. Recall the semantic entailment first. M is a *model* of an $\mathbf{L}$-set T of FAIs if $T(A \Rightarrow B) \leq ||A \Rightarrow B||_M$ for all $A, B \in L^Y$. Denoting the set of all models of T by $\mathrm{Mod}(T)$, a degree $||A \Rightarrow B||_T$ to which $A \Rightarrow B$ *semantically follows from* T can be defined as follows:

$$||A \Rightarrow B||_T = \textstyle\bigwedge_{M \in \mathrm{Mod}(T)} ||A \Rightarrow B||_M. \tag{2.26}$$

Now, I will recall also a syntactic characterization of $||\cdots||_T$ for FAIs, see [12]. The following deduction rules are usually considered:

$$\text{(Ax):}\frac{}{A{\cup}B \Rightarrow A}, \qquad \text{(Mul):}\frac{A \Rightarrow B}{c^{*}{\otimes}A \Rightarrow c^{*}{\otimes}B}, \qquad \text{(Cut):}\frac{A \Rightarrow B,\ B{\cup}C \Rightarrow D}{A{\cup}C \Rightarrow D},$$

where $A, B, C, D \in L^Y$, and $c \in L$. The meaning of these rules is "infer $A{\cup}B \Rightarrow A$", "from $A \Rightarrow B$ infer $c^{*}{\otimes}A \Rightarrow c^{*}{\otimes}B$" and "from $A \Rightarrow B$ and $B{\cup}C \Rightarrow D$ infer $A{\cup}C \Rightarrow D$". Note also that if the **L**-sets in (Ax) and (Cut) are replaced by ordinary sets, then the rules become rules from [31] which are equivalent to the well-known Armstrong rules [1]. Notice that the axiom infers exactly all FAIs which are true to degree 1 in all $M \in L^Y$. The results on graded completeness of FAL (in sense of Pavelka's abstract logic [29, 48]) can be also found in [12]. In addition, there is an alternative characterization of entailment degrees using least models, see [17].

2.6 Fuzzy Logic Programming

I recall here the standard notions of fuzzy logic programming (shortly FLP) used in [40, 46, 53] and depart from the standard notation only in cases when it simplifies formulation of the subsequent results. According to [53], consider a complete lattice **L** with L being the real unit interval with its genuine ordering $\leq$ of real numbers. The approach in [53] uses multiple adjoint operations on **L**.

Programs are considered as particular formulas written in a *language* $\mathcal{L}$ which is given by a finite nonempty set R of *relation symbols* and a finite set F of *function symbols*. Each $r \in R$ and $\mathsf{f} \in F$ is given its *arity* denoted by $\mathrm{ar}(r)$ and $\mathrm{ar}(\mathsf{f})$, respectively. Furthermore, I will assume that F contains at least one symbol for a *constant* (i.e., a function symbol f with $\mathrm{ar}(\mathsf{f}) = 0$) and R is nonempty or that R contains at least one *propositional symbol* (i.e., a relation symbol p with $\mathrm{ar}(p) = 0$). Moreover, assume a denumerable set of *variables* which will be denoted by $\mathbb{X}, \mathbb{Y}, \mathbb{X}_i, \ldots$ As usual, *terms* are defined recursively using variables (as the base cases) and function symbols. An *atomic formula* is any expression $r(t_1, \ldots, t_k)$ such that $r \in R$, $\mathrm{ar}(r) = k$, and $t_1, \ldots, t_k$ are terms. Moreover, *formulas* are defined recursively using atomic formulas (as the base cases) and symbols for binary logical connectives $\mathbin{\mathbb{\wedge}}_1, \mathbin{\mathbb{\wedge}}_2, \ldots$ (fuzzy conjunctions), $\mathbin{\mathbb{\vee}}_1, \mathbin{\mathbb{\vee}}_2, \ldots$ (fuzzy disjunctions), $\Rightarrow_1, \Rightarrow_2, \ldots$ (fuzzy implications), and symbols for *aggregations* $\mathbb{ag}_1, \mathbb{ag}_2, \ldots$ I accept the usual rules on the omission of parentheses and write $\varphi \Leftarrow \psi$ to denote $\psi \Rightarrow \varphi$ as it is usual in logic programming [40]. Since quantifiers are not used in FLP, all occurrences of variables in formulas are free.

Each atomic formula is called a *head*. Each formula that is free of symbols for fuzzy implications is called a *tail*. According to [53], a *theory* is a map which assigns to each formula of the language $\mathcal{L}$ a degree from $[0, 1]$. Moreover, a *definite program* is a theory such that

(i) there are only finitely many formulas that are assigned a nonzero degree,

(ii) all the assigned degrees are rational numbers from the unit interval,

(iii) each formula which is assigned a nonzero degree is either a head (a fact) or a formula of the form $\psi \Leftarrow \varphi$ (a rule), where ψ is a head, φ is a tail, and $\Leftarrow$ is an symbol for arbitrary implication.

The declarative meaning of programs is defined using substitutions and models. A *substitution* θ is a set of pairs denoted $\theta = \{\mathbb{X}_1/t_1, \ldots, \mathbb{X}_n/t_n\}$ where each t_i is a term and

each $\mathbb{X}_i$ a variable such that $\mathbb{X}_i \neq t_i$ and $\mathbb{X}_i \neq \mathbb{X}_j$ if $i \neq j$. Term/formula ψ results by application of θ from φ if ψ is obtained from φ by simultaneously replacing t_i for every free occurrence of $\mathbb{X}_i$ in φ. Then, denote ψ as $\varphi\theta$ and call it an *instance* of φ. An instance $\varphi\theta$ is called *ground* if $\varphi\theta$ does not have any (free) occurrences of variables. For substitutions $\theta = \{\mathbb{X}_1/s_1, \ldots, \mathbb{X}_m/s_m\}$ and $\eta = \{\mathbb{Y}_1/t_1, \ldots, \mathbb{Y}_n/t_n\}$, the *composition* $\theta\eta$ is a substitution obtained from $\eta \cup \{\mathbb{X}_1/s_1\eta, \ldots, \mathbb{X}_m/s_m\eta\}$ by removing all $\mathbb{X}_i/s_i\eta$ for which $\mathbb{X}_i = s_i\eta$ and by removing all $\mathbb{Y}_j/t_j$ for which $\mathbb{Y}_j \in \{\mathbb{X}_1, \ldots, \mathbb{X}_m\}$. Obviously, the composition is a monoidal operation on the set of all substitutions [46] with the neutral element being the identity substitution $\emptyset$.
Let P be a definite program formalized in language $\mathcal{L}$. The set of all ground terms of $\mathcal{L}$ is called a *Herbrand universe* of P and denoted by $\mathcal{U}_P$. The set of all ground atomic formulas of $\mathcal{L}$ is called a *Herbrand base* of P and denoted by $\mathcal{B}_P$. Due to my assumptions on $\mathcal{L}$, $\mathcal{B}_P$ is nonempty. A *structure* for P is any **L**-set in $\mathcal{B}_P$. If M is a structure for P, $M(\chi)$ is interpreted as a degree to which the atomic ground formula χ is true under M. The notion of a formula being true in M can be extended to all formulas. Let $M^\sharp$ be an **L**-set of ground formulas defined by

(i) $M^\sharp(\varphi) = M(\varphi)$ if φ is a ground atomic formula;

(ii) $M^\sharp(\psi \Leftarrow \varphi) = M^\sharp(\varphi) \to M^\sharp(\psi)$, where both φ and ψ are ground and $\to$ is a truth function (a residuum) interpreting $\Rightarrow$; analogously for the other binary connectives $\mathbb{A}_1, \mathbb{A}_2, \ldots$ and $\mathbb{V}_1, \mathbb{V}_2, \ldots$ and the corresponding truth functions;

(iii) $M^\sharp\big(\mathbb{ag}(\varphi_1, \ldots, \varphi_n)\big) = ag\big(M^\sharp(\varphi_1), \ldots, M^\sharp(\varphi_n)\big)$, where all φ_i are ground and $\mathbb{ag}$ is an n-ary symbol for aggregation which is interpreted by a monotone function $ag\colon [0,1]^n \to [0,1]$ which preserves $\{0\}^n$ and $\{1\}^n$.

Furthermore,$M^\sharp_\forall$ is defined to extend the notion for all formulas:

$$M^\sharp_\forall(\varphi) = \bigwedge\{M^\sharp(\varphi\theta) \mid \theta \text{ is a substitution such that } \varphi\theta \text{ is ground}\}. \qquad (2.27)$$

Structure M is called a *model* for theory T if $T(\chi) \leq M^\sharp_\forall(\chi)$ for each formula χ of the language $\mathcal{L}$. The collection of all models of T will be denoted by $\mathrm{Mod}(T)$. A pair $\langle a, \theta\rangle$ consisting of $a \in (0,1]$ and a substitution θ is called a *correct answer* for a definite program P and an atomic formula φ (called a query) if $M^\sharp_\forall(\varphi\theta) \geq a$ for each $M \in \mathrm{Mod}(P)$.

Chapter 3

Confuence and termination of fuzzy relations

This chapter summarizes our results in development of the notions related to the idea of rewriting with respect to a given fuzzy relation. The graded counterparts of the notions of reducibility, convertibility, convergence, divergence, confluence, and the Church-Roser property are introduced in Section 3.1. Furthermore, basic properties of these notions are also investigated in that Section. In Section 3.2, we focus on a reformulation of the notions mentioned above, which will respect not only a given reduction (fuzzy relation) but also a given indistinguishability relation (fuzzy equivalence). Moreover, Section 3.3 presents a collection of inequalities which provide us estimations of degrees of the investigated properties while a reduction and/or an indistinguishability relation is replaced by a similar one. We have also established estimation formulas for the properties of derived fuzzy relations (a-multiples and a-shifts), these can be found in Subsection 3.3.3. Finally, Section 3.4 describes a possible fuzzification of the notions of termination, well-foundedness, Noetherian property, and normal form for a given fuzzy reduction relation. Properties of the introduced notions are also investigated there.

3.1 Confluence and Related Properties of Fuzzy Relations

This section aims on developing notions related to the confluence of fuzzy relations and investigating their properties. Since we have been strongly inspired by the classical notions, the properties of fuzzy relations introduced in this section can be seen as direct generalizations of the classical ones.

The results summarized in this section have been published in [8].

3.1.1 Reducibility

In the following, we denote by $\rightharpoonup$ a binary **L**-relation on X. Given $\rightharpoonup$, we define a degree to which a sequence of elements from X is a reduction with respect to $\rightharpoonup$:

Definition 1. For $x_0, \ldots, x_n \in X$ we define a degree $\mathrm{re}(x_0, \ldots, x_n) \in L$ by

$$\mathrm{re}(x_0, \ldots, x_n) = \begin{cases} 1, & \text{if } n = 0, \\ \mathrm{re}(x_0, \ldots, x_{n-1}) \otimes x_{n-1} \rightharpoonup x_n, & \text{otherwise.} \end{cases}$$

$\mathrm{re}(x_0, \ldots, x_n)$ is called a degree to which $x_0, \ldots, x_n$ is a ***reduction*** (w.r.t. $\rightharpoonup$).

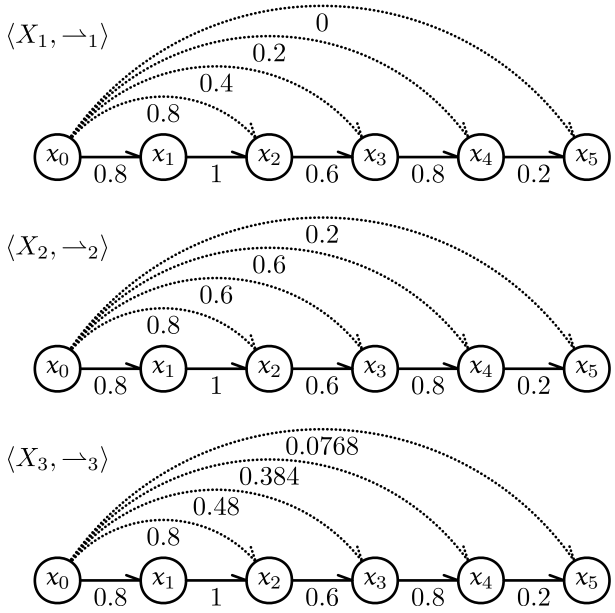

Figure 3.1: Reductions.

Remark 2. Using Definition 1, $\text{re}(x_0, \ldots, x_n) = x_0 \rightharpoonup x_1 \otimes x_1 \rightharpoonup x_2 \otimes \cdots \otimes x_{n-1} \rightharpoonup x_n$. Thus, if $x \rightharpoonup y$ is interpreted as a degree to which x reduces to y then $\text{re}(x_0, \ldots, x_n)$ can be seen as a degree to which "x_0 reduces to x_1 and x_1 reduces to x_2 and, ..., and x_{n-1} reduces to x_n". Clearly, if $\mathbf{L} = \mathbf{2}$, $\text{re}(x_0, \ldots, x_n) = 1$ iff $x_0, \ldots, x_n$ is a reduction in the usual sense.

Example 3. We are going to use weighted oriented graphs to depict **L**-relations. In Fig. 3.1, a solid arrow from an element x_i to an element x_j means that these two elements are in the **L**-relation represented by the corresponding graph to a nonzero degree. The degree is represented by the weight of the $x_i x_j$ edge. Weights of dotted arrows in Fig. 3.1 represent the degrees $\text{re}(x_0, \ldots, x_i)$ to which the element x_0 can be reduced to the element x_i. These values are computed according to the definition as conjunctions of weights of all edges in the reduction. Obviously, the values depend also on the used complete residuated lattice. In our case, we assume that $\rightharpoonup_1$ uses the standard Łukasiewicz algebra of truth degrees and that $\rightharpoonup_2$ and $\rightharpoonup_3$ use the Gödel and Goguen conjunctions, respectively. The degree $\text{re}(x_0, \ldots, x_5) = 0$ which appears in the graph of $\rightharpoonup_1$ means that the element x_0 cannot be reduced to the element x_5, i.e. the element x_5 cannot substitute the element x_0 to a nonzero degree. If we use Gödel or Goguen conjunctions, $\text{re}(x_0, \ldots, x_i) = 0$ iff $x_j \rightharpoonup x_{j+1} = 0$ for some $j \in \{0, \ldots, i-1\}$. When using Łukasiewicz conjunction, which has zero-divisors, there may be reductions with $\text{re}(x_0, \ldots, x_i) = 0$ and $x_j \rightharpoonup x_{j+1} \neq 0$ for each $j \in \{0, \ldots, i-1\}$ as shown in Fig. 3.1 (top).

Definition 4. For $x, y \in X$, the degree to which x is ***reducible*** to y is defined by $x \rightharpoonup^* y$ where $\rightharpoonup^*$ denotes the reflexive and transitive closure of $\rightharpoonup$.

Sometimes, it may be useful to expressed the reducibility degree by

$$x \rightharpoonup^* y = \begin{cases} 1, & \text{if } x = y, \\ \bigvee_{\langle z_1, z_2, \ldots, z_k \rangle \in X^*} \text{re}(x, z_1, \ldots, z_k, y), & \text{otherwise,} \end{cases}$$

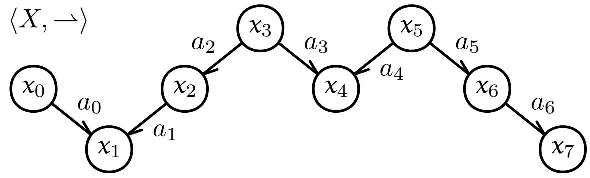

Figure 3.2: Illustration of Graded Convertibility.

where $X^* = \bigcup_{n \in N_0} X^n$, i.e. X^* is a union of all Cartesian powers of the set X (recall that $X^0 = \{\emptyset\}$, i.e. $X^* = \{\emptyset\} \cup X \cup (X \times X) \cup (X \times X \times X) \cup \cdots$). Alternatively, we can also write $\rightharpoonup^* = \bigcup_{n=0}^{\infty} \rightharpoonup^n$, where $\rightharpoonup^n$ is defined by $\rightharpoonup^n \;=\; \rightharpoonup \circ \rightharpoonup^{n-1}$ $(n \geq 1)$, $\rightharpoonup^0$ being the identity **L**-relation.

3.1.2 Convergence and Church-Rosser Property

In this subsection, we define convergence of elements and Church-Rosser property of binary **L**-relations. These notions will be defined directly as graded generalizations of their classical counterparts. Recall that in the classical case, elements x and y are convergent if they are reducible (according to a binary relation $\rightsquigarrow$) to a common element. That means there is z such that $x \rightsquigarrow^* z$ and $y \rightsquigarrow^* z$. The graded Church-Rosser property will be based on graded convergence and convertibility:

Definition 5. For $x, y \in X$ we define degree $x \downharpoonleft y \in L$ by

$$x \downharpoonleft y = \bigvee_{z \in X} (x \rightharpoonup^* z \otimes y \rightharpoonup^* z).$$

$x \downharpoonleft y$ is called the ***degree of convergence*** of x and y. We also say that x and y are convergent to degree $x \downharpoonleft y$. If $x \downharpoonleft y = 1$ we say that x and y are convergent. For x and y, $x \rightleftharpoons^* y$ is called the *degree to which x and y are **convertible***. The degree $\mathrm{CR}(\rightharpoonup)$ to which $\rightharpoonup$ has the ***Church-Rosser property*** is defined by $\mathrm{CR}(\rightharpoonup) = S(\rightleftharpoons^*, \downharpoonleft)$. If $\mathrm{CR}(\rightharpoonup) = 1$ we say that $\rightharpoonup$ has the Church-Rosser property.

Remark 6. Directly from definitions, $\downharpoonleft \;=\; \rightharpoonup^* \circ \leftharpoondown^*$ i.e. $\downharpoonleft$ is a $\circ$-composition of $\rightharpoonup^*$ and $\leftharpoondown^*$, see (2.18). By definition, $x \downharpoonleft y$ is a degree to which there is z such that $x \rightharpoonup^* z$ and $y \rightharpoonup^* z$. According to the definition of the degree of subsethood, we have

$$\mathrm{CR}(\rightharpoonup) = \bigwedge_{x,y \in X} (x \rightleftharpoons^* y \;\rightarrow\; x \downharpoonleft y).$$

Note that if $\mathbf{L} = \mathbf{2}$ then (i) $x \downharpoonleft y = 1$ iff x and y are convergent in the usual sense and (ii) $\mathrm{CR}(\rightharpoonup) = 1$ iff $\rightharpoonup$ has the (ordinary) Church-Rosser property.
The converbility is illustrated in Fig. 3.2. In this particular case, the degree $x_0 \rightleftharpoons^* x_7$ to which the elements x_0 and x_7 are convertible is computed as a conjunction of weights of all arrows on the only path from x_0 to x_7 regardless of directions of the arrows, i.e. $x_0 \rightleftharpoons^* x_7 = a_0 \otimes a_1 \otimes a_2 \otimes a_3 \otimes a_4 \otimes a_5 \otimes a_6$.

The following assertion shows that degrees to which a binary **L**-relation $\rightharpoonup$ has the Church-Rosser propery equals to the degree to which the convertibility **L**-relation $\rightleftharpoons^*$ equals the convergence **L**-relation $\downharpoonleft$.

Theorem 7. $\mathrm{CR}(\rightharpoonup) \;= E(\rightleftharpoons^*, \downharpoonleft)$.

Proof. Since $\mathrm{CR}(\rightharpoonup) = S(\rightleftharpoons^*, \downharpoonleft)$ and $E(\rightleftharpoons^*, \downharpoonleft) = S(\rightleftharpoons^*, \downharpoonleft) \wedge S(\downharpoonleft, \rightleftharpoons^*)$, in order to prove $\mathrm{CR}(\rightharpoonup) = E(\rightleftharpoons^*, \downharpoonleft)$, we have to check that $S(\downharpoonleft, \rightleftharpoons^*) = 1$, i.e. $\downharpoonleft \subseteq \rightleftharpoons^*$.
First, observe that $\rightharpoonup \subseteq \rightleftharpoons$ and $\leftharpoondown \subseteq \rightleftharpoons$, i.e. $x \rightharpoonup y \leq x \rightleftharpoons y$ and $x \leftharpoondown y \leq x \rightleftharpoons y$ are true for every $x, y \in X$. The latter inequalities and the monotony of the reflexive and transitive closure * yield $\rightharpoonup^* \subseteq \rightleftharpoons^*$ and $\leftharpoondown^* \subseteq \rightleftharpoons^*$.
Furthermore, since $\rightleftharpoons^*$ is transitive, we have $\rightleftharpoons^* \circ \rightleftharpoons^* \subseteq \rightleftharpoons^*$. Using $\downharpoonleft = \rightharpoonup^* \circ \leftharpoondown^*$ (see Remark 6) we get $\downharpoonleft = \rightharpoonup^* \circ \leftharpoondown^* \subseteq \rightleftharpoons^* \circ \rightleftharpoons^* \subseteq \rightleftharpoons^*$, proving the claim. □

Remark 8. Theorem 7 generalizes the classical theorem which is well known in the theory of abstract rewriting systems [2, 54] in the following way. If we let $\mathbf{L} = \mathbf{2}$, Theorem 7 is equivalent to saying that $\mathrm{CR}(\rightharpoonup) = 1$ iff $\rightleftharpoons^* = \downharpoonleft$, i.e. $\rightharpoonup$ has the Church-Rosser property iff the convertibility and convergence relations $\rightleftharpoons^*$ and $\downharpoonleft$ coincide.

3.1.3 Divergence and Confluence

In this subsection, we first introduce a graded divergence and then a graded confluence of binary $\mathbf{L}$-relations and investigate their properties. The main goal of the subsection is to show a correspondence between the graded confluence and the graded Church-Rosser property.

Definition 9. For $x, y \in X$ we define degree $x \upharpoonleft y \in L$ by

$$x \upharpoonleft y = \bigvee_{z \in X}(z \rightharpoonup^* x \otimes z \rightharpoonup^* y).$$

$x \upharpoonleft y$ is called the ***degree of divergence*** of x and y. We also say that x and y are divergent to degree $x \upharpoonleft y$. If $x \upharpoonleft y = 1$ we say that x and y are divergent. The degree $\mathrm{CFL}(\rightharpoonup)$ to which $\rightharpoonup$ is ***confluent*** is defined by $\mathrm{CFL}(\rightharpoonup) = S(\upharpoonleft, \downharpoonleft)$. If $\mathrm{CFL}(\rightharpoonup) = 1$ we say that $\rightharpoonup$ is confluent.

Remark 10. Analogously as in case of convergence, we have $\upharpoonleft = \leftharpoondown^* \circ \rightharpoonup^*$, i.e. $\upharpoonleft$ is a $\circ$-composition of $\leftharpoondown^*$ and $\rightharpoonup^*$. The degree $x \upharpoonleft y$ can be interpreted as a degree to which there is z such that $z \rightharpoonup^* x$ and $z \rightharpoonup^* y$. Using graded subsethood,

$$\mathrm{CFL}(\rightharpoonup) = \bigwedge_{x,y \in X}(x \upharpoonleft y \rightarrow x \downharpoonleft y).$$

As a consequence, if $\rightharpoonup$ is confluent then $x \upharpoonleft y \leq x \downharpoonleft y$ for all $x, y \in X$, i.e. $\upharpoonleft \subseteq \downharpoonleft$. Described verbally, the degree of confluence is a degree to which the following is true: "if any x and y are divergent then x and y are convergent".

The following assertion says that under an additional condition of idempotency, the degree $\mathrm{CFL}(\rightharpoonup)$ to which $\rightharpoonup$ is confluent equals to the degree $\mathrm{CR}(\rightharpoonup)$ to which $\rightharpoonup$ has the Church-Rosser property.

Theorem 11. *If* $\mathrm{CFL}(\rightharpoonup)$ *is an idempotent element of* $\mathbf{L}$ *then* $\mathrm{CR}(\rightharpoonup) = \mathrm{CFL}(\rightharpoonup)$.

Proof. "$\mathrm{CR}(\rightharpoonup) \leq \mathrm{CFL}(\rightharpoonup)$": We have to show $S(\rightleftharpoons^*, \downharpoonleft) \leq S(\upharpoonleft, \downharpoonleft)$. This follows easily from the antitony of the graded subsethood (2.19) in its first argument and from $\upharpoonleft = \leftharpoondown^* \circ \rightharpoonup^* \subseteq \rightleftharpoons^*$.
"$\mathrm{CFL}(\rightharpoonup) \leq \mathrm{CR}(\rightharpoonup)$": We have to show $S(\upharpoonleft, \downharpoonleft) \leq S(\rightleftharpoons^*, \downharpoonleft)$ which is true due to the adjointness iff for each $x, y \in X$ we have $x \rightleftharpoons^* y \otimes S(\upharpoonleft, \downharpoonleft) \leq x \downharpoonleft y$. By definition of $\rightleftharpoons^*$, the latter is true iff for each $z_1, \ldots, z_k \in X$ we have

$$x \rightleftharpoons z_1 \otimes z_1 \rightleftharpoons z_2 \otimes \cdots \otimes z_k \rightleftharpoons y \otimes S(\upharpoonleft, \downharpoonleft) \leq x \rightharpoonup^* \circ \leftharpoondown^* y.$$

Due to the definition of $\rightleftharpoons$ we have

$$\bigvee_{*_1\in\{\rightharpoonup,\leftharpoondown\}}(x *_1 z_1) \otimes \bigvee_{*_2\in\{\rightharpoonup,\leftharpoondown\}}(z_1 *_2 z_2) \otimes \cdots \otimes \bigvee_{*_{k+1}\in\{\rightharpoonup,\leftharpoondown\}}(z_k *_{k+1} y) \otimes S(\upharpoonleft, \downharpoonleft) \leq \\ \leq x \rightharpoonup^* \circ \leftharpoondown^* y$$

which is equivalent to

$$\bigvee_{*_1,\ldots,*_{k+1}\in\{\rightharpoonup,\leftharpoondown\}}\big((x *_1 z_1) \otimes (z_1 *_2 z_2) \otimes \cdots \otimes (z_k *_{k+1} y)\big) \otimes S(\upharpoonleft, \downharpoonleft) \leq x \rightharpoonup^* \circ \leftharpoondown^* y$$

which holds true iff for every $*_1, \ldots, *_{k+1} \in \{\rightharpoonup, \leftharpoondown\}$ we have

$$(x *_1 z_1) \otimes (z_1 *_2 z_2) \otimes \cdots \otimes (z_k *_{k+1} y) \otimes S(\upharpoonleft, \downharpoonleft) \leq x \rightharpoonup^* \circ \leftharpoondown^* y. \tag{3.1}$$

We verify this inequality for $*_1$ being $\leftharpoondown$ and $*_{k+1}$ being $\rightharpoonup$ (the other cases are either analogous or easy extensions of our case). Indicating the consecutive sequences of $\leftharpoondown$'s and $\rightharpoonup$'s in the left-hand side of inequality (3.1), we can write

$$(x *_1 z_1) \otimes (z_1 *_2 z_2) \otimes \cdots \otimes (z_k *_{k+1} y) \otimes S(\upharpoonleft, \downharpoonleft) = \\ = \big((x \leftharpoondown z_{12}) \otimes \cdots \otimes (z_{1l_1} \leftharpoondown z_{21})\big) \otimes \big((z_{21} \rightharpoonup z_{22}) \otimes \cdots \otimes (z_{2l_2} \rightharpoonup z_{31})\big) \otimes \\ \otimes \big((z_{31} \leftharpoondown z_{32}) \otimes \cdots \otimes (z_{3l_3} \leftharpoondown z_{41})\big) \otimes \cdots \otimes \big((z_{2m,1} \rightharpoonup z_{2m,2}) \otimes \cdots \otimes (z_{2m,l_{2m}} \rightharpoonup y)\big) \otimes \\ \otimes S(\upharpoonleft, \downharpoonleft),$$

where $2m$ is the number of the sequences and l_i (for $i \in \{1, 2, \ldots, 2m\}$) is the length of the i-th sequence. Notice that since we have initially assumed that $*_1$ is $\leftharpoondown$ and $*_{k+1}$ is $\rightharpoonup$, we indeed have an even number of sequences therefore the notation $2m$. So, the left-hand side of the inequality (3.1) is less than or equal to

$$(x \leftharpoondown^* z_{21}) \otimes (z_{21} \rightharpoonup^* z_{31}) \otimes (z_{31} \leftharpoondown^* z_{41}) \otimes \cdots \otimes (z_{2m,1} \rightharpoonup^* y) \otimes S(\upharpoonleft, \downharpoonleft).$$

Using the definition of the $\circ$-composition, the latter is less than or equal to

$$(x \leftharpoondown^* \circ \rightharpoonup^* z_{31}) \otimes (z_{31} \leftharpoondown^* \circ \rightharpoonup^* z_{51}) \otimes \cdots \otimes (z_{2m-1,1} \leftharpoondown^* \circ \rightharpoonup^* y) \otimes S(\upharpoonleft, \downharpoonleft).$$

Therefore, in order to show the required inequality, it is enough to verify that

$$(x \leftharpoondown^* \circ \rightharpoonup^* z_1) \otimes (z_1 \leftharpoondown^* \circ \rightharpoonup^* z_2) \otimes \cdots \otimes (z_n \leftharpoondown^* \circ \rightharpoonup^* y) \otimes S(\upharpoonleft, \downharpoonleft) \leq x \rightharpoonup^* \circ \leftharpoondown^* y$$

is true for any $z_1, z_2, \ldots, z_n \in X$. We show this by induction over n.
For $n = 0$ we have to show

$$(x \leftharpoondown^* \circ \rightharpoonup^* y) \otimes S(\upharpoonleft, \downharpoonleft) \leq x \rightharpoonup^* \circ \leftharpoondown^* y.$$

According to the definition of the degree of subsethood (2.19) and using (2.2), we have

$$(x \leftharpoondown^* \circ \rightharpoonup^* y) \otimes S(\upharpoonleft, \downharpoonleft) = \\ = (x \leftharpoondown^* \circ \rightharpoonup^* y) \otimes \bigwedge_{x,y\in X}((x \leftharpoondown^* \circ \rightharpoonup^* y) \to (x \rightharpoonup^* \circ \leftharpoondown^* y)) \leq \\ \leq (x \leftharpoondown^* \circ \rightharpoonup^* y) \otimes ((x \leftharpoondown^* \circ \rightharpoonup^* y) \to (x \rightharpoonup^* \circ \leftharpoondown^* y)) \leq \\ \leq x \rightharpoonup^* \circ \leftharpoondown^* y,$$

which proves the inequality for $n = 0$.

For $n+1$, provided the assertion is valid for n and using the idempotency of $\mathrm{CFL}(\rightharpoonup) = S(\upharpoonleft, \downharpoonleft)$, we have

$$
\begin{aligned}
&(x \leftharpoondown^* \circ \rightharpoonup^* z_1) \otimes \cdots \otimes (z_n \leftharpoondown^* \circ \rightharpoonup^* z_{n+1}) \otimes (z_{n+1} \leftharpoondown^* \circ \rightharpoonup^* y) \otimes S(\upharpoonleft, \downharpoonleft) = \\
&= (x \leftharpoondown^* \circ \rightharpoonup^* z_1) \otimes \cdots \otimes (z_n \leftharpoondown^* \circ \rightharpoonup^* z_{n+1}) \otimes S(\upharpoonleft, \downharpoonleft) \otimes (z_{n+1} \leftharpoondown^* \circ \rightharpoonup^* y) \otimes S(\upharpoonleft, \downharpoonleft) \leq \\
&\leq (x \rightharpoonup^* \circ \leftharpoondown^* z_{n+1}) \otimes (z_{n+1} \leftharpoondown^* \circ \rightharpoonup^* y) \otimes S(\upharpoonleft, \downharpoonleft).
\end{aligned}
$$

Using the definition of the $\circ$-composition,

$$
\begin{aligned}
&(x \rightharpoonup^* \circ \leftharpoondown^* z_{n+1}) \otimes (z_{n+1} \leftharpoondown^* \circ \rightharpoonup^* y) \otimes S(\upharpoonleft, \downharpoonleft) = \\
&= \textstyle\bigvee_{u \in X} \big((x \rightharpoonup^* u) \otimes (u \leftharpoondown^* z_{n+1})\big) \otimes \bigvee_{v \in X} \big((z_{n+1} \leftharpoondown^* v) \otimes (v \rightharpoonup^* y)\big) \otimes S(\upharpoonleft, \downharpoonleft) = \\
&= \textstyle\bigvee_{u,v \in X} \big((x \rightharpoonup^* u) \otimes (u \leftharpoondown^* z_{n+1}) \otimes (z_{n+1} \leftharpoondown^* v) \otimes (v \rightharpoonup^* y)\big) \otimes S(\upharpoonleft, \downharpoonleft) \leq \\
&\leq \textstyle\bigvee_{u,v \in X} \big((x \rightharpoonup^* u) \otimes (u \leftharpoondown^* v) \otimes (v \rightharpoonup^* y)\big) \otimes S(\upharpoonleft, \downharpoonleft) = \\
&= \textstyle\bigvee_{u \in X} \big((x \rightharpoonup^* u) \otimes \bigvee_{v \in X} \big((u \leftharpoondown^* v) \otimes (v \rightharpoonup^* y)\big)\big) \otimes S(\upharpoonleft, \downharpoonleft) = \\
&= \textstyle\bigvee_{u \in X} \big((x \rightharpoonup^* u) \otimes (u \leftharpoondown^* \circ \rightharpoonup^* y)\big) \otimes S(\upharpoonleft, \downharpoonleft).
\end{aligned}
$$

Using properties of the subsethood and the facts that $\rightharpoonup^*$ is transitive and (2.2),

$$
\begin{aligned}
&\textstyle\bigvee_{u \in X} \big((x \rightharpoonup^* u) \otimes (u \leftharpoondown^* \circ \rightharpoonup^* y)\big) \otimes S(\upharpoonleft, \downharpoonleft) = \\
&= \textstyle\bigvee_{u \in X} \big((x \rightharpoonup^* u) \otimes (u \leftharpoondown^* \circ \rightharpoonup^* y) \otimes S(\upharpoonleft, \downharpoonleft)\big) \leq \\
&\leq \textstyle\bigvee_{u \in X} \big((x \rightharpoonup^* u) \otimes (u \leftharpoondown^* \circ \rightharpoonup^* y) \otimes ((u \leftharpoondown^* \circ \rightharpoonup^* y) \to (u \rightharpoonup^* \circ \leftharpoondown^* y))\big) \leq \\
&\leq \textstyle\bigvee_{u \in X} \big((x \rightharpoonup^* u) \otimes (u \rightharpoonup^* \circ \leftharpoondown^* y)\big) = \\
&= \textstyle\bigvee_{u \in X} \big((x \rightharpoonup^* u) \otimes \bigvee_{w \in X} \big((u \rightharpoonup^* w) \otimes (w \leftharpoondown^* y)\big)\big) = \\
&= \textstyle\bigvee_{u,w \in X} \big((x \rightharpoonup^* u) \otimes (u \rightharpoonup^* w) \otimes (w \leftharpoondown^* y)\big) = \\
&= \textstyle\bigvee_{w \in X} \big(\bigvee_{u \in X} \big((x \rightharpoonup^* u) \otimes (u \rightharpoonup^* w)\big) \otimes (w \leftharpoondown^* y)\big) \leq \\
&\leq \textstyle\bigvee_{w \in X} \big((x \rightharpoonup^* w) \otimes (w \leftharpoondown^* y)\big) = (x \rightharpoonup^* \circ \leftharpoondown^* y).
\end{aligned}
$$

Altogether, $(x \rightharpoonup^* \circ \leftharpoondown^* z_{n+1}) \otimes (z_{n+1} \leftharpoondown^* \circ \rightharpoonup^* y) \otimes S(\upharpoonleft, \downharpoonleft) \leq x \rightharpoonup^* \circ \leftharpoondown^* y$ which concludes the proof. □

Corollary 12. *If in L we have $\otimes = \wedge$, then* $\mathrm{CR}(\rightharpoonup) = \mathrm{CFL}(\rightharpoonup)$.

Proof. Directly by Theorem 11 using the fact that if $\otimes = \wedge$ then each $a \in L$ is idempotent. □

Theorem 13. $\rightharpoonup$ *has the Church-Rosser property iff it is confluent.*

Proof. Directly by Theorem 11 using the fact that 1 is idempotent in each **L**. □

Example 14. Fig. 3.3 contains diagrams of three **L**-relations $\rightharpoonup_1$, $\rightharpoonup_2$, and $\rightharpoonup_3$. Let **L** be the standard Goguen algebra of truth degrees. Then, one can show that $\mathrm{CFL}(\rightharpoonup_1) = 1$, i.e. that $\rightharpoonup_1$ is confluent. In case of $\rightharpoonup_2$, we have $\mathrm{CFL}(\rightharpoonup_2) = 0.75$, i.e. we can say that $\rightharpoonup_2$ is "more or less confluent". On the other hand, $\mathrm{CFL}(\rightharpoonup_3) = 0.084$, i.e. $\rightharpoonup_3$ is practically not confluent at all.

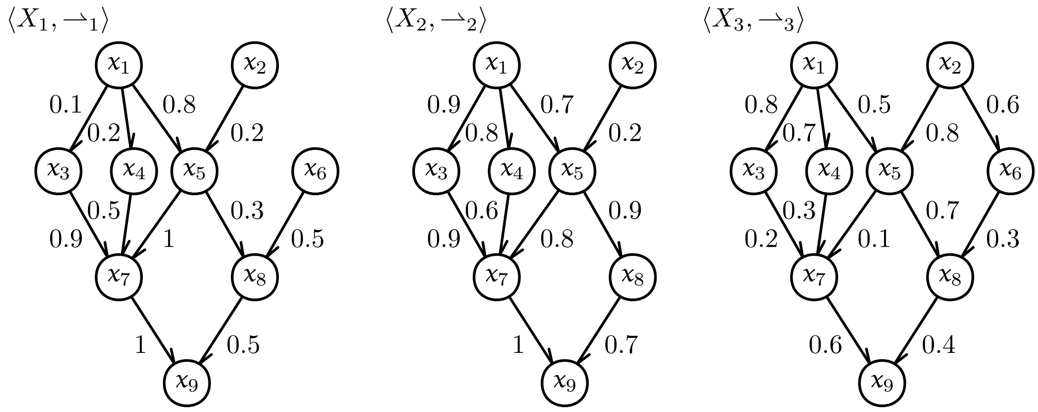

Figure 3.3: Relations with different degree of confluence values.

3.2 Confluence on Similarity and Pseudometric Spaces

Let us further generalize the notions related to rewriting which were introduced in Section 3.1. The main motivation is the fact that the universe of discourse is often equipped with an indistinguishability relation (an equivalence in the crisp or similarity in the fuzzy setting). This relation represents some background knowledge which should not be ignored in a rewriting process. This idea leads straightforward to introducing new notions which will respect a given reduction as well a given similarity. Moreover, it is a well-known fact [4] that the indistinguishability can be also expressed by means of a given pseudometric. Subsection 3.2.2 proposes a possible way of defining confluence and the related properties with a respect to a generalized pseudometric. We also present a link between the properties of reductions on similarity and pseudometric spaces.

The results summarized in this section have been published in [37].

3.2.1 Confluence of Fuzzy Relations over Similarity Spaces

In this subsection, we develop a theory of confluence of fuzzy relations over similarity spaces. We start by introducing a generalization of the notion of a reduction and investigate its properties. Then, we present graded versions of divergence, convergence, and convertibility which will respect a given similarity relation. Using these notions, we generalize two important properties related to rewriting: confluence and Church-Rosser property. The main result in this subsection shows a relationship between the degrees of confluence of a fuzzy relation and the degrees to which the fuzzy relation possesses the Church-Rosser property.
In what follows, $\langle X, \approx \rangle$ is an **L**-similarity space representing a universe X of all elements that can be used for substitution together with the indistinguishability relation $\approx$. In addition to that, we consider a binary **L**-relation on X which represents the substitutability relation. Given $\rightharpoonup$ and $\approx$, we define a degree to which a element from X can be reduced to another element from X with respect to $\rightharpoonup$ and $\approx$:

Definition 15. Given $\rightharpoonup$ and $\approx$, we define an **L**-relation $\rightharpoonup^*_\approx$ by

$$x \rightharpoonup^*_\approx y = \bigvee\nolimits_{\langle z_1, z_2, \ldots, z_{2k} \rangle \in X^{2\mathbb{N}_0}} \big(x \approx z_1 \otimes z_1 \rightharpoonup z_2 \otimes z_2 \approx z_3 \otimes \cdots \otimes z_{2k-1} \rightharpoonup z_{2k} \otimes z_{2k} \approx y\big),$$

where $x, y \in X$ and $X^{2\mathbb{N}_0} = \bigcup_{n \in \mathbb{N}_0} X^{2n}$, i.e. $X^{2\mathbb{N}_0}$ is a union of all even Cartesian powers of X. The **L**-relation $\rightharpoonup^*_\approx$ is called ***reducibility*** (induced by $\rightharpoonup$ and $\approx$). The degree $x \rightharpoonup^*_\approx y \in L$ is a degree to which x can be reduced to y with respect to $\approx$.

According to Definition 15, the degree $x \rightharpoonup^*_\approx y$ can be seen as a degree to which "there are some elements $z_1, z_2, \dots, z_{2k}$ in X such that x is similar to z_1 and z_1 reduces to z_2 and z_2 is similar to z_3 and, ..., and z_{2k-1} reduces to z_{2k} and z_{2k} is similar to y".
Note that Definition 15 generalizes the notion of reducibility $\rightharpoonup^*$ that has been introduced in Section 3.1. Namely, if one considers a trivial **L**-similarity space $\langle X, \approx \rangle$ where $\approx$ is the crisp equality (i.e., the identity) then obviously $x \rightharpoonup^*_\approx y = x \rightharpoonup^* y$, where $\rightharpoonup^*$ is a reflexive and transitive closure of $\rightharpoonup$. Moreover, if $\mathbf{L} = \mathbf{2}$ and if $\approx$ is the crisp equality relation then $x \rightharpoonup^*_\approx y = 1$ iff $x = z_1, z_2 = z_3, \dots, z_{2k} = y$ is a reduction in the usual sense, see [2].
Let us stress that "$^*_\approx$" can be seen as an operator $^*_\approx : L^{X\times X} \to L^{X\times X}$ which maps each binary **L**-relation $\rightharpoonup$ on X to the corresponding reducibility **L**-relation $\rightharpoonup^*_\approx$. Next, we investigate properties of such operator and conclude that it is a particular closure operator.

Theorem 16. *For any $\rightharpoonup$, $\approx$ and the corresponding $\rightharpoonup^*_\approx$, the following are true:*

(*i*) $\approx \subseteq \rightharpoonup^*_\approx$,

(*ii*) $\rightharpoonup^* \subseteq \rightharpoonup^*_\approx$,

(*iii*) $\rightharpoonup^*_\approx \circ \rightharpoonup^*_\approx \subseteq \rightharpoonup^*_\approx$,

(*iv*) $\rightharpoonup^*_\approx$ *is compatible with* $\approx$.

Proof. (*i*): Directly from Definition 15, using the fact that $X^0 \subseteq X^{2\mathbb{N}_0}$, we can see that $x \approx y \leq x \rightharpoonup^*_\approx y$ for all $x, y \in X$. Thus, $\approx \subseteq \rightharpoonup^*_\approx$.
(*ii*): Due to the reflexivity of $\approx$, for arbitrary $x = z_1, z_2, \dots, z_k = y$ from X, we have

$$\begin{aligned} & z_1 \rightharpoonup z_2 \otimes z_2 \rightharpoonup z_3 \otimes \cdots \otimes z_{k-1} \rightharpoonup z_k = \\ = & z_1 \approx z_1 \otimes z_1 \rightharpoonup z_2 \otimes z_2 \approx z_2 \otimes z_2 \rightharpoonup z_3 \otimes z_3 \approx z_3 \otimes \cdots \\ & \cdots \otimes z_{k-1} \approx z_{k-1} \otimes z_{k-1} \rightharpoonup z_k \otimes z_k \approx z_k \leq z_1 \rightharpoonup^*_\approx z_k = x \rightharpoonup^*_\approx y. \end{aligned}$$

Since $x = z_1, z_2, \dots, z_k = y$ have been taken arbitrarily, we get $x \rightharpoonup^* y \leq x \rightharpoonup^*_\approx y$ for all $x, y \in X$, showing $\rightharpoonup^* \subseteq \rightharpoonup^*_\approx$.
(*iii*): Analogously as in the previous case, for any $z_1, \dots, z_{2k}$ and $z'_1, \dots, z'_{2l}$ from X, the $\otimes$-transitivity of $\approx$ yields

$$\begin{aligned} & \big(x \approx z_1 \otimes z_1 \rightharpoonup z_2 \otimes \cdots \otimes z_{2k} \approx w\big) \otimes \big(w \approx z'_1 \otimes z'_1 \rightharpoonup z'_2 \otimes \cdots \otimes z'_{2l} \approx y\big) = \\ = & \big(x \approx z_1 \otimes \cdots \otimes z_{2k-1} \rightharpoonup z_{2k}\big) \otimes \big(z_{2k} \approx w \otimes w \approx z'_1\big) \otimes \big(z'_1 \rightharpoonup z'_2 \otimes \cdots \otimes z'_{2l} \approx y\big) \leq \\ \leq & \big(x \approx z_1 \otimes \cdots \otimes z_{2k-1} \rightharpoonup z_{2k}\big) \otimes z_{2k} \approx z'_1 \otimes \big(z'_1 \rightharpoonup z'_2 \otimes \cdots \otimes z'_{2l} \approx y\big) \leq x \rightharpoonup^*_\approx y \end{aligned}$$

for each $x, y, w \in X$. Therefore, $x \rightharpoonup^*_\approx w \otimes w \rightharpoonup^*_\approx y \leq x \rightharpoonup^*_\approx y$, proving $\rightharpoonup^*_\approx \circ \rightharpoonup^*_\approx \subseteq \rightharpoonup^*_\approx$.
(*iv*): Take any $x_1, x_2, y_1, y_2 \in X$. Using the reflexivity of $\approx$ together with (i) and (iii), we get

$$\begin{aligned} & x_1 \approx x_2 \otimes y_1 \approx y_2 \otimes x_1 \rightharpoonup^*_\approx y_1 = x_2 \approx x_1 \otimes y_1 \approx y_2 \otimes x_1 \rightharpoonup^*_\approx y_1 \leq \\ \leq & x_2 \rightharpoonup^*_\approx x_1 \otimes y_1 \rightharpoonup^*_\approx y_2 \otimes x_1 \rightharpoonup^*_\approx y_1 \leq x_2 \rightharpoonup^*_\approx y_2, \end{aligned}$$

showing the compatibility of $\rightharpoonup^*_\approx$ with $\approx$. □

The following assertion characterizes the closure properties of $\rightharpoonup^*_\approx$:

Theorem 17. $\rightharpoonup^*_\approx$ *is a reflexive, $\otimes$-transitive, and $\approx$-extensional closure of* $\rightharpoonup$.

Proof. Using Theorem 16, $\rightharpoonup^*_{\approx}$ is a reflexive and $\otimes$-transitive **L**-relation which is compatible with $\approx$ and contains $\rightharpoonup$. Indeed, the reflexivity is a consequence of Theorem 16 (i). The transitivity is a consequence of Theorem 16 (iii). Moreover, $\rightharpoonup \subseteq \rightharpoonup^*_{\approx}$ due to Theorem 16 (ii) since $\rightharpoonup \subseteq \rightharpoonup^*$. Furthermore, Theorem 16 (iv) yields that $\rightharpoonup^*_{\approx}$ is compatible with $\approx$. Thus, it remains to check that $\rightharpoonup^*_{\approx}$ is the least **L**-relation having this property. So, suppose that $\rightsquigarrow^*_{\approx}$ is an **L**-relation which is reflexive, $\otimes$-transitive, compatible with $\approx$ such that $\rightharpoonup \subseteq \rightsquigarrow^*_{\approx}$. Then, for any $x, z_1, \ldots, z_{2k}, y$ from X, the $\otimes$-transitivity of $\rightsquigarrow^*_{\approx}$ together with the facts that $\rightsquigarrow^*_{\approx}$ is compatible with $\approx$ and $\rightsquigarrow^*_{\approx}$ extends $\rightharpoonup$ give

$$\begin{aligned}
&x \approx z_1 \otimes z_1 \rightharpoonup z_2 \otimes z_2 \approx z_3 \otimes z_3 \rightharpoonup z_4 \otimes \cdots \otimes z_{2k-1} \rightharpoonup z_{2k} \otimes z_{2k} \approx y \leq \\
\leq &x \approx z_1 \otimes z_1 \rightsquigarrow^*_{\approx} z_2 \otimes z_2 \approx z_3 \otimes z_3 \rightsquigarrow^*_{\approx} z_4 \otimes \cdots \otimes z_{2k-1} \rightsquigarrow^*_{\approx} z_{2k} \otimes z_{2k} \approx y \leq \\
\leq &x \rightsquigarrow^*_{\approx} z_3 \otimes z_3 \rightsquigarrow^*_{\approx} z_5 \otimes \cdots \otimes z_{2k-1} \rightsquigarrow^*_{\approx} y \leq x \rightsquigarrow^*_{\approx} y.
\end{aligned}$$

Hence, $x \rightharpoonup^*_{\approx} y \leq x \rightsquigarrow^*_{\approx} y$ for all $x, y \in X$, meaning $\rightharpoonup^*_{\approx} \subseteq \rightsquigarrow^*_{\approx}$, finishing the proof. □

Remark 18. As a consequence of Theorem 17, the operator ${}^*_{\approx} \colon L^{X\times X} \to L^{X\times X}$ is extensional, idempotent and monotone. In this context, idempotency means that $(\rightharpoonup^*_{\approx})^*_{\approx} = \rightharpoonup^*_{\approx}$. Monotony means that $\rightharpoonup_1 \subseteq \rightharpoonup_2$ implies $(\rightharpoonup_1)^*_{\approx} \subseteq (\rightharpoonup_2)^*_{\approx}$. Moreover, Theorem 17 yields that $\rightharpoonup^*_{\approx}$ is compatible with $\approx$. As a consequence, we get

$$\approx \circ \rightharpoonup^*_{\approx} \circ \approx \; = \; \approx \circ \rightharpoonup^*_{\approx} \; = \; \rightharpoonup^*_{\approx} \circ \approx \; = \; \rightharpoonup^*_{\approx}, \tag{3.2}$$

i.e., the $\approx$-extensional closure of $\rightharpoonup^*_{\approx}$ equals to $\rightharpoonup^*_{\approx}$.

Recall that in the classic theory of abstract rewriting systems [2], $\rightharpoonup^*$ is a reflexive and transitive closure of relation $\rightharpoonup$. The same applies for reductions in graded setting as they were introduced in Section 3.1, where $\rightharpoonup^*$ is a reflexive and $\otimes$-transitive closure of $\rightharpoonup$. This is quite natural since we are interested in reachability (substitutability) in multiple steps. In case of similarity spaces, $\rightharpoonup^*_{\approx}$ introduced in Definition 15 turned out to be a reflexive, $\otimes$-transitive and $\approx$-extensional closure of $\rightharpoonup$, see Theorem 17. This should also be regarded as natural since we are interested in reachability (substitutability) in multiple steps with the possibility to "jump over" elements similar according to $\approx$. Nevertheless, an open question is, whether $\rightharpoonup^*$ can be seen as a reflexive and a $\otimes$-transitive closure of a new **L**-relation derived from $\rightharpoonup$. The answer is given by the following assertion which shows that it is sufficient to take the $\approx$-extensional closure $\approx \circ \rightharpoonup \circ \approx$ of $\rightharpoonup$.

Theorem 19. *For any $\rightharpoonup$, $\approx$ and $\rightharpoonup_\circ$ defined by $\rightharpoonup_\circ = \approx \circ \rightharpoonup \circ \approx$, we have $\rightharpoonup^*_{\approx} = (\rightharpoonup_\circ)^*$, where the operator * denotes the reflexive and $\otimes$-transitive closure.*

Proof. Since $\rightharpoonup^*_{\approx}$ is compatible with $\approx$ and contains $\rightharpoonup$, we get $\rightharpoonup_\circ \subseteq \rightharpoonup^*_{\approx}$ because $\rightharpoonup_\circ$ is the least **L**-relation which is compatible with $\approx$ and contains $\rightharpoonup$. Furthermore, since $\rightharpoonup^*_{\approx}$ is reflexive, $\otimes$-transitive and contains $\rightharpoonup_\circ$, we get that $(\rightharpoonup_\circ)^* \subseteq \rightharpoonup^*_{\approx}$ because $(\rightharpoonup_\circ)^*$ is the least reflexive and $\otimes$-transitive **L**-relation containing $\rightharpoonup_\circ$. Thus, it suffices to prove the converse inclusion.
In order to prove $\rightharpoonup^*_{\approx} \subseteq (\rightharpoonup_\circ)^*$, it suffices to check that $(\rightharpoonup_\circ)^*$ is compatible with $\approx$. Then, $\rightharpoonup^*_{\approx} \subseteq (\rightharpoonup_\circ)^*$ will readily follow since $\rightharpoonup^*_{\approx}$ is the least **L**-relation containing $\rightharpoonup$ which is reflexive, $\otimes$-transitive, and compatible with $\approx$. Thus, take any elements $x_1, x_2, y_1, y_2 \in X$. First, observe that for each $z_1 \in X$, the $\otimes$-transitivity of $\approx$ yields

$$\begin{aligned}
&x_1 \approx x_2 \otimes x_1 \rightharpoonup_\circ z_1 = x_1 \approx x_2 \otimes \textstyle\bigvee_{u_1,u_2 \in X} \big(x_1 \approx u_1 \otimes u_1 \rightharpoonup_\circ u_2 \otimes u_2 \approx z_1\big) = \\
= &\textstyle\bigvee_{u_1,u_2 \in X} \big(x_2 \approx x_1 \otimes x_1 \approx u_1 \otimes u_1 \rightharpoonup u_2 \otimes u_2 \approx z_1\big) \leq \\
\leq &\textstyle\bigvee_{u_1,u_2 \in X} \big(x_2 \approx u_1 \otimes u_1 \rightharpoonup u_2 \otimes u_2 \approx z_1\big) = x_2 \rightharpoonup_\circ z_1.
\end{aligned}$$

Analogously, we have $z_k \rightharpoonup_\circ y_1 \otimes y_1 \approx y_2$ for each $z_k \in X$. Using the latter observations,

$$\begin{aligned}
&x_1 \approx x_2 \otimes y_1 \approx y_2 \otimes x_1 \ (\rightharpoonup_\circ)^* \ y_1 = \\
=&x_1 \approx x_2 \otimes y_1 \approx y_2 \otimes \bigvee_{\langle z_1,z_2,\ldots,z_k\rangle \in X^{\mathbb{N}_0}} \big(x_1 \rightharpoonup_\circ z_1 \otimes z_1 \rightharpoonup_\circ z_2 \otimes \cdots \otimes z_k \rightharpoonup_\circ y_1\big) = \\
=&\bigvee_{\langle z_1,z_2,\ldots,z_k\rangle \in X^{\mathbb{N}_0}} \big(x_2 \approx x_1 \otimes x_1 \rightharpoonup_\circ z_1 \otimes z_1 \rightharpoonup_\circ z_2 \otimes \cdots \otimes z_k \rightharpoonup_\circ y_1 \otimes y_1 \approx y_2\big) \leq \\
\leq&\bigvee_{\langle z_1,z_2,\ldots,z_k\rangle \in X^{\mathbb{N}_0}} \big(x_2 \rightharpoonup_\circ z_1 \otimes z_1 \rightharpoonup_\circ z_2 \otimes \cdots \otimes z_k \rightharpoonup_\circ y_2\big) = x_2 \ (\rightharpoonup_\circ)^* \ y_2,
\end{aligned}$$

proving that $(\rightharpoonup_\circ)^*$ is compatible with $\approx$. As a consequence, $\rightharpoonup^*_\approx \subseteq (\rightharpoonup_\circ)^*$. □

Remark 20. Theorem 17 and Theorem 19 yield that if $\rightharpoonup$ is compatible with $\approx$, then $\rightharpoonup_\circ$ equals $\rightharpoonup$ and, hence, $\rightharpoonup^*_\approx$ becomes $\rightharpoonup^*$ which is the reflexive and $\otimes$-transitive closure of $\rightharpoonup$. Thus, in case of $\rightharpoonup$ compatible with $\approx$, the reducibility $\rightharpoonup^*_\approx$ induced by $\rightharpoonup$ and $\approx$ can be seen as reducibility $\rightharpoonup^*$ as it has been introduced in Section 3.1, i.e., without considering $\approx$.
From the epistemic point of view, the latter observation should be interpreted so that if $\rightharpoonup$ is given and one has an additional background knowledge about X formalized by similarity $\approx$ of elements on X, then from the point of view of reductions, the background knowledge is nontrivial if it is not already contained in $\rightharpoonup$, i.e., if $\rightharpoonup$ is not compatible with $\approx$.

We now explore **L**-relations connected to the idea or rewriting which are derived from given $\rightharpoonup$ with respect to $\approx$. We first introduce the notions of convergence and divergence, respectively.

Definition 21. Given $\rightharpoonup$ and $\approx$, we define **L**-relations $\downharpoonleft_\approx$ and $\upharpoonleft_\approx$ by

$$\begin{aligned}
x \downharpoonleft_\approx y &= \textstyle\bigvee_{z \in X}(x \rightharpoonup^*_\approx z \otimes y \rightharpoonup^*_\approx z), \\
x \upharpoonleft_\approx y &= \textstyle\bigvee_{z \in X}(z \rightharpoonup^*_\approx x \otimes z \rightharpoonup^*_\approx y),
\end{aligned}$$

for all $x, y \in X$. The **L**-relations $\downharpoonleft_\approx$ and $\upharpoonleft_\approx$ are called ***convergence*** and ***divergence*** (induced by $\rightharpoonup$ and $\approx$), respectively. The degrees $x \downharpoonleft_\approx y$ and $x \upharpoonleft_\approx y$ are called the degrees to which x and y are convergent and divergent (according to $\rightharpoonup$ with respect to $\approx$), respectively.

Remark 22. (1) Notice that if **L** is a two-valued Boolean algebra and if $\approx$ is the identity, then $\downharpoonleft_\approx$ and $\upharpoonleft_\approx$ become the ordinary convergence and divergence, respectively, see [2, 54].
(2) Note that it might be tempting to define $x \downharpoonleft_\approx y$ as a degree to which x reduces to z_1, y reduces to z_2, and to which z_1 and z_2 are similar. That is:

$$\textstyle\bigvee_{z_1,z_2 \in X}(x \rightharpoonup^*_\approx z_1 \otimes y \rightharpoonup^*_\approx z_2 \otimes z_1 \approx z_2). \tag{3.3}$$

At first sight, (3.3) better agrees with the idea of rewriting under similarity. A second glimpse at (3.3) shows that the degree is exactly $x \downharpoonleft_\approx y$. Indeed,

$$\begin{aligned}
&\textstyle\bigvee_{z_1,z_2 \in X}(x \rightharpoonup^*_\approx z_1 \otimes y \rightharpoonup^*_\approx z_2 \otimes z_1 \approx z_2) = \\
=&\textstyle\bigvee_{z_1 \in X}\big(x \rightharpoonup^*_\approx z_1 \otimes \bigvee_{z_2 \in X}(y \rightharpoonup^*_\approx z_2 \otimes z_1 \approx z_2)\big) = \\
=&\textstyle\bigvee_{z_1 \in X}\big(x \rightharpoonup^*_\approx z_1 \otimes y \rightharpoonup^*_\approx \circ \approx z_1\big) = \bigvee_{z_1 \in X}\big(x \rightharpoonup^*_\approx z_1 \otimes y \rightharpoonup^*_\approx z_1\big) = x \downharpoonleft_\approx y,
\end{aligned}$$

which is a consequence of (3.2). Analogous observation can be made for $\upharpoonleft_\approx$.
(3) Obviously, for $\rightharpoonup$ and $\rightharpoonup_\circ$ defined as in Theorem 19, we have $\downharpoonleft_\circ = \downharpoonleft_\approx$, i.e., the degree $x \downharpoonleft_\circ y$ of convergence without considering any **L**-similarity is equal to the degree $x \downharpoonleft_\approx y$ of convergence with respect to $\approx$. Since $\rightharpoonup_\circ$ is the $\approx$-extensional closure of $\rightharpoonup$, the latter

observation says that the convergence induced by $\rightharpoonup$ with respect to $\approx$ agrees with the convergence induced by the $\approx$-extensional closure of $\rightharpoonup$ in sense of Section 3.1. In addition to that, if $\rightharpoonup$ is compatible with $\approx$, then $\rightharpoonup \, = \, \rightharpoonup_\circ$ and thus $\downharpoonleft \, = \, \downharpoonleft_\approx$. Hence, if $\rightharpoonup$ is compatible with $\approx$, the notion of convergence with respect to $\approx$ becomes the notion of convergence without considering the similarity $\approx$. Analogous observations can be made for $\upharpoonleft_\circ$ and $\upharpoonleft_\approx$.
(4) From the point of view of $\circ$-compositions of **L**-relations, $\downharpoonleft_\approx$ is a $\circ$-composition of $\rightharpoonup^*_\approx$ and its inverse $(\rightharpoonup^*_\approx)^{-1}$ while $\upharpoonleft_\approx$ is a $\circ$-composition of the inverse $(\rightharpoonup^*_\approx)^{-1}$ with $\rightharpoonup^*_\approx$. Thus,

$$\begin{aligned}
\downharpoonleft_\approx &= \rightharpoonup^*_\approx \circ (\rightharpoonup^*_\approx)^{-1} = {\rightharpoonup_\circ}^* \circ ({\rightharpoonup_\circ}^*)^{-1} = {\rightharpoonup_\circ}^* \circ {\leftharpoondown_\circ}^* = \rightharpoonup^*_\approx \circ \leftharpoondown^*_\approx, \\
\upharpoonleft_\approx &= (\rightharpoonup^*_\approx)^{-1} \circ \rightharpoonup^*_\approx = ({\rightharpoonup_\circ}^*)^{-1} \circ {\rightharpoonup_\circ}^* = {\leftharpoondown_\circ}^* \circ {\rightharpoonup_\circ}^* = \leftharpoondown^*_\approx \circ \rightharpoonup^*_\approx,
\end{aligned}$$

on account of Theorem 19.

In order to study properties of **L**-relations related to rewriting, we need a notion of convertibility. The convertibility is in fact a reducibility relation induced by a symmetric closure of $\rightharpoonup$.

Definition 23. Given $\rightharpoonup$ and $\approx$, the **L**-relation $\rightleftharpoons^*_\approx$, where $\rightleftharpoons$ is defined by $\rightharpoonup \cup \rightharpoonup^{-1}$, is called a ***convertibility*** (induced by $\rightharpoonup$ and $\approx$). The degree $x \rightleftharpoons^*_\approx y \in L$ is a degree to which x and y are convertible with respect to $\rightharpoonup$ and $\approx$.

Remark 24. (1) Following Definition 15 and Definition 23, the degree $x \rightleftharpoons^*_\approx y$ to which x and y are convertible with respect to $\rightharpoonup$ and $\approx$ is given by

$$x \rightleftharpoons^*_\approx y = \bigvee_{\langle z_1, z_2, \dots, z_{2k}\rangle \in X^{2\mathbb{N}_0}} \big(x \approx z_1 \otimes z_1 \rightleftharpoons z_2 \otimes z_2 \approx z_3 \otimes \cdots \otimes z_{2k-1} \rightleftharpoons z_{2k} \otimes z_{2k} \approx y\big),$$

where $X^{2\mathbb{N}_0} = \bigcup_{n \in \mathbb{N}_0} X^{2n}$, i.e. $X^{2\mathbb{N}_0}$ is a union of all even Cartesian powers of the set X.
(2) As in cases of reducibility, convergence, and divergence, the notion of convertibility introduced in Definition 23 generalizes the corresponding notion known in abstract rewriting systems [2, 54] as well as the generalizations shown in Section 3.1. Indeed, if $\approx$ is identity, $\rightleftharpoons^*_\approx$ becomes $\rightleftharpoons^*$. In addition, if **L** is a two-valued Boolean algebra, $\rightleftharpoons^*_\approx$ becomes the ordinary convertibility which is the least equivalence relation induced by $\rightharpoonup$.

Closure properties of $\rightleftharpoons^*_\approx$ are characterized by the following assertion.

Theorem 25. *$\rightleftharpoons^*_\approx$ is a reflexive, symmetric, $\otimes$-transitive, and $\approx$-extensional closure of $\rightharpoonup$. As a consequence, $\rightleftharpoons^*_\approx$ is the least* **L***-similarity containing $\rightharpoonup$ which is compatible with $\approx$.*

Proof. According to Theorem 17, $\rightleftharpoons^*_\approx$ is a reflexive, $\otimes$-transitive, and $\approx$-extensional closure of $\rightleftharpoons$. Thus, it is sufficient to show that $\rightleftharpoons^*_\approx$ is symmetric. The symmetry of $\rightleftharpoons^*_\approx$ is but a direct consequence of the facts that both $\approx$ and $\rightleftharpoons$ are symmetric. In a more detail, take any $x, z_1, z_2, \dots, z_{2k}, y \in X$ and observe that

$$\begin{aligned}
&x \approx z_1 \otimes z_1 \rightleftharpoons z_2 \otimes z_2 \approx z_3 \otimes \cdots \otimes z_{2k-1} \rightleftharpoons z_{2k} \otimes z_{2k} \approx y = \\
&= y \approx z_{2k} \otimes z_{2k} \rightleftharpoons z_{2k-1} \otimes \cdots \otimes z_3 \approx z_2 \otimes z_2 \rightleftharpoons z_1 \otimes z_1 \approx x,
\end{aligned}$$

which follows by the symmetry of $\approx$ and $\rightleftharpoons$ together with the fact that $\otimes$ is commutative and associative. Therefore, $x \rightleftharpoons^*_\approx y = y \rightleftharpoons^*_\approx x$ for all $x, y \in X$, showing $\rightleftharpoons^*_\approx = (\rightleftharpoons^*_\approx)^{-1}$, i.e. $\rightleftharpoons$ is symmetric. Thus, $\rightleftharpoons^*_\approx$ is the least **L**-similarity containing $\rightharpoonup$ which is compatible with $\approx$. □

Corollary 26. *For any* $\rightharpoonup$, $\approx$ *and* $\rightleftharpoons_\circ$ *defined by* $\rightleftharpoons_\circ = (\approx \circ \rightharpoonup \circ \approx) \cup (\approx \circ \leftharpoondown \circ \approx)$, *we have* $\rightleftharpoons^*_\approx = (\rightleftharpoons_\circ)^*$.

Proof. According to distributivity of $\circ$ over $\cup$ [5], we immediately get $(\approx \circ \rightharpoonup \circ \approx) \cup (\approx \circ \leftharpoondown \circ \approx) = \approx \circ [(\rightharpoonup \circ \approx) \cup (\leftharpoondown \circ \approx)] = \approx \circ (\rightharpoonup \cup \leftharpoondown) \circ \approx = \approx \circ \rightleftharpoons \circ \approx$. The rest follows from Theorem 19 for $\rightharpoonup$ being $\rightleftharpoons$. $\square$

Similar observations as in Remark 22 (2) can be made about the convertibility degrees $\rightleftharpoons^*_\approx$ and $(\rightleftharpoons_\circ)^*$.

So far, we have introduced derived **L**-relations based on $\rightharpoonup$ and $\approx$. From now on, we turn our attention to properties of $\rightharpoonup$ determined from the derived **L**-relations.

Definition 27. The degree $\mathrm{CFL}(\rightharpoonup)_\approx$ to which $\rightharpoonup$ is ***confluent*** with respect $\approx$ is defined by $\mathrm{CFL}(\rightharpoonup)_\approx = S(\upharpoonleft_\approx, \downharpoonleft_\approx)$. The degree $\mathrm{CR}(\rightharpoonup)_\approx$ to which $\rightharpoonup$ has the ***Church-Rosser property*** with respect to $\approx$ is defined by $\mathrm{CR}(\rightharpoonup)_\approx = S(\rightleftharpoons^*_\approx, \downharpoonleft_\approx)$.

Remark 28. Due to Remark 22 (2) and Corollary 26, we have $\mathrm{CFL}(\rightharpoonup)_\approx = \mathrm{CFL}(\rightharpoonup_\circ)$ and $\mathrm{CR}(\rightharpoonup)_\approx = \mathrm{CR}(\rightharpoonup_\circ)$, where $\mathrm{CFL}(\rightharpoonup_\circ)$ denotes the degree of confluence of $\rightharpoonup_\circ$ and $\mathrm{CR}(\rightharpoonup_\circ)$ denotes the degree to which $\rightharpoonup_\circ$ has the Church-Rosser property without considering a similarity on X. As a further consequence of these observations, if $\rightharpoonup$ is compatible with $\approx$, $\mathrm{CFL}(\rightharpoonup)_\approx = \mathrm{CFL}(\rightharpoonup)$ and $\mathrm{CR}(\rightharpoonup)_\approx = \mathrm{CR}(\rightharpoonup)$.

Further properties of CR and CFL can be proved by a reductionist approach using the fact that instead of considering $\rightharpoonup$ over $\langle X, \approx \rangle$ one may take $\rightharpoonup_\circ = \approx \circ \rightharpoonup \circ \approx$ over X. Using this approach, we can prove that the degree to which $\rightharpoonup$ has the Church-Rosser property with respect to $\approx$ is the degree to which convertibility $\rightleftharpoons^*_\approx$ is equal to convergence $\downharpoonleft_\approx$.

Theorem 29. *For each* $\rightharpoonup$ *and* $\approx$, *we get* $\mathrm{CR}(\rightharpoonup)_\approx = E(\rightleftharpoons^*_\approx, \downharpoonleft_\approx)$.

Proof. Take $\rightharpoonup_\circ$ and apply observations from Remark 28 together with Theorem 7 and observe that $\mathrm{CR}(\rightharpoonup)_\approx = \mathrm{CR}(\rightharpoonup_\circ) = E(\rightleftharpoons^*_\circ, \downharpoonleft_\circ) = E(\rightleftharpoons^*_\approx, \downharpoonleft_\approx)$, proving the claim. $\square$

The relationship between degrees of confluence and degrees to which **L**-relations relations have the Church-Rosser property can be characterized as follows:

Theorem 30. *If* $\mathrm{CFL}(\rightharpoonup)_\approx$ *is an idempotent element of* **L** *then* $\mathrm{CR}(\rightharpoonup)_\approx = \mathrm{CFL}(\rightharpoonup)_\approx$.

Proof. Using Remark 28, we get $\mathrm{CR}(\rightharpoonup)_\approx = \mathrm{CR}(\rightharpoonup_\circ)$ as well as $\mathrm{CFL}(\rightharpoonup)_\approx = \mathrm{CFL}(\rightharpoonup_\circ)$. The rest follows by Theorem 11 applied to $\mathrm{CR}(\rightharpoonup_\circ)$ and $\mathrm{CFL}(\rightharpoonup_\circ)$. Indeed, $\mathrm{CR}(\rightharpoonup)_\approx = \mathrm{CR}(\rightharpoonup_\circ) = \mathrm{CFL}(\rightharpoonup_\circ) = \mathrm{CFL}(\rightharpoonup)_\approx$, proving the claim. $\square$

If $\otimes = \wedge$ then $\mathrm{CR}(\rightharpoonup)_\approx = \mathrm{CFL}(\rightharpoonup)_\approx$ which is an obvious corollary of Theorem 30. Moreover, one can claim that an **L**-relation $\rightharpoonup$ has the Church-Rosser property with respect to an **L**-similarity $\approx$ iff it is confluent with respect to $\approx$. This follows immediately from the fact that $1 \in L$ is idempotent.

3.2.2 Substitutability on Generalized Pseudometric Spaces

In this subsection, we introduce an alternative semantics of rewriting under similarity. Unlike the approach from the previous subsection, we consider a distance-based similarity which is formalized by considering generalized pseudometric spaces as a universe of discourse. We introduce notions related to reducibility by allowing to "jump" to elements based on their distance. Furthermore, we show a connection between the notions and their counterparts from the previous subsection. Throughout this subsection, we let $\langle X, \delta\rangle$ be a generalized pseudometric space and $\rightsquigarrow$ be a (classical) reduction relation on X.

We first introduce the notion of reducibility:

Definition 31. An element $x \in X$ is said to be ***reducible*** *to* $y \in X$ *by* $\rightsquigarrow$ *with the* ***cumulative jump distance*** $d < \infty$ *with respect to* δ if there is a sequence of elements $z_1, \ldots, z_{2k} \in X$ such that $z_1 \rightsquigarrow z_2$ and $z_3 \rightsquigarrow z_4$ and …and $z_{2k-1} \rightsquigarrow z_{2k}$ and $\delta(x, z_1) + \delta(z_2, z_3) + \cdots + \delta(z_{2k}, y) = d$. This fact will be briefly denoted by $x \rightsquigarrow_\delta^d y$.

Remark 32. (1) It can be easily seen that the notion of reducibility introduced by Definition 31 generalizes the classic notion of reducibility. Namely, if x is reducible to y in the usual sense, then one can put $z_{2i} = z_{2i+1}$ for all i and thus $\delta(z_{2i}, z_{2i+1}) = 0$ gives that x is reducible to y with the cumulative jump distance 0. The converse is not true in general, i.e., x can be reducible to y with the cumulative jump distance 0 while x is not reducible to y in the ordinary sense since distinct elements of the universe can have a zero distance. (2) The presence of jumps reflects the idea of approximate rewriting if one is allowed to substitute an element by another one which is sufficiently near (note that according to Definition 31, $\delta(x, y) = \infty$ means that a jump from x to y is not possible). Needless to say, there may be other reasonable ways to capture such requirement (e.g., one can restrict the distance of a single jump between applications of the given reduction relation). We elaborate here on the approach from Definition 31 and leave other possible extensions to interested readers.

Definition 33. Elements $x, y \in X$ are said to be ***convergent*** *with the cumulative jump distance* d *with respect to* δ if there is $z \in X$ such that $x \rightsquigarrow_\delta^{d_1} z$, $y \rightsquigarrow_\delta^{d_2} z$, and $d_1 + d_2 = d$. We denote the fact shortly by $x \downarrow_\delta^d y$. Analogously, $x, y \in X$ are said to be ***divergent*** *with the cumulative jump distance* d *with respect to* δ (written $x \uparrow_\delta^d y$) if there is $z \in X$ such that $z \rightsquigarrow_\delta^{d_1} x$, $z \rightsquigarrow_\delta^{d_2} y$, and $d_1 + d_2 = d$.

Notice that all the notions introduced so far are bivalent. Given a threshold d, either x is reducible to y with the cumulative jump distance d or not. Analogously in cases of convergence and divergence. Using divergence and convergence, we can introduce confluence in a fairly standard way:

Definition 34. A relation $\rightsquigarrow$ is called ***confluent*** *with respect to* δ if for each $x, y \in X$, $x \uparrow_\delta^{d_1} y$ implies $x \downarrow_\delta^{d_2} y$ with $d_1 \geq d_2$.

Described verbally, $\rightsquigarrow$ is confluent with respect to δ whenever any x and y which are divergent with the cumulative jump distance d are convergent with a cumulative jump distance at most d. Thus, the notion from Definition 34 generalizes the ordinary notion of confluence. The notions of convertibility and the Church-Roser property can be introduced in a similar way. Now we could examine properties of the proposed notions and their mutual relationship but we leave it for interested readers and deliberately go in a different direction: we show a connection between the notions introduced in this subsection for an ordinary relation $\rightsquigarrow$ on a generalized pseudometric space $\langle X, \delta\rangle$ and the notions from Subsection 3.2.1 which are dealing with an $\mathbf{L}$-relation on a similarity space $\langle X, \approx\rangle$.
In the rest of this subsection, we let $\mathbf{L} = \langle [0, 1], \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ be a complete residuated lattice defined on the real unit interval with $\otimes$ being a continuous Archimedean t-norm with a continuous additive generator f. Denote the pseudoinverse of f by $f^{(-1)}$. Let $\langle X, \delta\rangle$ be a generalized pseudometric space and denote by $\approx$ the similarity $\mathbf{L}$-relation corresponding to δ in the sense of [4, 5]. This way, $\langle X, \approx\rangle$ is a similarity space corresponding to $\langle X, \delta\rangle$.

Furthermore, $\rightharpoonup$ be an ordinary relation on X. As it is usual, we can consider $\rightharpoonup$ as a crisp **L**-relation on X, i.e., $x \rightharpoonup y \in \{0,1\}$ for all $x, y \in X$.

The following two assertions chartacterize the basic relationship between the notions of reducibility that appear in the proposed generalizations:

Theorem 35. *For any $x, y \in X$, $x \rightharpoonup_{\delta}^{d} y$ implies $x \rightharpoonup_{\approx}^{*} y \geq f^{(-1)}(d)$.*

Proof. By definition of $x \rightharpoonup_{\delta}^{d} y$, there are elements $z_1, \dots, z_{2k} \in X$ such that $z_1 \rightharpoonup z_2$ and $z_3 \rightharpoonup z_4$ and, ..., and $z_{2k-1} \rightharpoonup z_{2k}$ and $\delta(x, z_1) + \delta(z_2, z_3) + \cdots + \delta(z_{2k}, y) = d$.
Thus, for $z_1, \dots, z_{2k} \in X$, we immediately have

$$\begin{aligned}
&x \rightharpoonup_{\approx}^{*} y = \textstyle\bigvee_{\langle v_1,\dots,v_{2m}\rangle \in X^{2\mathbb{N}_0}} x \approx v_1 \otimes v_1 \rightharpoonup v_2 \otimes \cdots \otimes v_{2m-1} \rightharpoonup v_{2m} \otimes v_{2m} \approx y \geq \\
&\geq x \approx z_1 \otimes z_1 \rightharpoonup z_2 \otimes \cdots \otimes z_{2k-1} \rightharpoonup z_{2k} \otimes z_{2k} \approx y = \\
&= x \approx z_1 \otimes z_2 \approx z_3 \otimes \cdots \otimes z_{2k-2} \approx z_{2k-1} \otimes z_{2k} \approx y.
\end{aligned}$$

Using the basic properties of the additive generator f and its pseudoinverse $f^{(-1)}$, we can rewrite $\otimes$ and $\approx$ in the foregoing expression to get

$$\begin{aligned}
&x \approx z_1 \otimes z_2 \approx z_3 \otimes \cdots \otimes z_{2k-2} \approx z_{2k-1} \otimes z_{2k} \approx y = \\
&= f^{(-1)}\left[f\left[f^{(-1)}\left(\cdots\left(f\left[f^{(-1)}\left(f\left(x \approx z_1\right) + f\left(z_2 \approx z_3\right)\right)\right] + f\left(z_4 \approx z_5\right)\right)\cdots\right)\right] + \\
&\quad + f\left(z_{2k} \approx y\right)\right] = f^{(-1)}\left[f\left(x \approx z_1\right) + f\left(z_2 \approx z_3\right) + f\left(z_4 \approx z_5\right) + \cdots + f\left(z_{2k} \approx y\right)\right] = \\
&= f^{(-1)}\left[f\left(f^{(-1)}\left(\delta(x, z_1)\right)\right) + f\left(f^{(-1)}\left(\delta(z_2, z_3)\right)\right) + \cdots + f\left(f^{(-1)}\left(\delta(z_{2k}, y)\right)\right)\right] = \\
&= f^{(-1)}\left[\delta(x, z_1) + \delta(z_2, z_3) + \cdots + \delta(z_{2k}, y)\right] = f^{(-1)}(d)
\end{aligned}$$

which concludes the proof. □

Theorem 36. *For any $x, y \in X$, $x \rightharpoonup_{\approx}^{*} y = a > 0$ implies that there is a nonincreasing sequence $\{d_n\}_{n=1}^{\infty}$ of distances such that $f(a) = \lim_{n\to\infty} d_n$ and $x \rightharpoonup_{\delta}^{d_n} y$ for each $n \in \mathbb{N}$.*

Proof. By definition of $\rightharpoonup_{\approx}^{*}$ and the assumption $x \rightharpoonup_{\approx}^{*} y > 0$, we immediately get that $x \rightharpoonup_{\approx}^{*} y$ is a supremum of a nonempty set $\{c_i \,|\, i \in I\}$ of nonzero degrees of the from

$$c_i = x \approx z_{i,1} \otimes z_{i,1} \rightharpoonup z_{i,2} \otimes \cdots \otimes z_{i,2k_i} \approx y > 0$$

for $\langle z_{i,1}, z_{i,2}, \dots, z_{i,2k_i}\rangle \in X^{2\mathbb{N}_0}$ $(i \in I)$. Furthermore, using the fact that $\rightharpoonup$ is a crisp **L**-relation, each c_i simplifies to $c_i = x \approx z_{i,1} \otimes z_{i,2} \approx z_{i,3} \otimes \cdots \otimes z_{i,2k_i} \approx y$. As in the previous proof,

$$\begin{aligned}
c_i &= x \approx z_{i,1} \otimes z_{i,2} \approx z_{i,3} \otimes \cdots \otimes z_{i,2k_i} \approx y = \\
&= f^{(-1)}\big(\delta(x, z_{i,1}) + \delta(z_{i,2}, z_{i,3}) + \cdots + \delta(z_{i,2k_i}, y)\big) = f^{(-1)}(d_i)
\end{aligned}$$

for $d_i = \delta(x, z_{i,1}) + \delta(z_{i,2}, z_{i,3}) + \cdots + \delta(z_{i,2k_i}, y)$. As a consequence, $x \rightharpoonup_{\delta}^{d_i} y$ $(i \in I)$ and we can write $x \rightharpoonup_{\approx}^{*} y = a = \bigvee_{i\in I} f^{(-1)}(d_i)$. The latter equality gives $f(a) = f\big(\bigvee_{i\in I} f^{(-1)}(d_i)\big)$. Since f is decreasing and continuous, $f(a) = \bigwedge_{i\in I} f\big(f^{(-1)}(d_i)\big) = \bigwedge_{i\in I} d_i$. Now, by a standard argument, select from $\{d_i \,|\, i \in I\}$ a nonincreasing sequence of values such that $\lim_{n\to\infty} d_n = \bigwedge_{i\in I} d_i$ (trivial if $\{d_i \,|\, i \in I\}$ has a least element; otherwise for each natural n, take d_n such that $\bigwedge_{i\in I} d_i < d_n \leq \bigwedge_{i\in I} d_i + \frac{1}{n}$ and $d_{n+1} \leq d_n$). □

Remark 37. If X is finite, for each x and y satisfying $x \rightharpoonup_{\approx}^{*} y > 0$ there is a sequence $z_1, \ldots, z_{2k} \in X$ such that $x \rightharpoonup_{\approx}^{*} y = x \approx z_1 \otimes z_1 \rightharpoonup z_2 \otimes \cdots \otimes z_{2k-1} \rightharpoonup z_{2k} \otimes z_{2k} \approx y$ (recall that our $\mathbf{L}$ is linearly ordered). Therefore, Theorem 36 yields that from $x \rightharpoonup_{\approx}^{*} y = \bigvee\{f^{(-1)}(d) \mid x \rightharpoonup_{\delta}^{d} y\}$ we get that $x \rightharpoonup_{\approx}^{*} y = f^{(-1)}(d)$ such that d is the least distance for which $x \rightharpoonup_{\delta}^{d} y$.

Theorem 38. *Let $x, y \in X$ be arbitrary elements. Then, $x \upharpoonleft_{\delta}^{d} y$ implies $x \upharpoonleft_{\approx} y \geq f^{(-1)}(d)$ and $x \downharpoonleft_{\delta}^{d} y$ implies $x \downharpoonleft_{\approx} y \geq f^{(-1)}(d)$.*

Proof. Suppose that $x \upharpoonleft_{\delta}^{d} y$, i.e., there is an element $z \in X$ such that $z \rightharpoonup_{\delta}^{d_1} x$ and $z \rightharpoonup_{\delta}^{d_2} y$ and $d = d_1 + d_2$. Using Theorem 35, we immediately get

$$x \upharpoonleft_{\approx} y = \textstyle\bigvee_{w \in X} w \rightharpoonup_{\approx}^{*} x \otimes w \rightharpoonup_{\approx}^{*} y \geq z \rightharpoonup_{\approx}^{*} x \otimes z \rightharpoonup_{\approx}^{*} y \geq f^{(-1)}(d_1) \otimes f^{(-1)}(d_2) =$$
$$= f^{(-1)}\left(f\left(f^{(-1)}(d_1)\right) + f\left(f^{(-1)}(d_2)\right)\right) = f^{(-1)}(d_1 + d_2) = f^{(-1)}(d)$$

which concludes the first part of the proof. The second implication can be proved analogously. □

Theorem 39. *For arbitrary $x, y \in X$, if $x \upharpoonleft_{\approx} y = a > 0$ then there is a nonincreasing sequence of distances $\{d_n\}_{n=1}^{\infty}$ such that $f(a) = \lim_{n\to\infty} d_n$ and $x \upharpoonleft_{\delta}^{d_n} y$ for each $n \in \mathbb{N}$. Analogously for $\downharpoonleft_{\approx}$.*

Proof. Using Theorem 36 and basic properties of f and $f^{(-1)}$, we obtain

$$a = x \upharpoonleft_{\approx} y = \textstyle\bigvee_{i \in I} z_i \rightharpoonup_{\approx}^{*} x \otimes z_i \rightharpoonup_{\approx}^{*} y = \bigvee_{i \in I}\left(\bigvee_{j \in J} f^{(-1)}(g_{i,j})\right) \otimes \left(\bigvee_{k \in K} f^{(-1)}(h_{i,k})\right) =$$
$$= \textstyle\bigvee_{i \in I, j \in J, k \in K} f^{(-1)}(g_{i,j}) \otimes f^{(-1)}(h_{i,k}) = \bigvee_{i \in I, j \in J, k \in K} f^{(-1)}(g_{i,j} + h_{i,k}),$$

where $z \rightharpoonup_{\delta}^{g_{i,j}} x$ and $z \rightharpoonup_{\delta}^{h_{i,k}} y$ for each $i \in I$, $j \in J$, and $k \in K$. Therefore, we may write $a = \bigvee_{m \in I \times J \times K} f^{(-1)}(d_m)$ where $d_m = g_{i,j} + h_{i,k}$ and $x \upharpoonleft_{\delta}^{d_m} y$ for each $m = \langle i, j, k\rangle \in I \times J \times K$. Applying basic properties of f and $f^{(-1)}$, we may proceed as in the proof of Theorem 36: We get that $f(a) = \bigwedge_{m \in I \times J \times K} d_m$, i.e., we may pick from d_m ($m \in I \times J \times K$) a nonincreasing sequence $\{d_n\}_{n=1}^{\infty}$ such that $f(a) = \bigwedge_{m \in I \times J \times K} d_m = \lim_{n\to\infty} d_n$, finishing the proof. For $\downharpoonleft_{\approx}$, one proceeds analogously. □

Theorem 40. *If $\rightharpoonup$ is confluent with respect to δ then* $\mathrm{CFL}(\rightharpoonup)_{\approx} = 1$.

Proof. Let $\rightharpoonup$ be confluent with respect to δ and take arbitrary $x, y \in X$. If $x \upharpoonleft_{\approx} y = 0$ then we trivially get $x \upharpoonleft_{\approx} y \leq x \downharpoonleft_{\approx} y$. Now, assume that $x \upharpoonleft_{\approx} y = a > 0$. By Theorem 39, there are distances $\{d_n\}_{n=1}^{\infty}$ such that $x \upharpoonleft_{\delta}^{d_n} y$ for each $n \in \mathbb{N}$ and $a = \bigvee_{n \in \mathbb{N}} f^{(-1)}(d_n)$. Since $\rightharpoonup$ is confluent with respect to δ, for each $n \in \mathbb{N}$ there is some distance $d'_n \leq d_n$ such that $x \downharpoonleft_{\delta}^{d'_n} y$. As a consequence, $f^{(-1)}(d'_n) \geq f^{(-1)}(d_n)$ for each $n \in \mathbb{N}$. Due to Theorem 38, we immediately get $x \downharpoonleft_{\approx} y \geq f^{(-1)}(d'_n)$ for each $n \in \mathbb{N}$. Altogether, we obtain

$$x \upharpoonleft_{\approx} y = \textstyle\bigvee_{n \in \mathbb{N}} f^{(-1)}(d_n) \leq \bigvee_{n \in \mathbb{N}} f^{(-1)}(d'_n) \leq x \downharpoonleft_{\approx} y$$

for each $x, y \in X$, i.e., $\mathrm{CFL}(\rightharpoonup)_{\approx} = 1$. □

Theorem 41. *Let $\mathrm{CFL}(\rightharpoonup)_{\approx} = 1$. If $x \upharpoonleft_{\delta}^{d} y$ and $d < f(0)$, then there is a nonincreasing sequence $\{d_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} d_n \leq d$ and $x \downharpoonleft_{\delta}^{d_n} y$ for each $n \in \mathbb{N}$.*

Proof. Assume that $\mathrm{CFL}(\rightharpoonup)_{\approx}=1$, i.e., $x \upharpoonleft_{\approx} y \le x \downharpoonleft_{\approx} y$ for each $x, y \in X$. Take $x, y \in X$ such that $x \upharpoonleft_{\delta}^{d} y$ and $f(0) > d$. Now, observe that using properties of f and $f^{(-1)}$, we get $0 < f^{(-1)}(d)$. Hence, by Theorem 38 and the fact that $\mathrm{CFL}(\rightharpoonup)_{\approx}=1$, we get $0 < f^{(-1)}(d) \le x \upharpoonleft_{\approx} y \le x \downharpoonleft_{\approx} y$. Furthermore, properties of f and $f^{(-1)}$ yield $f(x \downharpoonleft_{\approx} y) \le d$. Applying Theorem 39, there is a nonincreasing sequence $\{d_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} d_n = f(x \downharpoonleft_{\approx} y) \le d$ and $x \downharpoonleft_{\delta}^{d_n} y$ for each $n \in \mathbb{N}$. □

Remark 42. Note that the value $f(0)$ can be seen as a distinguishability threshold while transferring the background knowledge on distance from $\langle X, \delta\rangle$ to $\langle X, \delta_{\approx}\rangle$. Theorem 41 shall be understood so that the full confluence of $\rightharpoonup$ with respect to $\approx$ implies confluence of $\rightharpoonup$ with respect to δ except the cases when the cumulative jump distance exceed the threshold distance $f(0)$. From this point of view, the property of $\rightharpoonup$ "being confluent with respect to δ" is stronger then the property "being fully confluent with respect to $\approx_{\delta}$", cf. Theorem 40 and Theorem 41.

Similar observations to Remark 37 can also be made for divergence, convergence and confluence. Moreover, one can easily develop further properties related to rewriting in generalized pseudometric spaces (convertibility, Church-Roser property, . . .).

3.3 Similarity Issues of Confluence of Fuzzy Relations

In Section 3.2, the notions of confluence and some other related properties which respect a fuzzy reduction relation and a fuzzy equivalence were introduced. Since all these notions are naturally graded, one can examine if usage of two similar reductions and two similar equivalences always yields to somehow similar degrees of the properties investigated in the first part of this chapter. Our results dealing with these similarity issues are presented in Subsection 3.3.2. Furthermore, we have also constructed some estimation formulas for the degrees of confluence of derived fuzzy reductions which are shown in Subsection 3.3.3.

The results summarized in this section have been published in [36].

3.3.1 Monotony of reducibility degrees

Let $\rightharpoonup$ be an **L**-relation on a similarity space $\langle X, \approx\rangle$. Recall from Section 3.2 that $\rightharpoonup_{\approx}^{*}$ is called a reducibility **L**-relation induced by $\rightharpoonup$ with respect to $\approx$. In this subsection, we deal with properties of the corresponding operator $^{*}_{\approx}$, i.e., the operator which associates to each $\rightharpoonup$ the reducibility $\rightharpoonup_{\approx}^{*}$ induced by $\rightharpoonup$ and $\approx$. In particular, we will focus on properties related to monotony here.

Directly from its definition, we can observe that $^{*}_{\approx}$ is monotone in the usual sense, i.e., $x \rightharpoonup_1 y \le x \rightharpoonup_2 y$ implies $x(\rightharpoonup_1)_{\approx}^{*} y \le x(\rightharpoonup_2)_{\approx}^{*} y$ for any **L**-relations $\rightharpoonup_1$, $\rightharpoonup_2$ on X and for any elements $x, y \in X$. Hence, $\rightharpoonup_1 \subseteq \rightharpoonup_2$ implies $(\rightharpoonup_1)_{\approx}^{*} \subseteq (\rightharpoonup_2)_{\approx}^{*}$ for any $\rightharpoonup_1$, $\rightharpoonup_2$.

A question is whether $^{*}_{\approx}$ satisfies any stronger from of monotony. It can be easily shown that $^{*}_{\approx}$ does not satisfy the graded monotony $S(\rightharpoonup_1, \rightharpoonup_2) \le S((\rightharpoonup_1)_{\approx}^{*}, (\rightharpoonup_2)_{\approx}^{*})$ which often appears in literature [5], see Example 43.

Example 43. Let $\langle X, \approx\rangle$ be a similarity space with the standard Łukasiewicz structure of truth degrees on the real unit interval with $\approx$ being the (fuzzy) identity relation ($x \approx y = 1$ if $x = y$ and $x \approx y = 0$ otherwise). Let $\rightharpoonup_1$ and $\rightharpoonup_2$ be **L**-relations defined as in Figure 3.4 by the degrees written abote the solid lines (the other degrees are zero). For instance, the value 1 at the solid arrow from x_1 to x_2 in the left part of the figure means that x_1 can be fully reduced to x_2 by $\rightharpoonup_1$, i.e. $x_1 \rightharpoonup_1 x_2 = 1$. If there is no arrow between elements

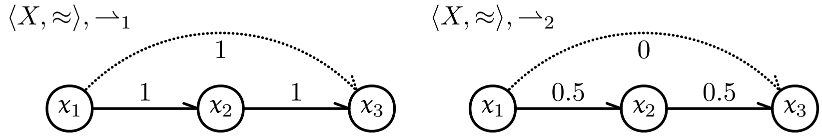

Figure 3.4: Illustration to Example 43.

then the corresponding degree is 0, e.g. $x_1 \rightharpoonup_1 x_3 = 0$. The dotted arrows show selected degrees of reducibility by $(\rightharpoonup_1)^*_\approx$ and $(\rightharpoonup_2)^*_\approx$. The degree of subsethood $\rightharpoonup_1$ in $\rightharpoonup_2$ can be calculated directly from its definition as follows:

$$\begin{aligned}
&S(\rightharpoonup_1, \rightharpoonup_2) = \textstyle\bigwedge_{x,y\in X}(x \rightharpoonup_1 y \rightarrow x \rightharpoonup_2 y) = \\
&= \min(x_1 \rightharpoonup_1 x_1 \rightarrow x_1 \rightharpoonup_2 x_1,\ x_1 \rightharpoonup_1 x_2 \rightarrow x_1 \rightharpoonup_2 x_2,\ x_1 \rightharpoonup_1 x_3 \rightarrow x_1 \rightharpoonup_2 x_3, \\
&\quad x_2 \rightharpoonup_1 x_1 \rightarrow x_2 \rightharpoonup_2 x_1,\ x_2 \rightharpoonup_1 x_2 \rightarrow x_2 \rightharpoonup_2 x_2,\ x_2 \rightharpoonup_1 x_3 \rightarrow x_2 \rightharpoonup_2 x_3, \\
&\quad x_3 \rightharpoonup_1 x_1 \rightarrow x_3 \rightharpoonup_2 x_1,\ x_3 \rightharpoonup_1 x_2 \rightarrow x_3 \rightharpoonup_2 x_2,\ x_3 \rightharpoonup_1 x_3 \rightarrow x_3 \rightharpoonup_2 x_3) = \\
&= \min(1,\ 0.5,\ 1,\ 1,\ 1,\ 0.5,\ 1,\ 1,\ 1) = 0.5\,.
\end{aligned}$$

In a similar way, we can compute $S((\rightharpoonup_1)^*_\approx, (\rightharpoonup_2)^*_\approx)$:

$$\begin{aligned}
&S((\rightharpoonup_1)^*_\approx, (\rightharpoonup_2)^*_\approx) = \textstyle\bigwedge_{x,y\in X}(x(\rightharpoonup_1)^*_\approx y \rightarrow x(\rightharpoonup_2)^*_\approx y) = \\
&= \min(x_1(\rightharpoonup_1)^*_\approx x_1 \rightarrow x_1(\rightharpoonup_2)^*_\approx x_1,\ x_1(\rightharpoonup_1)^*_\approx x_2 \rightarrow x_1(\rightharpoonup_2)^*_\approx x_2,\ x_1(\rightharpoonup_1)^*_\approx x_3 \rightarrow x_1(\rightharpoonup_2)^*_\approx x_3, \\
&\quad x_2(\rightharpoonup_1)^*_\approx x_1 \rightarrow x_2(\rightharpoonup_2)^*_\approx x_1,\ x_2(\rightharpoonup_1)^*_\approx x_2 \rightarrow x_2(\rightharpoonup_2)^*_\approx x_2,\ x_2(\rightharpoonup_1)^*_\approx x_3 \rightarrow x_2(\rightharpoonup_2)^*_\approx x_3, \\
&\quad x_3(\rightharpoonup_1)^*_\approx x_1 \rightarrow x_3(\rightharpoonup_2)^*_\approx x_1,\ x_3(\rightharpoonup_1)^*_\approx x_2 \rightarrow x_3(\rightharpoonup_2)^*_\approx x_2,\ x_3(\rightharpoonup_1)^*_\approx x_3 \rightarrow x_3(\rightharpoonup_2)^*_\approx x_3) = \\
&= \min(1,\ 0.5,\ 0,\ 1,\ 1,\ 0.5,\ 1,\ 1,\ 1) = 0\,.
\end{aligned}$$

In this particular case, we get $S(\rightharpoonup_1, \rightharpoonup_2) > S((\rightharpoonup_1)^*_\approx, (\rightharpoonup_2)^*_\approx)$. Hence, the inequality $S(\rightharpoonup_1, \rightharpoonup_2) \leq S((\rightharpoonup_1)^*_\approx, (\rightharpoonup_2)^*_\approx)$ does not hold in general.

Although the operator $^*_\approx$ does not satisfy the most common graded form of monotony as it is demonstrated by the foregoing example, a weaker form of graded monotony can be established which is shown by the following two assertions.

Theorem 44. *Let $\rightharpoonup_1$, $\rightharpoonup_2$ be* **L***-relations on X, $\approx_1$, $\approx_2$ be* **L***-similarities on X. The inequality $S(\approx_1, \approx_2)^{n+1} \otimes S(\rightharpoonup_1, \rightharpoonup_2)^n \leq S((\rightharpoonup_1)^n_{\approx_1}, (\rightharpoonup_2)^n_{\approx_2})$ holds for each $n \in \mathbb{N}_0$, where $(\rightharpoonup_i)^n_{\approx_j}$ (for $i, j \in \{1, 2\}$) denotes $\approx_j$ if $n = 0$ and $\approx_j \circ \rightharpoonup_i \circ (\rightharpoonup_i)^{n-1}_{\approx_j}$ otherwise.*

Proof. The inequality will be proved by induction. For $n = 0$, the inequality $S(\approx_1, \approx_2)^1 \otimes S(\rightharpoonup_1, \rightharpoonup_2)^0 = S(\approx_1, \approx_2) \otimes 1 = S(\approx_1, \approx_2) = S((\rightharpoonup_1)^0_{\approx_1}, (\rightharpoonup_2)^0_{\approx_2})$ holds trivially. Now, we suppose that the inequality is true for $n = k$ and we will prove it for $n = k + 1$. Using the assumption $S(\approx_1, \approx_2)^{k+1} \otimes S(\rightharpoonup_1, \rightharpoonup_2)^k \leq S((\rightharpoonup_1)^k_{\approx_1}, (\rightharpoonup_2)^k_{\approx_2})$, we get

$$\begin{aligned}
&S(\approx_1, \approx_2)^{k+2} \otimes S(\rightharpoonup_1, \rightharpoonup_2)^{k+1} = \\
&= S(\approx_1, \approx_2) \otimes S(\rightharpoonup_1, \rightharpoonup_2) \otimes S(\approx_1, \approx_2)^{k+1} \otimes S(\rightharpoonup_1, \rightharpoonup_2)^k \leq \\
&\leq S(\approx_1, \approx_2) \otimes S(\rightharpoonup_1, \rightharpoonup_2) \otimes S((\rightharpoonup_1)^k_{\approx_1}, (\rightharpoonup_2)^k_{\approx_2}) = \\
&= S(\approx_1, \approx_2) \otimes \textstyle\bigwedge_{w,x\in X}(w \rightharpoonup_1 x \rightarrow w \rightharpoonup_2 x) \otimes \bigwedge_{y,z\in X}\left(y(\rightharpoonup_1)^k_{\approx_1} z \rightarrow y(\rightharpoonup_2)^k_{\approx_2} z\right).
\end{aligned}$$

Using the inequalities (2.10) and (2.6), the latter expression is less than or equal to

$$
\begin{aligned}
&S(\approx_1,\approx_2)\otimes\bigwedge\nolimits_{w,x,y,z\in X}\left[(w\rightharpoonup_1 x\to w\rightharpoonup_2 x)\otimes\left(y(\rightharpoonup_1)^k_{\approx_1}z\to y(\rightharpoonup_2)^k_{\approx_2}z\right)\right]\leq\\
\leq&S(\approx_1,\approx_2)\otimes\bigwedge\nolimits_{w,x,y,z\in X}\left[\left(w\rightharpoonup_1 x\otimes y(\rightharpoonup_1)^k_{\approx_1}z\right)\to\left(w\rightharpoonup_2 x\otimes y(\rightharpoonup_2)^k_{\approx_2}z\right)\right]=\\
=&\bigwedge\nolimits_{u,v\in X}(u\approx_1 v\to u\approx_2 v)\otimes\\
&\otimes\bigwedge\nolimits_{w,x,y,z\in X}\left[\left(w\rightharpoonup_1 x\otimes y(\rightharpoonup_1)^k_{\approx_1}z\right)\to\left(w\rightharpoonup_2 x\otimes y(\rightharpoonup_2)^k_{\approx_2}z\right)\right]\leq\\
\leq&\bigwedge\nolimits_{u,v,w,x,y,z\in X}[(u\approx_1 v\to u\approx_2 v)\otimes\\
&\otimes\left((w\rightharpoonup_1 x\otimes y(\rightharpoonup_1)^k_{\approx_1}z)\to(w\rightharpoonup_2 x\otimes y(\rightharpoonup_2)^k_{\approx_2}z)\right)\Big]\leq\\
\leq&\bigwedge\nolimits_{u,v,w,x,y,z\in X}\left[\left(u\approx_1 v\otimes w\rightharpoonup_1 x\otimes y(\rightharpoonup_1)^k_{\approx_1}z\right)\to\left(u\approx_2 v\otimes w\rightharpoonup_2 x\otimes y(\rightharpoonup_2)^k_{\approx_2}z\right)\right].
\end{aligned}
$$

Now, we identify the elements $v=w$ and $x=y$ to get the following expression which is greater than or equal to the foregoing one. Then, we use the inequality (2.14) to conclude the proof.

$$
\begin{aligned}
&\bigwedge\nolimits_{u,v,x,z\in X}\left[\left(u\approx_1 v\otimes v\rightharpoonup_1 x\otimes x(\rightharpoonup_1)^k_{\approx_1}z\right)\to\left(u\approx_2 v\otimes v\rightharpoonup_2 x\otimes x(\rightharpoonup_2)^k_{\approx_2}z\right)\right]\leq\\
\leq&\bigwedge\nolimits_{u,z\in X}\Big[\bigvee\nolimits_{v,x\in X}\left(u\approx_1 v\otimes v\rightharpoonup_1 x\otimes x(\rightharpoonup_1)^k_{\approx_1}z\right)\to\\
&\to\bigvee\nolimits_{v,x\in X}\left(u\approx_2 v\otimes v\rightharpoonup_2 x\otimes x(\rightharpoonup_2)^k_{\approx_2}z\right)\Big]=\\
=&\bigwedge\nolimits_{u,z\in X}\left((u\approx_1\circ\rightharpoonup_1\circ(\rightharpoonup_1)^k_{\approx_1}z)\to(u\approx_2\circ\rightharpoonup_2\circ(\rightharpoonup_2)^k_{\approx_2}z)\right)=\\
=&\bigwedge\nolimits_{u,z\in X}\left(u(\rightharpoonup_1)^{k+1}_{\approx_1}z\to u(\rightharpoonup_2)^{k+1}_{\approx_2}z\right)=S((\rightharpoonup_1)^{k+1}_{\approx_1},(\rightharpoonup_2)^{k+1}_{\approx_2})
\end{aligned}
$$

Altogether, we have $S(\approx_1,\approx_2)^{n+1}\otimes S(\rightharpoonup_1,\rightharpoonup_2)^n\leq S((\rightharpoonup_1)^n_{\approx_1},(\rightharpoonup_2)^n_{\approx_2})$ for each $n\in\mathbb{N}_0$. □

Theorem 45. *For any* **L***-relations* $\rightharpoonup_1$, $\rightharpoonup_2$ *and for any* **L***-similarities* $\approx_1$, $\approx_2$ *on* X, *the following inequality holds.*

$$\bigwedge\nolimits_{n\in\mathbb{N}_0}\left[S(\approx_1,\approx_2)^{n+1}\otimes S(\rightharpoonup_1,\rightharpoonup_2)^n\right]\leq S((\rightharpoonup_1)^*_{\approx_1},(\rightharpoonup_2)^*_{\approx_2}) \tag{3.4}$$

Proof. The **L**-relations $(\rightharpoonup_i)^*_{\approx_j}$ $(i,j\in\{1,2\})$ can be alternatively defined by $\bigcup_{n\in\mathbb{N}_0}(\rightharpoonup_i)^n_{\approx_j}$, i.e. $x(\rightharpoonup_i)^*_{\approx_j}y=\bigvee_{n\in\mathbb{N}_0}x(\rightharpoonup_i)^n_{\approx_j}y$ for each $x,y\in X$. By Theorem 44, we also know that $S(\approx_1,\approx_2)^{n+1}\otimes S(\rightharpoonup_1,\rightharpoonup_2)^n\leq S((\rightharpoonup_1)^n_{\approx_1},(\rightharpoonup_2)^n_{\approx_2})$ holds for each $n\in\mathbb{N}_0$. Using these facts and (2.14), we get

$$
\begin{aligned}
&\bigwedge\nolimits_{n\in\mathbb{N}_0}\left[S(\approx_1,\approx_2)^{n+1}\otimes S(\rightharpoonup_1,\rightharpoonup_2)^n\right]\leq\bigwedge\nolimits_{n\in\mathbb{N}_0}S((\rightharpoonup_1)^n_{\approx_1},(\rightharpoonup_2)^n_{\approx_2})=\\
=&\bigwedge\nolimits_{n\in\mathbb{N}_0}\bigwedge\nolimits_{x,y\in X}\left[x(\rightharpoonup_1)^n_{\approx_1}y\to x(\rightharpoonup_2)^n_{\approx_2}y\right]\leq\\
\leq&\bigwedge\nolimits_{x,y\in X}\left[\bigvee\nolimits_{n\in\mathbb{N}_0}\left(x(\rightharpoonup_1)^n_{\approx_1}y\right)\to\bigvee\nolimits_{n\in\mathbb{N}_0}\left(x(\rightharpoonup_2)^n_{\approx_2}y\right)\right]=\\
=&\bigwedge\nolimits_{x,y\in X}\left[x(\rightharpoonup_1)^*_{\approx_1}y\to x(\rightharpoonup_2)^*_{\approx_2}y\right]=S((\rightharpoonup_1)^*_{\approx_1},(\rightharpoonup_2)^*_{\approx_2}).
\end{aligned}
$$

□

Remark 46. (1) Notice that the inequality in Theorem 45 is in a general form and gives a lower estimate of subsethood degrees of $(\rightharpoonup_1)^*_{\approx_1}$ in $(\rightharpoonup_2)^*_{\approx_2}$ by means of (n-th powers of)

subsethood degrees $S(\approx_1, \approx_2)$ and $S(\rightharpoonup_1, \rightharpoonup_2)$. Hence, it can be seen as a monotony condition which involves in general two different similarities $\approx_1$ and $\approx_2$ on the universe of discourse. In a special case, the inequality becomes $\bigwedge_{n\in\mathbb{N}_0} S(\rightharpoonup_1, \rightharpoonup_2)^n \leq S((\rightharpoonup_1)^*_{\approx_1}, (\rightharpoonup_2)^*_{\approx_1})$ whenever $\approx_1 = \approx_2$ and it becomes $\bigwedge_{n\in\mathbb{N}_0} S(\approx_1, \approx_2)^{n+1} \leq S((\rightharpoonup_1)^*_{\approx_1}, (\rightharpoonup_1)^*_{\approx_2})$ whenever $\rightharpoonup_1 = \rightharpoonup_2$.
(2) Note that if $\mathbf{L}$ is a complete BL-chain satisfying $a \otimes \bigwedge_{i\in I} b_i = \bigwedge_{i\in I}(a \otimes b_i)$ (e.g., if $\mathbf{L}$ is a complete BL-chain on the real unit interval with $\otimes$ being a continuous t-norm), then for each $a \in L$, $\bigwedge_{n\in\mathbb{N}_0} a^n$ is the greatest idempotent element which is less than or equal to a [13]. Thus, for an important class of structures of truth degrees (including the standard Łukasiewicz, Gödel and Goguen structures), the degree which appears on the left-hand side of (3.4) is an idempotent element.

3.3.2 Sensitivity of Confluence

We will now investigate how much the degrees of reducibility, convergence, divergence, confluence, convertibility and the Church-Roser property of an $\mathbf{L}$-relation $\rightharpoonup_1$ with respect to a similarity $\approx_1$ change if $\rightharpoonup_1$ is replaced by a (very) similar $\mathbf{L}$-relation $\rightharpoonup_2$ and/or $\approx_1$ is substituted by a (very) similar $\mathbf{L}$-equivalence $\approx_2$. We will provide some inequalities which can be used for estimations of the above mentioned degrees for $\rightharpoonup_2$ and $\approx_2$.
The first assertion shows that the degree of equality $E((\rightharpoonup_1)^*_{\approx_1}, (\rightharpoonup_2)^*_{\approx_2})$ of reducibility $\mathbf{L}$-relations $(\rightharpoonup_1)^*_{\approx_1}$ and $(\rightharpoonup_2)^*_{\approx_2}$ can be estimated from below in a similar way as in Theorem 45:

Theorem 47. *For any* $\mathbf{L}$*-relations* $\rightharpoonup_1$, $\rightharpoonup_2$ *and for any* $\mathbf{L}$*-similarities* $\approx_1$, $\approx_2$ *on* X*, the following inequality holds:*

$$\bigwedge_{n\in\mathbb{N}_0} \left(E(\approx_1, \approx_2)^{n+1} \otimes E(\rightharpoonup_1, \rightharpoonup_2)^n\right) \leq E((\rightharpoonup_1)^*_{\approx_1}, (\rightharpoonup_2)^*_{\approx_2})$$

Proof. By definition of $E(\cdots)$ and Theorem 45, we immediately get

$$\bigwedge_{n\in\mathbb{N}_0} \left(E(\approx_1, \approx_2)^{n+1} \otimes E(\rightharpoonup_1, \rightharpoonup_2)^n\right) \leq S((\rightharpoonup_1)^*_{\approx_1}, (\rightharpoonup_2)^*_{\approx_2}) \text{ and}$$
$$\bigwedge_{n\in\mathbb{N}_0} \left(E(\approx_1, \approx_2)^{n+1} \otimes E(\rightharpoonup_1, \rightharpoonup_2)^n\right) \leq S((\rightharpoonup_2)^*_{\approx_2}, (\rightharpoonup_1)^*_{\approx_1}).$$

Using these two inequalities, idempotency and isotony of $\wedge$, we obtain

$$\begin{aligned} &\bigwedge_{n\in\mathbb{N}_0} \left(E(\approx_1, \approx_2)^{n+1} \otimes E(\rightharpoonup_1, \rightharpoonup_2)^n\right) = \\ &= \bigwedge_{n\in\mathbb{N}_0} \left(E(\approx_1, \approx_2)^{n+1} \otimes E(\rightharpoonup_1, \rightharpoonup_2)^n\right) \wedge \bigwedge_{n\in\mathbb{N}_0} \left(E(\approx_1, \approx_2)^{n+1} \otimes E(\rightharpoonup_1, \rightharpoonup_2)^n\right) \leq \\ &\leq S((\rightharpoonup_1)^*_{\approx_1}, (\rightharpoonup_2)^*_{\approx_2}) \wedge S((\rightharpoonup_2)^*_{\approx_2}, (\rightharpoonup_1)^*_{\approx_1}) = E((\rightharpoonup_1)^*_{\approx_1}, (\rightharpoonup_2)^*_{\approx_2}), \end{aligned}$$

proving the claim. □

The following assertion shows a property that will be used in subsequent proofs.

Theorem 48. *For any* $\mathbf{L}$*-relations* $\rightharpoonup_1$, $\rightharpoonup_2$*, any* $\mathbf{L}$*-similarities* $\approx_1$, $\approx_2$ *on* X *and arbitrary* $u, v, x, y \in X$*, we always have*

$$\begin{aligned} &\left[\bigwedge_{n\in\mathbb{N}_0} \left(E(\approx_1, \approx_2)^{n+1} \otimes E(\rightharpoonup_1, \rightharpoonup_2)^n\right)\right]^2 \leq \\ &\leq \left[(u(\rightharpoonup_1)^*_{\approx_1} v) \otimes (x(\rightharpoonup_1)^*_{\approx_1} y)\right] \leftrightarrow \left[(u(\rightharpoonup_2)^*_{\approx_2} v) \otimes (x(\rightharpoonup_2)^*_{\approx_2} y)\right]. \end{aligned}$$

Proof. Using the isotony of $\otimes$ and Theorem 47, we get

$$
\begin{aligned}
&\left[\bigwedge_{n\in\mathbb{N}_0}\left(E(\approx_1,\approx_2)^{n+1}\otimes E(\rightharpoonup_1,\rightharpoonup_2)^n\right)\right]^2 \leq\\
\leq & E((\rightharpoonup_1)^*_{\approx_1},(\rightharpoonup_2)^*_{\approx_2})\otimes E((\rightharpoonup_1)^*_{\approx_1},(\rightharpoonup_2)^*_{\approx_2}) =\\
= & \left[\bigwedge_{u,v\in X}(u(\rightharpoonup_1)^*_{\approx_1}v \leftrightarrow u(\rightharpoonup_2)^*_{\approx_2}v)\right]\otimes\left[\bigwedge_{x,y\in X}(x(\rightharpoonup_1)^*_{\approx_1}y \leftrightarrow x(\rightharpoonup_2)^*_{\approx_2}y)\right].
\end{aligned}
$$

By definition of $\leftrightarrow$ and the inequality (2.9), we can continue as follows:

$$
\begin{aligned}
&\left[\bigwedge_{u,v\in X}(u(\rightharpoonup_1)^*_{\approx_1}v \leftrightarrow u(\rightharpoonup_2)^*_{\approx_2}v)\right]\otimes\left[\bigwedge_{x,y\in X}(x(\rightharpoonup_1)^*_{\approx_1}y \leftrightarrow x(\rightharpoonup_2)^*_{\approx_2}y)\right]\\
\leq & \bigwedge_{u,v,x,y\in X}\left[\left((u(\rightharpoonup_1)^*_{\approx_1}v \to u(\rightharpoonup_2)^*_{\approx_2}v)\wedge(u(\rightharpoonup_2)^*_{\approx_2}v \to u(\rightharpoonup_1)^*_{\approx_1}v)\right)\otimes\right.\\
& \left.\otimes\left((x(\rightharpoonup_1)^*_{\approx_1}y \to x(\rightharpoonup_2)^*_{\approx_2}y)\wedge(x(\rightharpoonup_2)^*_{\approx_2}y \to x(\rightharpoonup_1)^*_{\approx_1}y)\right)\right] \leq\\
\leq & \bigwedge_{u,v,x,y\in X}\left[\left((u(\rightharpoonup_1)^*_{\approx_1}v \to u(\rightharpoonup_2)^*_{\approx_2}v)\otimes(x(\rightharpoonup_1)^*_{\approx_1}y \to x(\rightharpoonup_2)^*_{\approx_2}y)\right)\wedge\right.\\
& \wedge\left((u(\rightharpoonup_1)^*_{\approx_1}v \to u(\rightharpoonup_2)^*_{\approx_2}v)\otimes(x(\rightharpoonup_2)^*_{\approx_2}y \to x(\rightharpoonup_1)^*_{\approx_1}y)\right)\wedge\\
& \wedge\left((u(\rightharpoonup_2)^*_{\approx_2}v \to u(\rightharpoonup_1)^*_{\approx_1}v)\otimes(x(\rightharpoonup_1)^*_{\approx_1}y \to x(\rightharpoonup_2)^*_{\approx_2}y)\right)\wedge\\
& \left.\wedge\left((u(\rightharpoonup_2)^*_{\approx_2}v \to u(\rightharpoonup_1)^*_{\approx_1}v)\otimes(x(\rightharpoonup_2)^*_{\approx_2}y \to x(\rightharpoonup_1)^*_{\approx_1}y)\right)\right].
\end{aligned}
$$

Now, we interchange $\to$ and $\otimes$ using (2.6) and then we reduce the amount of operands for the meet operation (by “skipping 2 lines”). Thus, we get that the last term in the foregoing inequality is less than or equal to

$$
\begin{aligned}
& \bigwedge_{u,v,x,y\in X}\left[\left((u(\rightharpoonup_1)^*_{\approx_1}v\otimes x(\rightharpoonup_1)^*_{\approx_1}y)\to(u(\rightharpoonup_2)^*_{\approx_2}v\otimes x(\rightharpoonup_2)^*_{\approx_2}y)\right)\wedge\right.\\
& \wedge\left((u(\rightharpoonup_1)^*_{\approx_1}v\otimes x(\rightharpoonup_2)^*_{\approx_2}y)\to(u(\rightharpoonup_2)^*_{\approx_2}v\otimes x(\rightharpoonup_1)^*_{\approx_1}y)\right)\wedge\\
& \wedge\left((u(\rightharpoonup_2)^*_{\approx_2}v\otimes x(\rightharpoonup_1)^*_{\approx_1}y)\to(u(\rightharpoonup_1)^*_{\approx_1}v\otimes x(\rightharpoonup_2)^*_{\approx_2}y)\right)\wedge\\
& \left.\wedge\left((u(\rightharpoonup_2)^*_{\approx_2}v\otimes x(\rightharpoonup_2)^*_{\approx_2}y)\to(u(\rightharpoonup_1)^*_{\approx_1}v\otimes x(\rightharpoonup_1)^*_{\approx_1}y)\right)\right]\leq\\
\leq & \bigwedge_{u,v,x,y\in X}\left[\left((u(\rightharpoonup_1)^*_{\approx_1}v\otimes x(\rightharpoonup_1)^*_{\approx_1}y)\to(u(\rightharpoonup_2)^*_{\approx_2}v\otimes x(\rightharpoonup_2)^*_{\approx_2}y)\right)\wedge\right.\\
& \left.\wedge\left((u(\rightharpoonup_2)^*_{\approx_2}v\otimes x(\rightharpoonup_2)^*_{\approx_2}y)\to(u(\rightharpoonup_1)^*_{\approx_1}v\otimes x(\rightharpoonup_1)^*_{\approx_1}y)\right)\right]=\\
= & \bigwedge_{u,v,x,y\in X}\left[(u(\rightharpoonup_1)^*_{\approx_1}v\otimes x(\rightharpoonup_1)^*_{\approx_1}y)\leftrightarrow(u(\rightharpoonup_2)^*_{\approx_2}v\otimes x(\rightharpoonup_2)^*_{\approx_2}y)\right].
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\left[\bigwedge_{n\in\mathbb{N}_0}\left(E(\approx_1,\approx_2)^{n+1}\otimes E(\rightharpoonup_1,\rightharpoonup_2)^n\right)\right]^2 \leq\\
\leq & \left[(u(\rightharpoonup_1)^*_{\approx_1}v)\otimes(x(\rightharpoonup_1)^*_{\approx_1}y)\right]\leftrightarrow\left[(u(\rightharpoonup_2)^*_{\approx_2}v)\otimes(x(\rightharpoonup_2)^*_{\approx_2}y)\right],
\end{aligned}
$$

which concludes the proof. □

Using Theorem 48, we can express lower estimates of equality degrees of divergence, convergence, and confluence.

Theorem 49. *For any* $\mathbf{L}$*-relations* $\rightharpoonup_1$, $\rightharpoonup_2$ *and for any* $\mathbf{L}$*-similarities* $\approx_1$, $\approx_2$ *on* X*, the following inequalities hold.*

$$
\begin{aligned}
&\left[\bigwedge_{n\in\mathbb{N}_0}\left(E(\approx_1,\approx_2)^{n+1}\otimes E(\rightharpoonup_1,\rightharpoonup_2)^n\right)\right]^2 \leq E(\upharpoonleft^{\rightharpoonup_1}_{\approx_1},\upharpoonleft^{\rightharpoonup_2}_{\approx_2})\\
&\left[\bigwedge_{n\in\mathbb{N}_0}\left(E(\approx_1,\approx_2)^{n+1}\otimes E(\rightharpoonup_1,\rightharpoonup_2)^n\right)\right]^2 \leq E(\downharpoonleft^{\rightharpoonup_1}_{\approx_1},\downharpoonleft^{\rightharpoonup_2}_{\approx_2})
\end{aligned}
$$

Proof. We start with the inequality from Theorem 48. First, we reduce the amount of operands for the meet operation by identifying u and x on the right-hand side of this inequality. Furthermore, we use (2.17) to get

$$
\begin{aligned}
&\left[\bigwedge_{n\in\mathbb{N}_0} E(\rightharpoonup_1,\rightharpoonup_2)^n\right]^2 \leq \\
\leq &\bigwedge_{u,v,y\in X}\left[\left((u(\rightharpoonup_1)^*_{\approx_1}v)\otimes(u(\rightharpoonup_1)^*_{\approx_1}y)\right)\leftrightarrow\left((u(\rightharpoonup_2)^*_{\approx_2}v)\otimes(u(\rightharpoonup_2)^*_{\approx_2}y)\right)\right] \leq \\
\leq &\bigwedge_{v,y\in X}\left[\left(\bigvee_{u\in X}(u(\rightharpoonup_1)^*_{\approx_1}v)\otimes(u(\rightharpoonup_1)^*_{\approx_1}y)\right)\leftrightarrow\left(\bigvee_{u\in X}(u(\rightharpoonup_2)^*_{\approx_2}v)\otimes(u(\rightharpoonup_2)^*_{\approx_2}y)\right)\right] = \\
= &\bigwedge_{v,y\in X}\left[(v\upharpoonleft^{\rightharpoonup_1}_{\approx_1}y))\leftrightarrow(v\upharpoonleft^{\rightharpoonup_2}_{\approx_2}y))\right] = E(\upharpoonleft^{\rightharpoonup_1}_{\approx_1},\upharpoonleft^{\rightharpoonup_2}_{\approx_2}),
\end{aligned}
$$

finishing the proof of the first inequality.
The second inequality can be proved analogously, just start with identifying v and y. ☐

Theorem 50. *For any* **L***-relations* $\rightharpoonup_1$, $\rightharpoonup_2$ *and any* **L***-similarities* $\approx_1$, $\approx_2$ *on* X, *the following inequlality holds.*

$$\left[\bigwedge_{n\in\mathbb{N}_0}\left(E(\approx_1,\approx_2)^{n+1}\otimes E(\rightharpoonup_1,\rightharpoonup_2)^n\right)\right]^4 \leq \mathrm{CFL}(\rightharpoonup_1)_{\approx_1}\leftrightarrow \mathrm{CFL}(\rightharpoonup_2)_{\approx_2}$$

Proof. Using the isotony of $\otimes$, Theorem 49, we get

$$
\begin{aligned}
&\left[\bigwedge_{n\in\mathbb{N}_0}\left(E(\approx_1,\approx_2)^{n+1}\otimes E(\rightharpoonup_1,\rightharpoonup_2)^n\right)\right]^4 = \\
= &\left[\bigwedge_{n\in\mathbb{N}_0}\left(E(\approx_1,\approx_2)^{n+1}\otimes E(\rightharpoonup_1,\rightharpoonup_2)^n\right)\right]^2\otimes\left[\bigwedge_{n\in\mathbb{N}_0}\left(E(\approx_1,\approx_2)^{n+1}\otimes E(\rightharpoonup_1,\rightharpoonup_2)^n\right)\right]^2 \leq \\
&\leq E(\upharpoonleft^{\rightharpoonup_1}_{\approx_1},\upharpoonleft^{\rightharpoonup_2}_{\approx_2})\otimes E(\downharpoonleft^{\rightharpoonup_1}_{\approx_1},\downharpoonleft^{\rightharpoonup_2}_{\approx_2}) = \\
= &\left[\bigwedge_{u,v\in X}(u\upharpoonleft^{\rightharpoonup_1}_{\approx_1}v\leftrightarrow u\upharpoonleft^{\rightharpoonup_2}_{\approx_2}v)\right]\otimes\left[\bigwedge_{x,y\in X}(x\downharpoonleft^{\rightharpoonup_1}_{\approx_1}y\leftrightarrow x\downharpoonleft^{\rightharpoonup_2}_{\approx_12}y)\right].
\end{aligned}
$$

Applying (2.9) and (2.15), we get

$$
\begin{aligned}
&\left[\bigwedge_{u,v\in X}(u\upharpoonleft^{\rightharpoonup_1}_{\approx_1}v\leftrightarrow u\upharpoonleft^{\rightharpoonup_2}_{\approx_2}v)\right]\otimes\left[\bigwedge_{x,y\in X}(x\downharpoonleft^{\rightharpoonup_1}_{\approx_1}y\leftrightarrow x\downharpoonleft^{\rightharpoonup_2}_{\approx_2}y)\right] \leq \\
&\bigwedge_{u,v,x,y\in X}\left[(u\upharpoonleft^{\rightharpoonup_1}_{\approx_1}v\leftrightarrow u\upharpoonleft^{\rightharpoonup_2}_{\approx_2}v)\otimes(x\downharpoonleft^{\rightharpoonup_1}_{\approx_1}y\leftrightarrow x\downharpoonleft^{\rightharpoonup_2}_{\approx_2}y)\right] \leq \\
\leq &\bigwedge_{u,v,x,y\in X}\left[(u\upharpoonleft^{\rightharpoonup_1}_{\approx_1}v\to x\downharpoonleft^{\rightharpoonup_1}_{\approx_1}y)\leftrightarrow(u\upharpoonleft^{\rightharpoonup_2}_{\approx_2}ev\to x\downharpoonleft^{\rightharpoonup_2}_{\approx_2}y)\right].
\end{aligned}
$$

Now, we reduce the amount of operands for the meet operation by identifying x with u and y with v. Using (2.16), we further get

$$
\begin{aligned}
&\bigwedge_{u,v,x,y\in X}\left[(u\upharpoonleft^{\rightharpoonup_1}_{\approx_1}v\to x\downharpoonleft^{\rightharpoonup_1}_{\approx_1}y)\leftrightarrow(u\upharpoonleft^{\rightharpoonup_2}_{\approx_2}v\to x\downharpoonleft^{\rightharpoonup_2}_{\approx_2}y)\right] \leq \\
&\bigwedge_{u,v\in X}\left[(u\upharpoonleft^{\rightharpoonup_1}_{\approx_1}v\to x\downharpoonleft^{\rightharpoonup_1}_{\approx_1}y)\leftrightarrow(u\upharpoonleft^{\rightharpoonup_2}_{\approx_2}v\to x\downharpoonleft^{\rightharpoonup_2}_{\approx_2}y)\right] \leq \\
\leq &\left[\bigwedge_{u,v\in X}(u\upharpoonleft^{\rightharpoonup_1}_{\approx_1}v\to u\downharpoonleft^{\rightharpoonup_1}_{\approx_1}v)\right]\leftrightarrow\left[\bigwedge_{u,v\in X}(u\upharpoonleft^{\rightharpoonup_2}_{\approx_2}v\to u\downharpoonleft^{\rightharpoonup_2}_{\approx_2}v)\right] = \\
= &\,\mathrm{CFL}(\rightharpoonup_1)_{\approx_1}\leftrightarrow\mathrm{CFL}(\rightharpoonup_2)_{\approx_2}.
\end{aligned}
$$

Putting previous inequalities together, we get the desired inequality. ☐

Similar inequlalities can be proved also for convertibility degrees as well as for degrees of the Church-Rosser property:

Theorem 51. *For given* **L***-relations* $\rightharpoonup_1$, $\rightharpoonup_2$, *the corresponding* $\rightleftharpoons_1$, $\rightleftharpoons_2$, *and given* **L***-similarities* $\approx_1$, $\approx_2$, *the following inequality holds:*

$$\bigwedge_{n\in\mathbb{N}_0}\left[E(\approx_1,\approx_2)^{n+1}\otimes E(\rightharpoonup_1,\rightharpoonup_2)^n\right] \leq E((\rightleftharpoons_1)^*_{\approx_1},(\rightleftharpoons_2)^*_{\approx_2}).$$

Proof. Directly from definitions of $\rightleftharpoons_1$ and $\rightleftharpoons_2$, we have $\rightharpoonup_1 \subseteq \rightleftharpoons_1$, $\leftharpoondown_1 \subseteq \rightleftharpoons_1$, $\rightharpoonup_2 \subseteq \rightleftharpoons_2$ and $\leftharpoondown_2 \subseteq \rightleftharpoons_2$,i.e., $S(\rightharpoonup_1, \rightleftharpoons_1) = S(\leftharpoondown_1, \rightleftharpoons_1) = S(\rightharpoonup_2, \rightleftharpoons_2) = S(\leftharpoondown_2, \rightleftharpoons_2) = 1$. Using the definition of $E(\cdots)$ and transitivity of $S(\cdots)$, we also get:

$$E(\rightharpoonup_1, \rightharpoonup_2) \leq S(\rightharpoonup_1, \rightharpoonup_2) = S(\rightharpoonup_1, \rightharpoonup_2) \otimes S(\rightharpoonup_2, \rightleftharpoons_2) \leq S(\rightharpoonup_1, \rightleftharpoons_2), \tag{3.5}$$

$$E(\rightharpoonup_1, \rightharpoonup_2) = E(\leftharpoondown_1, \leftharpoondown_2) \leq S(\leftharpoondown_1, \leftharpoondown_2) = S(\leftharpoondown_1, \leftharpoondown_2) \otimes S(\leftharpoondown_2, \rightleftharpoons_2) \leq S(\leftharpoondown_1, \rightleftharpoons_2), \tag{3.6}$$

$$E(\rightharpoonup_1, \rightharpoonup_2) = E(\rightharpoonup_2, \rightharpoonup_1) \leq S(\rightharpoonup_2, \rightharpoonup_1) = S(\rightharpoonup_2, \rightharpoonup_1) \otimes S(\rightharpoonup_1, \rightleftharpoons_1) \leq S(\rightharpoonup_2, \rightleftharpoons_1), \tag{3.7}$$

$$E(\rightharpoonup_1, \rightharpoonup_2) = E(\leftharpoondown_2, \leftharpoondown_1) \leq S(\leftharpoondown_2, \leftharpoondown_1) = S(\leftharpoondown_2, \leftharpoondown_1) \otimes S(\leftharpoondown_1, \rightleftharpoons_1) \leq S(\leftharpoondown_2, \rightleftharpoons_1). \tag{3.8}$$

According to inequalities (3.5) and (3.6) and the associativity of $\wedge$, the following holds.

$$\begin{aligned} &E(\rightharpoonup_1, \rightharpoonup_2) \leq S(\rightharpoonup_1, \rightleftharpoons_2) \wedge S(\leftharpoondown_1, \rightleftharpoons_2) = \\ &= \Big[\bigwedge_{u,v\in X}(u \rightharpoonup_1 v \to u \rightleftharpoons_2 v)\Big] \wedge \Big[\bigwedge_{x,y\in X}(x \leftharpoondown_1 y \to x \rightleftharpoons_2 y)\Big] = \\ &= \bigwedge_{u,v,x,y\in X} [(u \rightharpoonup_1 v \to u \rightleftharpoons_2 v) \wedge (x \leftharpoondown_1 y \to x \rightleftharpoons_2 y)] \end{aligned}$$

Then, we identify elements $u = x$ and $v = y$ and use (2.11) to get

$$\begin{aligned} &E(\rightharpoonup_1, \rightharpoonup_2) \leq \bigwedge_{x,y\in X} [(x \rightharpoonup_1 y \to x \rightleftharpoons_2 y) \wedge (x \leftharpoondown_1 y \to x \rightleftharpoons_2 y)] = \\ &= \bigwedge_{x,y\in X} [(x \rightharpoonup_1 y \vee x \leftharpoondown_1 y) \to x \rightleftharpoons_2 y] = \\ &= \bigwedge_{x,y\in X} (x \rightleftharpoons_1 y \to x \rightleftharpoons_2 y) = S(\rightleftharpoons_1, \rightleftharpoons_2). \end{aligned}$$

Starting with inequalities (3.7) and (3.8), we can prove inequality $E(\rightharpoonup_1, \rightharpoonup_2) \leq S(\rightleftharpoons_2, \rightleftharpoons_1)$ in the same way. Using these 2 inequalities together, we immediately get $E(\rightharpoonup_1, \rightharpoonup_2) \leq S(\rightleftharpoons_1, \rightleftharpoons_2) \wedge S(\rightleftharpoons_2, \rightleftharpoons_1) = E(\rightleftharpoons_1, \rightleftharpoons_2)$.

Due to Theorem 45 for $\rightharpoonup$ being $\rightleftharpoons$, we also have

$$\bigwedge_{n\in\mathbb{N}_0} \left[S(\approx_1, \approx_2)^{n+1} \otimes S(\rightleftharpoons_1, \rightleftharpoons_2)^n\right] \leq S((\rightleftharpoons_1)^*_{\approx_1}, (\rightleftharpoons_2)^*_{\approx_2}).$$

By definitions of $E(\cdots)$ and $S(\cdots)$ and the isotony of $\otimes$, we also get

$$\begin{aligned} &\bigwedge_{n\in\mathbb{N}_0} \left[E(\approx_1, \approx_2)^{n+1} \otimes E(\rightleftharpoons_1, \rightleftharpoons_2)^n\right] \leq S((\rightleftharpoons_1)^*_{\approx_1}, (\rightleftharpoons_2)^*_{\approx_2}), \\ &\bigwedge_{n\in\mathbb{N}_0} \left[E(\approx_1, \approx_2)^{n+1} \otimes E(\rightleftharpoons_1, \rightleftharpoons_2)^n\right] \leq S((\rightleftharpoons_2)^*_{\approx_2}, (\rightleftharpoons_1)^*_{\approx_1}). \end{aligned}$$

Altogether, using the inequality $E(\rightharpoonup_1, \rightharpoonup_2) \leq E(\rightleftharpoons_1, \rightleftharpoons_2)$, we obtain

$$\begin{aligned} &\bigwedge_{n\in\mathbb{N}_0} \left[E(\approx_1, \approx_2)^{n+1} \otimes E(\rightharpoonup_1, \rightharpoonup_2)^n\right] \leq \bigwedge_{n\in\mathbb{N}_0} \left[E(\approx_1, \approx_2)^{n+1} \otimes E(\rightleftharpoons_1, \rightleftharpoons_2)^n\right] \leq \\ &\leq S((\rightleftharpoons_1)^*_{\approx_1}, (\rightleftharpoons_2)^*_{\approx_2}) \wedge S((\rightleftharpoons_2)^*_{\approx_2}, (\rightleftharpoons_1)^*_{\approx_1}) = E((\rightleftharpoons_1)^*_{\approx_1}, (\rightleftharpoons_2)^*_{\approx_2}) \end{aligned}$$

which concludes the proof. □

Theorem 52. *For arbitrary* **L***-relations* $\rightharpoonup_1$, $\rightharpoonup_2$ *and* **L***-similarities* $\approx_1$, $\approx_2$ *on* X*, the following inequality holds.*

$$\left[\bigwedge_{n\in\mathbb{N}_0} \left(E(\approx_1, \approx_2)^{n+1} \otimes E(\rightharpoonup_1, \rightharpoonup_2)^n\right)\right]^3 \leq \mathrm{CR}(\rightharpoonup_1)_{\approx_1} \leftrightarrow \mathrm{CR}(\rightharpoonup_2)_{\approx_2}$$

Proof. Using the isotony of $\otimes$, Theorem 49 and Theorem 51, we get

$$\begin{aligned} &\left[\bigwedge_{n\in\mathbb{N}_0} \left(E(\approx_1, \approx_2)^{n+1} \otimes E(\rightharpoonup_1, \rightharpoonup_2)^n\right)\right]^3 = \\ &= \left[\bigwedge_{n\in\mathbb{N}_0} \left(E(\approx_1, \approx_2)^{n+1} \otimes E(\rightharpoonup_1, \rightharpoonup_2)^n\right)\right] \otimes \left[\bigwedge_{n\in\mathbb{N}_0} \left(E(\approx_1, \approx_2)^{n+1} \otimes E(\rightharpoonup_1, \rightharpoonup_2)^n\right)\right]^2 \leq \\ &\leq E((\rightleftharpoons_1)^*_{\approx_1}, (\rightleftharpoons_2)^*_{\approx_2}) \otimes E(\downharpoonleft^{\rightharpoonup_1}_{\approx_1}, \downharpoonleft^{\rightharpoonup_2}_{\approx_2}) = \\ &= \Big[\bigwedge_{u,v\in X}(u(\rightleftharpoons_1)^*_{\approx_1} v \leftrightarrow u(\rightleftharpoons_2)^*_{\approx_2} v)\Big] \otimes \Big[\bigwedge_{x,y\in X}(x \downharpoonleft^{\rightharpoonup_1}_{\approx_1} y \leftrightarrow x \downharpoonleft^{\rightharpoonup_2}_{\approx_2} y)\Big]. \end{aligned}$$

Applying the inequalities (2.9) and (2.15), we get

$$\begin{aligned}
&\left[\bigwedge_{u,v\in X}(u(\rightleftharpoons_1)^*_{\approx_1}v \leftrightarrow u(\rightleftharpoons_2)^*_{\approx_2}v)\right] \otimes \left[\bigwedge_{x,y\in X}(x \downharpoonleft^{\rightharpoonup_1}_{\approx_1} y \leftrightarrow x \downharpoonleft^{\rightharpoonup_2}_{\approx_2} y)\right] \leq \\
&\leq \bigwedge_{u,v,x,y\in X}\left[(u(\rightleftharpoons_1)^*_{\approx_1}v \leftrightarrow u(\rightleftharpoons_2)^*_{\approx_2}v) \otimes (x \downharpoonleft^{\rightharpoonup_1}_{\approx_1} y \leftrightarrow x \downharpoonleft^{\rightharpoonup_2}_{\approx_2} y)\right] \leq \\
&\leq \bigwedge_{u,v,x,y\in X}\left[(u(\rightleftharpoons_1)^*_{\approx_1}v \to x \downharpoonleft^{\rightharpoonup_1}_{\approx_1} y) \leftrightarrow (u(\rightleftharpoons_2)^*_{\approx_2}v \to x \downharpoonleft^{\rightharpoonup_2}_{\approx_2} y)\right].
\end{aligned}$$

Now, we reduce the amount of operands for the meet operation by identifying $x = u$ and $y = v$. Using (2.16), we further get

$$\begin{aligned}
&\bigwedge_{u,v,x,y\in X}\left[(u(\rightleftharpoons_1)^*_{\approx_1}v \to x \downharpoonleft^{\rightharpoonup_1}_{\approx_1} y) \leftrightarrow (u(\rightleftharpoons_2)^*_{\approx_2}v \to x \downharpoonleft^{\rightharpoonup_2}_{\approx_2} y)\right] \leq \\
&\bigwedge_{u,v\in X}\left[(u(\rightleftharpoons_1)^*_{\approx_1}v \to u \downharpoonleft^{\rightharpoonup_1}_{\approx_1} v) \leftrightarrow (u(\rightleftharpoons_2)^*_{\approx_2}v \to u \downharpoonleft^{\rightharpoonup_2}_{\approx_2} v)\right] \leq \\
&\leq \left[\bigwedge_{u,v\in X}(u(\rightleftharpoons_1)^*_{\approx_1}v \to u \downharpoonleft^{\rightharpoonup_1}_{\approx_1} v)\right] \leftrightarrow \left[\bigwedge_{u,v\in X}(u(\rightleftharpoons_2)^*_{\approx_2}v \to u \downharpoonleft^{\rightharpoonup_2}_{\approx_2} v)\right] = \\
&= S((\rightleftharpoons_1)^*_{\approx_1}, \downharpoonleft^{\rightharpoonup_1}_{\approx_1}) \leftrightarrow S((\rightleftharpoons_2)^*_{\approx_2}, \downharpoonleft^{\rightharpoonup_2}_{\approx_2}) = \mathrm{CR}(\rightharpoonup_1)_{\approx_1} \leftrightarrow \mathrm{CR}(\rightharpoonup_2)_{\approx_2}.
\end{aligned}$$

Putting all these inequalities together, we get the desired inequality. □

Remark 53. The similarity estimation formulas provided in this subsection can be simplified if $\rightharpoonup_1 = \rightharpoonup_2$ or $\approx_1 = \approx_2$, see Remark 46 (1). Another simplification of the estimations is obtained under the assumption that either $E(\approx_1, \approx_2)$ or $E(\rightharpoonup_1, \rightharpoonup_2)$ is an idempotent element of **L**.

There are important cases in which $\rightharpoonup_1$ and $\rightharpoonup_2$ are similar to a degree which is an idempotent element of **L**. For instance, if **L** is a BL-algebra on $[0, 1]$ then an idempotent similarity degree can be obtained in the following situation. Suppose that $\rightharpoonup_1$ is an **L**-relation and one wants to replace $\rightharpoonup_1$ by $\rightharpoonup_2$ which is simpler in that the set of all degrees used by $\rightharpoonup_2$ is small (i.e., $\{u_1 \rightharpoonup_2 u_2 \mid u_1, u_2 \in X\}$ is finite and sufficiently small) and represents $\rightharpoonup_1$ sufficiently well. One way to get $\rightharpoonup_2$ from $\rightharpoonup_1$ is by "rounding" truth degrees, e.g., one selects a finite subset of L, say $K = \{0, 0.25, 0.5, 0.75, 1\}$, and defines $u_1 \rightharpoonup_2 u_2$ as the greatest element from K which is less than or equal to $u_1 \rightharpoonup_1 u_2$. If each element in K is an idempotent of **L** and $\rightharpoonup_1$ and $\rightharpoonup_2$ satisfy the following condition: for all $u_1, u_2 \in X$, $u_1 \rightharpoonup_1 u_2$ is an idempotent or there is an idempotent $c \in L$ such that $u_1 \rightharpoonup_2 u_2 < c < u_1 \rightharpoonup_1 u_2$, then $E(\rightharpoonup_1, \rightharpoonup_2) \in L$ is idempotent. This is a consequence of the well-known Mostert-Shields representation [34, 39, 44] of BL-algebras saying that **L** can be seen as an ordinal sum of isomorphic copies of standard Łukasiewicz and product algebras mutually separated by idempotent elements. Also note that for any finite $\rightharpoonup_1$ and $\rightharpoonup_2$ using degrees from the $[0, 1]$ interval and finite $K \subseteq [0, 1]$, one can always take (infinitely many) **L** satisfying the above-described condition (again, this is a consequence of the Mostert-Shields representation).

Example 54. Let **L** be ordinal sum of 3 isomorphic copies of standard Łukasiewicz structures and 2 isomorphic copies of product algebras mutually separated by idempotent elements, i.e., $\mathbf{L} = \bigoplus_{i\in I} \mathbf{L}_i$, where $I = \{1, 2, \ldots, 5\}$, $\mathbf{L}_1, \mathbf{L}_3, \mathbf{L}_5$ are standard Łukasiewicz algebras and $\mathbf{L}_2, \mathbf{L}_4$ are standard product structures of truth degrees.
Let $\rightharpoonup_1$ and $\rightharpoonup_2$ be **L**-relations on $X = \{x_0, x_1, \ldots, x_6\}$ which are depicted in Figure 3.5 and let $\approx$ be the identity. By direct computation using the definition of $E(\cdots)$, we get $E(\rightharpoonup_1, \rightharpoonup_2) = 0.2$. By Theorem 50, we can estimate the degrees of confluence of $\rightharpoonup_1$ and $\rightharpoonup_2$ by $\mathrm{CFL}(\rightharpoonup_1)_{\approx} \leftrightarrow \mathrm{CFL}(\rightharpoonup_2)_{\approx} \geq 0.2$. Thus, after determining $\mathrm{CFL}(\rightharpoonup_1)_{\approx} = 0.9$, we immediately know from the estimate that the value $\mathrm{CFL}(\rightharpoonup_2)_{\approx}$ will be somewhere in the interval from 0.2 to 1. The estimate can be useful because the exact computation may

be very demanding. In this particular case, by direct computation of $\mathrm{CFL}(\rightharpoonup_2)_{\approx}$, we get $\mathrm{CFL}(\rightharpoonup_2)_{\approx} = 0.6$.

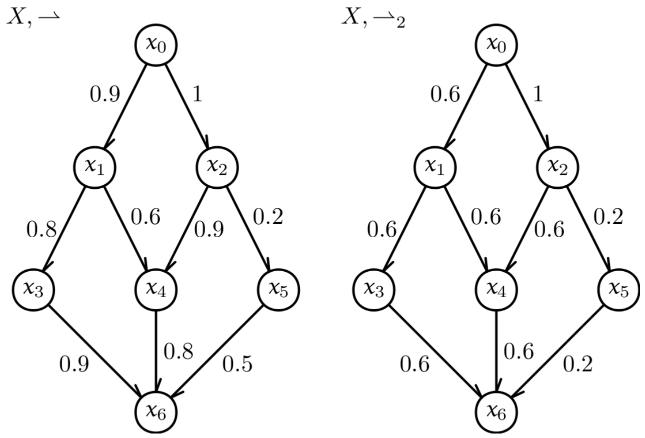

Figure 3.5: Illustration to Example 54.

3.3.3 Confluence of Derived Fuzzy Relations

In this subsection, we focus on properties of derived **L**-relations. In particular, we focus on **L**-relations which result by so-called a-multiples and a-shifts from other **L**-relations. We show lower and upper estimates of degrees of divergence, convergence, and confluence of the derived **L**-relations based on the corresponding properties of the original **L**-relations. For sake of simplicity, we will use here the notions developed in Section 3.1, i.e., notions without considering a similarity.

From now on, let $a \in L$ and let $\rightharpoonup$ be an arbitrary **L**-relation on X. Moreover, we consider an **L**-relation $\rightharpoonup_2$ which is the a-multiple of $\rightharpoonup$, i.e., $\rightharpoonup_2 = a \otimes \rightharpoonup$.

Theorem 55. *For each $x, y \in X$, the following inequalities hold:*

$$\textstyle\bigwedge_{n\in\mathbb{N}} a^n \otimes x \rightharpoonup^* y \le x\,(\rightharpoonup_2)^*\,y \le x \rightharpoonup^* y. \tag{3.9}$$

Proof. The first inequality can be proved using (2.8) and the fact that $\bigwedge_{n\in\mathbb{N}} a^n \le a^{k+1}$ for every $a \in L$ and $k \in \mathbb{N}_0$. Indeed, we get

$$\begin{aligned}
&\textstyle\bigwedge_{n\in\mathbb{N}} a^n \otimes x \rightharpoonup^* y = \bigwedge_{n\in\mathbb{N}} a^n \otimes \bigvee_{\langle z_1,\ldots,z_k\rangle\in X^*} (x \rightharpoonup z_1 \otimes \cdots \otimes z_k \rightharpoonup y) = \\
&= \textstyle\bigvee_{\langle z_1,\ldots,z_k\rangle\in X^*} \left[\bigwedge_{n\in\mathbb{N}} a^n \otimes (x \rightharpoonup z_1 \otimes \cdots \otimes z_k \rightharpoonup y)\right] \le \\
&\le \textstyle\bigvee_{\langle z_1,\ldots,z_k\rangle\in X^*} \left[a^{k+1} \otimes (x \rightharpoonup z_1 \otimes \cdots \otimes z_k \rightharpoonup y)\right] = \\
&= \textstyle\bigvee_{\langle z_1,\ldots,z_k\rangle\in X^*} \left[(a \otimes x \rightharpoonup z_1) \otimes \cdots \otimes (a \otimes z_k \rightharpoonup y)\right] = x\,(a\otimes\rightharpoonup)^*\,y = x\,(\rightharpoonup_2)^*\,y.
\end{aligned}$$

The second inequality follows directly from the isotony of $\bigvee$ and fact that $a\otimes\rightharpoonup \subseteq \rightharpoonup$, i.e., $x\,(a\otimes\rightharpoonup)\,y \le x \rightharpoonup y$ for each $x, y \in X$:

$$\begin{aligned}
&x\,(\rightharpoonup_2)^*\,y = x\,(a\otimes\rightharpoonup)^*\,y = \textstyle\bigvee_{\langle z_1,\ldots,z_k\rangle\in X^*} (x\,(a\otimes\rightharpoonup)\,z_1 \otimes \cdots \otimes z_k\,(a\otimes\rightharpoonup)\,y) \le \\
&\le \textstyle\bigvee_{\langle z_1,\ldots,z_k\rangle\in X^*} (x \rightharpoonup z_1 \otimes \cdots \otimes z_k \rightharpoonup y) \le x \rightharpoonup^* y,
\end{aligned}$$

showing both inequalities. □

Theorem 56. *For each $x, y \in X$, the following inequalities hold:*

$$\left(\bigwedge_{n\in\mathbb{N}} a^n\right)^2 \otimes x \upharpoonleft y \leq x \upharpoonleft_2 y \leq x \upharpoonleft y, \tag{3.10}$$
$$\left(\bigwedge_{n\in\mathbb{N}} a^n\right)^2 \otimes x \downharpoonleft y \leq x \downharpoonleft_2 y \leq x \downharpoonleft y. \tag{3.11}$$

Proof. Using Theorem 55 and (2.8), the first inequality in (3.10) can be proved as follows:

$$\begin{aligned}&\left(\bigwedge_{n\in\mathbb{N}} a^n\right)^2 \otimes x \upharpoonleft y = \left(\bigwedge_{n\in\mathbb{N}} a^n\right)^2 \otimes \bigvee_{z\in X} (z \rightharpoonup^* x \otimes z \rightharpoonup^* y) = \\ = &\bigvee_{z\in X} \left(\bigwedge_{n\in\mathbb{N}} a^n \otimes z \rightharpoonup^* x \otimes \bigwedge_{n\in\mathbb{N}} a^n \otimes z \rightharpoonup^* y\right) \leq \\ \leq &\bigvee_{z\in X} (z\,(a\otimes\rightharpoonup)^* x \otimes z\,(a\otimes\rightharpoonup)^* y) = x \upharpoonleft_2 y.\end{aligned}$$

The remaining part of (3.10) follows directly from Theorem 55:

$$x \upharpoonleft_2 y = \bigvee_{z\in X} (z\,(a\otimes\rightharpoonup)^* x \otimes z\,(a\otimes\rightharpoonup)^* y) \leq \bigvee_{z\in X} (z \rightharpoonup^* x \otimes z \rightharpoonup^* y) = x \upharpoonleft y.$$

The inequalities in (3.11) can be shown analogously. □

Now, we can express lower and upper bounds for degrees of confluence of the a-multiple of $\rightharpoonup$ based on $a \in L$ and the degrees of confluence of $\rightharpoonup$ as follows:

Theorem 57. $\left(\bigwedge_{n\in\mathbb{N}} a^n\right)^2 \otimes \mathrm{CFL}(\rightharpoonup) \leq \mathrm{CFL}(\rightharpoonup_2) \leq \left(\bigwedge_{n\in\mathbb{N}} a^n\right)^2 \to \mathrm{CFL}(\rightharpoonup)$.

Proof. Due to Theorem 56, using the inequalities (2.5) and (2.9), the antitony of $\to$ in its first argument and the isotony of $\to$ in the second argument, we obtain

$$\begin{aligned}&\left(\bigwedge_{n\in\mathbb{N}} a^n\right)^2 \otimes \mathrm{CFL}(\rightharpoonup) = \left(\bigwedge_{n\in\mathbb{N}} a^n\right)^2 \otimes \bigwedge_{x,y\in X} (x \upharpoonleft y \to x \downharpoonleft y) \leq \\ \leq &\bigwedge_{x,y\in X} \left[\left(\bigwedge_{n\in\mathbb{N}} a^n\right)^2 \otimes (x \upharpoonleft y \to x \downharpoonleft y)\right] \leq \bigwedge_{x,y\in X} \left[x \upharpoonleft y \to \left(\left(\bigwedge_{n\in\mathbb{N}} a^n\right)^2 \otimes x \downharpoonleft y\right)\right] \leq \\ \leq &\bigwedge_{x,y\in X} \left[x \upharpoonleft_2 y \to \left(\left(\bigwedge_{n\in\mathbb{N}} a^n\right)^2 \otimes x \downharpoonleft y\right)\right] \leq \bigwedge_{x,y\in X} (x \upharpoonleft_2 y \to x \downharpoonleft_2 y) = \mathrm{CFL}(\rightharpoonup_2),\end{aligned}$$

which proves the first part of the inequality. In order to show the remaining inequality, we use Theorem 56 together with (2.3), (2.10), and the antitony (isotony) of $\to$ in the first (second) argument:

$$\begin{aligned}&\mathrm{CFL}(\rightharpoonup_2) = \bigwedge_{x,y\in X} (x \upharpoonleft_2 y \to x \downharpoonleft_2 y) \leq \bigwedge_{x,y\in X} \left[\left(\left(\bigwedge_{n\in\mathbb{N}} a^n\right)^2 \otimes x \upharpoonleft y\right) \to x \downharpoonleft y\right] \leq \\ \leq &\bigwedge_{x,y\in X} \left[\left(\bigwedge_{n\in\mathbb{N}} a^n\right)^2 \to (x \upharpoonleft y \to x \downharpoonleft y)\right] = \left(\bigwedge_{n\in\mathbb{N}} a^n\right)^2 \to \bigwedge_{x,y\in X} (x \upharpoonleft y \to x \downharpoonleft y) = \\ = &\left(\bigwedge_{n\in\mathbb{N}} a^n\right)^2 \to \mathrm{CFL}(\rightharpoonup),\end{aligned}$$

which concludes the proof. □

While considering only idempotent elements $a \in L$, the inequalities in Theorem 55, Theorem 56, and Theorem 57 can be simplified.

Corollary 58. *Let $a \in L$ be idempotent. Then, for each $x, y \in X$, we get*

$$x \rightharpoonup_2^* y = a \otimes x \rightharpoonup^* y, \tag{3.12}$$
$$x \upharpoonleft_2 y = a \otimes x \upharpoonleft y, \tag{3.13}$$
$$x \downharpoonleft_2 y = a \otimes x \downharpoonleft y. \tag{3.14}$$

Proof. The equality (3.12) can be proved similarly to the first inequality in Theorem 55:

$$\begin{aligned} x\,(\rightharpoonup_2)^*\,y &= x\,(a\otimes\rightharpoonup)^*\,y = \bigvee_{\langle z_1,\dots,z_k\rangle\in X^*}[(a\otimes x\rightharpoonup z_1)\otimes\cdots\otimes(a\otimes z_k\rightharpoonup y)] = \\ &= \bigvee_{\langle z_1,\dots,z_k\rangle\in X^*}\left[a^{k+1}\otimes(x\rightharpoonup z_1\otimes\cdots\otimes z_k\rightharpoonup y)\right] = \\ &= \bigvee_{\langle z_1,\dots,z_k\rangle\in X^*}[a\otimes(x\rightharpoonup z_1\otimes\cdots\otimes z_k\rightharpoonup y)] = \\ &= a\otimes\bigvee_{\langle z_1,\dots,z_k\rangle\in X^*}(x\rightharpoonup z_1\otimes\cdots\otimes z_k\rightharpoonup y) = a\otimes x\rightharpoonup^* y. \end{aligned}$$

The proof of (3.13) is based on (3.12) and the assumption that a is an idempotent element of $\mathbf{L}$. This proof is done similarly to the first part of the proof of Theorem 56:

$$\begin{aligned} x\upharpoonleft_2 y &= \bigvee_{z\in X}(z\,(a\otimes\rightharpoonup)^*\,x\otimes z\,(a\otimes\rightharpoonup)^*\,y) = \\ &= \bigvee_{z\in X}(a\otimes z\rightharpoonup^* x\otimes a\otimes z\rightharpoonup^* y) = \\ &= a^2\otimes\bigvee_{z\in X}(z\rightharpoonup^* x\otimes z\rightharpoonup^* y) = a\otimes x\upharpoonleft y. \end{aligned}$$

The proof of (3.14) is very similar and is therefore omitted. □

Corollary 59. *If $a\in L$ is idempotent, then* $\mathrm{CFL}(\rightharpoonup)\le\mathrm{CFL}(\rightharpoonup_2)\le a\to\mathrm{CFL}(\rightharpoonup)$.

Proof. Corollary 58 together with (2.6) yields

$$\begin{aligned} \mathrm{CFL}(\rightharpoonup_2) &= \bigwedge_{x,y}[(x\upharpoonleft_2 y)\to(x\downharpoonleft_2 y)] = \bigwedge_{x,y}[(a\otimes x\upharpoonleft y)\to(a\otimes x\downharpoonleft y)]\ge \\ &\ge \bigwedge_{x,y}[(a\to a)\otimes(x\upharpoonleft y\to x\downharpoonleft y)] = \bigwedge_{x,y}[1\otimes(x\upharpoonleft y\to x\downharpoonleft y)] = \\ &= \bigwedge_{x,y}(x\upharpoonleft y\to x\downharpoonleft y) = \mathrm{CFL}(\rightharpoonup). \end{aligned}$$

The rest can be shown using Theorem 57 and the fact that $\left(\bigwedge_{n\in\mathbb{N}}a^n\right)^2 = a$. □

Example 60. Let $\mathbf{L}$ be a complete residuated lattice with Gödel structure of truth degrees [5, 29], i.e., the structure on the real unit interval with $\vee$ being maximum, $\wedge$ and $\otimes$ being both minimum, and $\to$ is defined by $a\to b=b$ (for $a>b$) and $a\to b=1$ otherwise. We also recall that all elements of $\mathbf{L}$ are idempotent.
Let $\rightharpoonup$ be $\mathbf{L}$-relation on X which is depicted in the left part of Figure 3.6. Using direct computation, we get $\mathrm{CFL}(\rightharpoonup)=0.6$. Now, consider the derived $\mathbf{L}$-relation $0.7\otimes\rightharpoonup$, which is depicted in the middle part of Figure 3.6. By Corollary 59, we immediately get $0.6\le\mathrm{CFL}(0.7\otimes\rightharpoonup)\le 0.7\to 0.6=0.6$, i.e. $\mathrm{CFL}(0.7\otimes\rightharpoonup)=0.6$, without its demanding direct computation.

We now turn our attention to properties of $\mathbf{L}$-relations which result by a-shifts. In the sequel, we denote by $\rightharpoonup_3$ an $\mathbf{L}$-relation which results from $\rightharpoonup$ using an a-shift, i.e., $\rightharpoonup_3 = a\to\rightharpoonup$ for a given $a\in L$.

Theorem 61. *For each $x,y\in X$, the following inequalities hold:*

$$x\rightharpoonup^* y\le x\,(\rightharpoonup_3)^*\,y\le\bigwedge_{n\in\mathbb{N}}a^n\to x\rightharpoonup^* y. \qquad (3.15)$$

Proof. The first inequality follows easily since $\rightharpoonup\subseteq a\to\rightharpoonup$ for each $a\in L$. We now prove the remaining inequality. Using commutativity and associativity of $\bigvee$, we immediately get

$$\begin{aligned} x\rightharpoonup^* y &= \bigvee_{\langle z_1,\dots,z_k\rangle\in X^*}\left(x\rightharpoonup z_1\otimes z_1\rightharpoonup z_2\otimes\cdots\otimes z_k\rightharpoonup y\right) = \\ &= \bigvee_{k\in\mathbb{N}_0}\bigvee_{z_1,\dots,z_k\in X}(x\rightharpoonup z_1\otimes z_1\rightharpoonup z_2\otimes\cdots\otimes z_k\rightharpoonup y). \end{aligned}$$

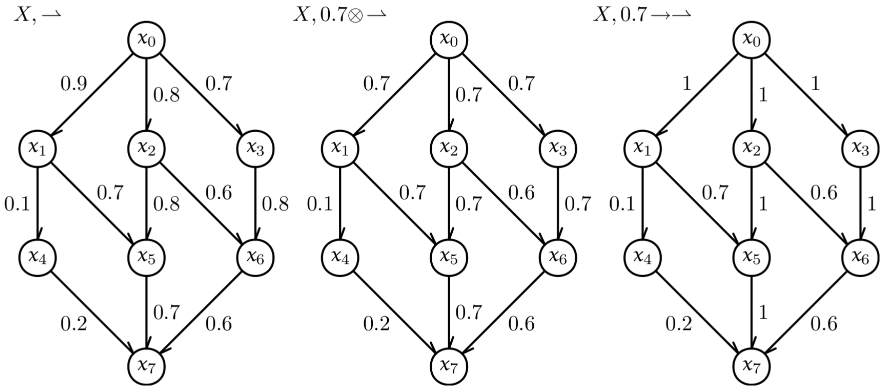

Figure 3.6: Illustration to Examples 60 and 65.

By applying (2.12) and (2.13), we acquire:

$$\begin{aligned}&\textstyle\bigwedge_{n\in\mathbb{N}_0} a^{n+1} \to x \rightharpoonup^* y \geq \bigvee_{n\in\mathbb{N}_0} \left(a^{n+1} \to x \rightharpoonup^* y\right) =\\ &= \textstyle\bigvee_{n\in\mathbb{N}_0} \left[a^{n+1} \to \bigvee_{k\in\mathbb{N}_0} \bigvee_{z_1,\dots,z_k\in X} (x \rightharpoonup z_1 \otimes \cdots \otimes z_k \rightharpoonup y)\right] \geq\\ &\geq \textstyle\bigvee_{n\in\mathbb{N}_0} \bigvee_{k\in\mathbb{N}_0} \bigvee_{z_1,\dots,z_k\in X} \left[a^{n+1} \to (x \rightharpoonup z_1 \otimes \cdots \otimes z_k \rightharpoonup y)\right].\end{aligned}$$

By identifying the numbers $k = n$, we get

$$\begin{aligned}&\textstyle\bigvee_{n\in\mathbb{N}_0} \bigvee_{k\in\mathbb{N}_0} \bigvee_{z_1,\dots,z_k\in X} \left[a^{n+1} \to (x \rightharpoonup z_1 \otimes \cdots \otimes z_k \rightharpoonup y)\right] \geq\\ &\geq \textstyle\bigvee_{k\in\mathbb{N}_0} \bigvee_{z_1,\dots,z_k\in X} \left[a^{k+1} \to (x \rightharpoonup z_1 \otimes \cdots \otimes z_k \rightharpoonup y)\right] =\\ &= \textstyle\bigvee_{k\in\mathbb{N}_0} \bigvee_{z_1,\dots,z_k\in X} \left[(a \otimes \cdots \otimes a) \to (x \rightharpoonup z_1 \otimes \cdots \otimes z_k \rightharpoonup y)\right] \geq\\ &\geq \textstyle\bigvee_{k\in\mathbb{N}_0} \bigvee_{z_1,\dots,z_k\in X} \left[(a \to x \rightharpoonup z_1) \otimes \cdots \otimes (a \to z_k \rightharpoonup y)\right] = x\,(a \to \rightharpoonup)^*\, y = x\,(\rightharpoonup_3)^*\, y,\end{aligned}$$

which follows using (2.7). □

Theorem 62. *For each $x, y \in X$, the following inequalities hold:*

$$x \upharpoonleft y \leq x \upharpoonleft_3 y, \leq \left(\textstyle\bigwedge_{n\in\mathbb{N}} a^n\right)^2 \to x \upharpoonleft y, \tag{3.16}$$

$$x \downharpoonleft y \leq x \downharpoonleft_3 y \leq \left(\textstyle\bigwedge_{n\in\mathbb{N}} a^n\right)^2 \to x \downharpoonleft y. \tag{3.17}$$

Proof. The first inequalities in (3.16) and (3.17) can be proved easily using the fact that $\rightharpoonup \subseteq a \to \rightharpoonup$ for each $a \in L$. The second part of (3.16) follows from Theorem 61 using (2.6) and (2.13):

$$\begin{aligned}&x \upharpoonleft_3 y = \textstyle\bigvee_{z\in X} \left[z\,(a \to \rightharpoonup)^*\, x \otimes z\,(a \to \rightharpoonup)^*\, y\right] \leq\\ &\leq \textstyle\bigvee_{z\in X} \left[\left(\bigwedge_{n\in\mathbb{N}} a^n \to z \rightharpoonup^* x\right) \otimes \left(\bigwedge_{n\in\mathbb{N}} a^n \to z \rightharpoonup^* y\right)\right] \leq\\ &\leq \textstyle\bigvee_{z\in X} \left[\left(\bigwedge_{n\in\mathbb{N}} a^n \otimes \bigwedge_{n\in\mathbb{N}} a^n\right) \to (z \rightharpoonup^* x \otimes z \rightharpoonup^* y)\right] \leq\\ &\leq \left(\textstyle\bigwedge_{n\in\mathbb{N}} a^n\right)^2 \to \textstyle\bigvee_{z\in X} (z \rightharpoonup^* x \otimes z \rightharpoonup^* y) = \left(\textstyle\bigwedge_{n\in\mathbb{N}} a^n\right)^2 \to x \upharpoonleft y.\end{aligned}$$

Finally, the remaining part of (3.17) can be proved analogously. □

Theorem 63. $\mathrm{CFL}(\rightharpoonup_3) \leq \left(\bigwedge_{n\in\mathbb{N}} a^n\right)^2 \to \mathrm{CFL}(\rightharpoonup)$.

Proof. According to Theorem 56 and using the antitony of $\to$ in the first argument and the isotony of $\to$ in the second argument, we get

$$\begin{aligned}\mathrm{CFL}(\rightharpoonup_3) = \bigwedge_{x,y\in X}(x \upharpoonleft_3 y \to x \downharpoonleft_3 y) \leq \bigwedge_{x,y\in X}(x \upharpoonleft y \to x \downharpoonleft_3 y) \leq \\ \leq \bigwedge_{x,y\in X}\Big[x \upharpoonleft y \to \Big(\big(\bigwedge_{n\in\mathbb{N}} a^n\big)^2 \to x \downharpoonleft y\Big)\Big].\end{aligned}$$

By applying the equalities (2.4) and (2.10), we further obtain

$$\begin{aligned}\mathrm{CFL}(\rightharpoonup_3) \leq \bigwedge_{x,y\in X}\Big[\big(\bigwedge_{n\in\mathbb{N}} a^n\big)^2 \to (x \upharpoonleft y \to x \downharpoonleft y)\Big] = \\ = \big(\bigwedge_{n\in\mathbb{N}} a^n\big)^2 \to \bigwedge_{x,y\in X}(x \upharpoonleft y \to x \downharpoonleft y) = \big(\bigwedge_{n\in\mathbb{N}} a^n\big)^2 \to \mathrm{CFL}(\rightharpoonup)\end{aligned}$$

which concludes the proof. □

As in case of a-multiples, the estimation formulas can be simplified if a is an idempotent element of **L**.

Corollary 64. *Let $a \in L$ be idempotent, then the following inequalities hold:*

$$\begin{aligned}x(\rightharpoonup_3)^* y &\leq a \to x \rightharpoonup^* y,\\ x \upharpoonleft_3 y &\leq a \to x \upharpoonleft y,\\ x \downharpoonleft_3 y &\leq a \to x \downharpoonleft y,\\ \mathrm{CFL}(\rightharpoonup_3) &\leq a \to \mathrm{CFL}(\rightharpoonup).\end{aligned}$$

Proof. Directly from Theorems 61, 62 and 63 using idempotency of a. □

Example 65. Let **L** be a complete residuated lattice with Gödel structure of truth degrees (see Example 60 for details). Let $\rightharpoonup$ be **L**-relation on X which is depicted in the left part of Figure 3.6. Using the definition of CFL(·), we can compute $\mathrm{CFL}(\rightharpoonup) = 0.6$. Now, let us consider the derived **L**-relation $0.7 \to\rightharpoonup$, which is depicted in the right part of Figure 3.6. By Corollary 64, we immediately get $\mathrm{CFL}(0.7 \to\rightharpoonup) \leq 0.6$, which provides a fast estimation without the need to compute the precise value $\mathrm{CFL}(0.7 \to\rightharpoonup)$, which is more demanding. The exact confluence degree of $0.7 \to\rightharpoonup$ is $\mathrm{CFL}(0.7 \to\rightharpoonup) = 0.6$.

3.4 Termination and Related Properties of Fuzzy Relation

This section aims on developing notions related to the termination of fuzzy relations and investigating their properties. Unlike in the previous sections, there is no indistinguishability relation taken into account here. Due to our inspiration by the bivalent notions, the ordinary properties of reductions can be seen as a particular case in our approach, i.e. the case when the underlying structure of truth degrees is the two-valued Boolean algebra.

The results summarized in this section have been published in [8].

3.4.1 Termination

In this subsection we turn our attention to termination which is considered an important property of abstract rewriting systems. Intuitively, termination can be seen as a natural property of "algorithms", saying that each reduction may terminate after finitely many steps. We would like to have the same intuitive interpretation of termination in case of fuzzy relations. Therefore, unlike the notions introduced in the previous sections which were naturally graded, termination seems to be a bivalent notion—a reduction either terminates or not. From the computational point of view, it is desirable that each reduction that is supposed to be handled algorithmically stops after finitely many steps. Following these motivations, we introduce several notions of termination and inspect their properties. We first introduce a new notation. For any sequence $x_0, \ldots, x_n$ of elements from X, let $\mathrm{nt}(x_0, \ldots, x_n) \in L$ denote a degree defined by

$$\mathrm{nt}(x_0, \ldots, x_n) = \bigvee_{y \in X} \mathrm{re}(x_0, \ldots, x_n, y). \tag{3.18}$$

Remark 66. Following the definition of $\mathrm{re}(x_0, \ldots, x_n)$, (3.18) is equivalent to

$$\mathrm{nt}(x_0, \ldots, x_n) = \mathrm{re}(x_0, \ldots, x_n) \otimes \bigvee_{y \in X}(x_n \rightharpoonup y).$$

Hence, $\mathrm{nt}(x_0, \ldots, x_n)$ can be seen as a degree to which the sequence $x_0, \ldots, x_n$ can be extended by an additional element y. If $x \rightharpoonup y$ is interpreted as a degree to which x reduces to y then $\mathrm{nt}(x_0, \ldots, x_n)$ is a degree to which "x_0 reduces to x_1 and, ..., and x_{n-1} reduces to x_n, and there is $y \in X$ such that x_n reduces to y". If $\mathrm{re}(x_0, \ldots, x_n) \neq 0$ and $\mathrm{nt}(x_0, \ldots, x_n) = 0$, we may say that $x_0, \ldots, x_n$ cannot be further extended with a nonzero degree of reduction. In a special case for $n = 0$, the latter is true if

$$\mathrm{nt}(x_0) = \bigvee_{y \in X} \mathrm{re}(x_0, y) = \bigvee_{y \in X}(x_0 \rightharpoonup y) = 0.$$

These observations motivate the following definition of termination.

Definition 67. An element $x \in X$ has a ***terminating reduction*** if there is a finite sequence $x = x_0, x_1, \ldots, x_n$ ($n \geq 0$) such that $\mathrm{re}(x_0, \ldots, x_n) \neq 0$, and $\mathrm{nt}(x_0, \ldots, x_n) = 0$. An element $x \in X$ is called ***irreducible*** if $\mathrm{nt}(x) = 0$. An element $x \in X$ has a ***strictly terminating reduction*** if there is a terminating reduction $x = x_0, x_1, \ldots, x_n$, where x_n is irreducible. An element $x \in X$ has a ***nonterminating reduction*** if there is an infinite sequence $x = x_0, x_1, \ldots$ such that for each $n \in \mathbb{N}_0$, $\mathrm{nt}(x_0, \ldots, x_n) \neq 0$.

Remark 68. (1) The first requirement in the definition of the terminating reduction, namely $\mathrm{re}(x_0, \ldots, x_n) \neq 0$, postulates that the element x_0 can be reduced to the element x_n to a nonzero degree. On the other hand, the second condition, i.e. $\mathrm{nt}(x_0, \ldots, x_n) = 0$, says that there is no element, which can further extend this reduction.
(2) Using (3.18), condition $\mathrm{nt}(x) = 0$ can be equivalently written as

$$\mathrm{re}(x) \otimes \bigvee_{y \in X} x \rightharpoonup y = 1 \otimes \bigvee_{y \in X} x \rightharpoonup y = \bigvee_{y \in X} x \rightharpoonup y = 0.$$

Hence, if x is an irreducible element then $x \rightharpoonup y = 0$ is true for each $y \in X$.
(3) An element $x \in X$ has a nonterminating reduction iff there is an infinite sequence $x = x_0, x_1, \ldots$ such that for each $n \in \mathbb{N}_0$, $\mathrm{re}(x_0, \ldots, x_n) \neq 0$. Intuitively, the element x can be reduced infinitely many times.
(4) Each strictly terminating reduction is terminating. The converse claim does not hold.

Example 69. Consider the $\mathbf{L}$-relation $\rightharpoonup$ from Fig. 3.7 and let $\mathbf{L}$ be the Łukasiewicz structure. The element x_0 has a reduction $x_0 x_1 x_2$ which is terminating, i.e. $\mathrm{nt}(x_0, x_1, x_2) = 0$. On the other hand, the element x_2 is not irreducible since $\mathrm{nt}(x_2) = 0.4 \neq 0$. Hence, this reduction is not strictly terminating.

Since termination and strict termination of fuzzy relations have been introduced as bivalent notions, we should investigate their relationship to the (ordinary) termination of bivalent relations. The following assertion shows that termination of fuzzy relations can be observed from their strong 0-cuts.

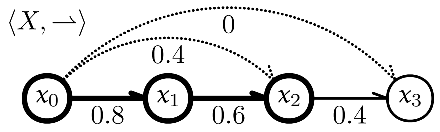

Figure 3.7: Terminating reduction which is not strictly terminating.

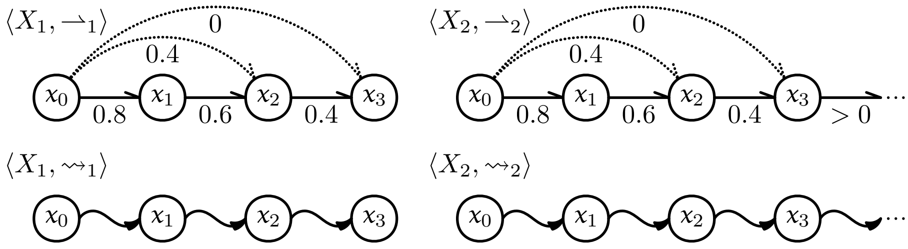

Figure 3.8: Counterexample to the converse claim to Theorem 70.

Theorem 70. *The following are true for any binary* **L***-relation* $\rightharpoonup$ *on* X*.*

(*i*) *If an element* x *has a strictly terminating reduction, then* x *has a terminating reduction in the strong* 0*-cut* $\rightsquigarrow$ *of* $\rightharpoonup$*.*

(*ii*) *If an element* x *has a nonterminationg reduction, then* x *has a nonterminating reduction in the strong* 0*-cut* $\rightsquigarrow$ *of* $\rightharpoonup$*.*

Proof. (*i*): Denote by $\rightsquigarrow$ the strong 0-cut of $\rightharpoonup$. Let x have a strictly terminating reduction $x = x_0, x_1, \ldots, x_n$. By definition, we have $\mathrm{re}(x_0, \ldots, x_n) \neq 0$ and $\mathrm{nt}(x_n) = 0$. The inequality $\mathrm{re}(x_0, \ldots, x_n) \neq 0$ yields $(x_0 \rightharpoonup x_1) \otimes \cdots \otimes (x_{n-1} \rightharpoonup x_n) \neq 0$, i.e. $x_i \rightharpoonup x_{i+1} \neq 0$ for each $i \in \{0, \ldots, n-1\}$. Therefore, in the strong 0-cut $\rightsquigarrow$ of $\rightharpoonup$, we have $x_i \rightsquigarrow x_{i+1}$ for each $i \in \{0, \ldots, n-1\}$, i.e. there is a reduction $x = x_0, x_1, \ldots, x_n$. From $\mathrm{nt}(x_n) = 0$ it follows that $x_n \rightharpoonup y = 0$ for each $y \in X$. Thus, there is no element $y \in X$ such that $x_n \rightsquigarrow y$, i.e. x_n is an irreducible element in the strong 0-cut $\rightsquigarrow$. Altogether, the reduction $x_0, x_1, \ldots, x_n$ in $\rightsquigarrow$ is terminating.
(*ii*): Let x has a nonterminating reduction $x = x_0, x_1, \ldots$ By Remark 68, for each $n \in \mathbb{N}_0$, $\mathrm{re}(x_0, \ldots, x_n) \neq 0$. Hence, $(x_0 \rightharpoonup x_1) \otimes (x_1 \rightharpoonup x_2) \otimes \cdots \neq 0$, i.e. $x_i \rightharpoonup x_{i+1} \neq 0$ is true for each $i \in \mathbb{N}_0$. Therefore, for each $i \in \mathbb{N}_0$, we get $x_i \rightsquigarrow x_{i+1}$, i.e. $x_0, x_1, \ldots$ is a nonterminating reduction in the strong 0-cut $\rightsquigarrow$. □

Example 71. The converse claim to Theorem 70 does not hold in general. Consider $\rightharpoonup_1$ and $\rightharpoonup_2$ from Fig. 3.8 and their strong 0-cuts $\rightsquigarrow_1$ and $\rightsquigarrow_2$, respectively. Suppose that **L** is the Łukasiewicz structure of truth degrees. In these two cases, $x_0 \in X_1$ has a terminating reduction $x_0x_1x_2x_3$ in the strong 0-cut but $\mathrm{re}(x_0, x_1, x_2, x_3) = 0$, i.e. it is not a reduction with respect to $\rightharpoonup_1$. Furthermore, $x_0 \in X_2$ has a nonterminating reduction $x_0x_1x_2\ldots$ in the strong 0-cut but it does not have any nonterminating reduction with respect to $\rightharpoonup_2$.

We now introduce (strict) termination as a property of **L**-relations:

Definition 72. An **L**-relation $\rightharpoonup$ on X is called ***terminating*** if no $x \in X$ has a nonterminating reduction. Moreover, $\rightharpoonup$ is called ***strictly terminating*** if (i) $\rightharpoonup$ is terminating, and (ii) for each $x \in X$: if x has a terminating reduction $x = x_0, \ldots, x_n$ then x_n is irreducible.

Remark 73. By definition, each strictly terminating **L**-relation is terminating, the converse does not hold in general (see Example 69).

Theorem 74. *If $\rightharpoonup$ is terminating then each $x \in X$ has a terminating reduction.*

Proof. The proof is done by showing that x has a terminating reduction $y_0, \ldots, y_k$. We construct the sequence $y_0, \ldots, y_k$ incrementally provided that in each j-th step we already have $y_0, \ldots, y_j$ with $\mathrm{re}(y_0, \ldots, y_j) \neq 0$.
Put $y_0 = x$. Directly from the definition, $\mathrm{re}(y_0) = 1$. Let us have $y_0, \ldots, y_j$ with $\mathrm{re}(y_0, \ldots, y_j) \neq 0$. If $\mathrm{nt}(y_0, \ldots, y_j) = 0$, we are done for $y_0, \ldots, y_j$ is a terminating reduction. In the other case, we have $\bigvee_{y \in X} \mathrm{re}(y_0, \ldots, y_j, y) \neq 0$, i.e. there is $y \in X$ such that $\mathrm{re}(y_0, \ldots, y_j, y) \neq 0$. We can put $y_{j+1} = y$. Clearly, we cannot repeat the above procedure infinitely many times without getting a sequence with $\mathrm{nt}(y_0, \ldots, y_k) = 0$, because x does not have any nonterminating reductions. $\square$

3.4.2 Inductive Properties and Well-Foundedness

We shall now investigate further properties of fuzzy relations which are closely related to termination. We are motivated by the fact that in the ordinary case, a relation terminates iff it is well-founded which is iff the relation obeys the principle of Noetherian induction. In this subsection we first present a generalization of inductive properties and well-foundedness and then investigate their relationship to our notions of termination. We conclude the subsection by presenting applications of well-foundedness. Namely, we introduce a notion of a local confluence and prove its relationship to confluence which is analogous to that known from the ordinary case.

Definition 75. An $\mathbf{L}$-set $\mathcal{P} \in L^X$ is called a ***property***. For any subset $B \subseteq X$ we define a degree $||\mathcal{P}||_B \in L$ to which $\mathcal{P}$ holds in B by $||\mathcal{P}||_B = \bigwedge_{b \in B} \mathcal{P}(b)$. An $\mathbf{L}$-set $\mathcal{P} \in L^X$ is called an ***inductive property*** with the respect to an $\mathbf{L}$-relation $\rightharpoonup$, if

$$\textstyle\bigwedge_{y \in X} \big(x \rightharpoonup y \to \mathcal{P}(y)\big) \leq \mathcal{P}(x), \qquad \text{for each } x \in X. \tag{3.19}$$

Remark 76. An inductive property as it is established in Definition 75 is a graded generalization of the classical inductive property. Indeed, the formula used in the previous definition corresponds to the first-order predicate formula

$$(\forall x)((\forall y)(r(x, y) \Rightarrow p(y)) \Rightarrow p(x)), \tag{3.20}$$

which occurs in the definition of the classical inductive property (denoted by symbol) p on a set X with the respect to a relation (denoted by symbol) r. If (3.20) is required to be true to degree 1 in a general complete residuated lattice $\mathbf{L}$, we obtain exactly the concept from Definition 75. Thus, (3.19) can be read as follows: “In order to prove that x has property $\mathcal{P}$, it suffices to show that each reduct of x has property $\mathcal{P}$”. A finer reading, using degrees of truth may be: “The degree to which x has $\mathcal{P}$ is at least the degree to which all reducts of x have $\mathcal{P}$”. Obviously, for $\mathbf{L} = \mathbf{2}$, we obtain the ordinary concept of an inductive property.

Definition 77. For $\emptyset \neq B \subseteq X$, an element $m \in B$ is called ***$\rightharpoonup$-minimal*** in B if, for each $x \in X$, $m \rightharpoonup x \neq 0$ implies $x \notin B$. Let $\min(B) = \{m \in B \mid m \text{ is } \rightharpoonup\text{-minimal in } B\}$. $\rightharpoonup$ is called ***well-founded*** $\mathbf{L}$-relation on X if, for each $\emptyset \neq B \subseteq X$, $\min(B) \neq \emptyset$.

Remark 78. Well-foundedness is again a generalization of the corresponding ordinary notion for fuzzy relations. In addition to that, the fact that $\rightharpoonup$ is well-founded can be observed from its strong 0-cut. Indeed, $\rightharpoonup$ is a well-founded $\mathbf{L}$-relation on X iff each $B \subseteq X$ has at least one $\rightharpoonup$-minimal element which is true iff each $B \subseteq X$ has at least one

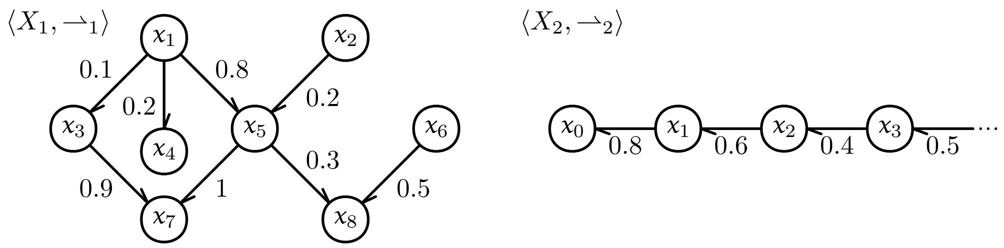

Figure 3.9: Well-founded **L**-relations.

$\rightsquigarrow$-minimal element in the strong 0-cut $\rightsquigarrow$ of $\rightharpoonup$. This follows directly from the fact that $m \rightharpoonup x \neq 0$ is true iff $m \rightsquigarrow x$, i.e. iff m is related with x in the strong 0-cut $\rightsquigarrow$ of $\rightharpoonup$. For **L** being **2**, Definition 77 yields the classical notion of well-foundedness.

Example 79. Take $\rightharpoonup_1$ and $\rightharpoonup_2$ from Fig. 3.9. Both $\rightharpoonup_1$ and $\rightharpoonup_2$ are well-founded. It is easy to show that every subset of X_1 has a $\rightharpoonup_1$-minimal element. The **L**-relation $\rightharpoonup_2$ is an example of a well-founded **L**-relation on an infinite set. Observe that the well-foundedness is not influenced by the choice of a structure of truth degrees. The **L**-relation $\rightharpoonup_1$ from Fig. 3.10 is an example of an **L**-relation which is not well-founded.

The following theorem generalizes the well-known principle of Noetherian induction:

Theorem 80. *Let $\rightharpoonup$ be well-founded. If $\mathcal{P}$ is an inductive property then $||\mathcal{P}||_X = 1$.*

Proof. Put $B = \{x \in X \mid \mathcal{P}(x) = 1\}$ and assume $X - B \neq \emptyset$. The well-foundedness of $\rightharpoonup$ yields that $\min(X - B) \neq \emptyset$, i.e. there is a $\rightharpoonup$-minimal element $m \in \min(X - B) \subseteq X - B$. Observe that if $m \rightharpoonup y = 0$, we get $m \rightharpoonup y \rightarrow \mathcal{P}(y) = 1$, because $0 \rightarrow a = 1$ is true for any $a \in L$. Suppose that $m \rightharpoonup y \neq 0$. Since m is a $\rightharpoonup$-minimal element in $X - B$, $m \rightharpoonup y \neq 0$ yields $y \notin X - B$, i.e. $y \in B$. By definition of B, we get $\mathcal{P}(y) = 1$. As a consequence, $m \rightharpoonup y \rightarrow \mathcal{P}(y) = 1$, because $a \rightarrow 1 = 1$ is true for any $a \in L$. Altogether, the equality $m \rightharpoonup y \rightarrow \mathcal{P}(y) = 1$ is true for any $y \in X$. Therefore, $\bigwedge_{y \in X}\big(m \rightharpoonup y \rightarrow \mathcal{P}(y)\big) = 1$. Since $\mathcal{P}$ is supposed to be an inductive property, $1 = \bigwedge_{y \in X}\big(m \rightharpoonup y \rightarrow \mathcal{P}(y)\big) \leq \mathcal{P}(m)$, i.e. $\mathcal{P}(m) = 1$ which violates $m \in X - B$. Thus, we have $B = X$ and hence $||\mathcal{P}||_X = 1$. □

We now investigate the relationship between well-foundedness, principle of Noetherian induction, and termination.

Theorem 81. *Each well-founded fuzzy relation is terminating.*

Proof. Let $\rightharpoonup$ be well-founded. By contradiction, suppose $\rightharpoonup$ is not terminating, i.e. there is $x \in X$, which has a nonterminating reduction $x = x_0, x_1, \ldots$ Since $B = \{x_i \mid i \in \mathbb{N}_0\} \neq \emptyset$ and $\rightharpoonup$ is well-founded, there is $j \in \mathbb{N}_0$ such that $x_j \in \min(B)$. Since $x = x_0, \ldots, x_j, \ldots$ is a nonterminating reduction, from $\mathrm{nt}(x_0, \ldots, x_j, x_{j+1}) \neq 0$ it follows that $x_j \rightharpoonup x_{j+1} \neq 0$, i.e. we have $x_{j+1} \notin B$ by $\rightharpoonup$-minimality of x_j which is a contradiction. □

Theorem 82. *Each strictly terminating fuzzy relation is well-founded.*

Proof. The claim follows from the properties of termination of the bivalent relations. If $\rightharpoonup$ is strictly terminating then the corresponding strong 0-cut $\rightsquigarrow$ is terminating in the classical sense. This follows from Theorem 70. As a consequence, $\rightsquigarrow$ is well-founded in the classical sense. By definition, for every subset $\emptyset \neq Y \subseteq X$, there is a $\rightsquigarrow$-minimal element m, i.e. $m \rightsquigarrow x$ implies $x \notin Y$. Therefore, $m \rightharpoonup x \neq 0$ implies $x \notin Y$, i.e. m is $\rightharpoonup$-minimal in Y and $\rightharpoonup$ is well-founded. □

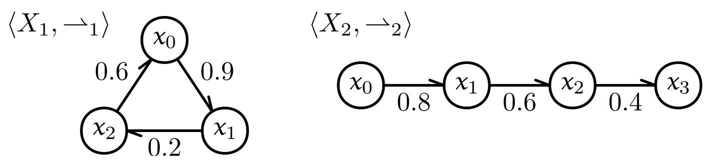

Figure 3.10: Terminating **L**-relation which is not well-founded, well-founded **L**-relation which is not strictly terminating (**L** is a Łukasiewicz structure).

Example 83. The converse claims to Theorem 81 and Theorem 82 do not hold in general. Counterexamples can be found in Fig. 3.10. For instance, consider $\rightharpoonup_2$ from Fig. 3.10 and the Łukasiewicz structure of truth degrees. We got that $\rightharpoonup_2$ is well-founded and is not strictly terminating. It is easy to show that every subset of X_2 has a $\rightharpoonup_2$-minimal element but terminating reduction $x_0 x_1 x_2$ is not strictly terminating.

The following corollary summarizes previous observations on termination of fuzzy relations and their corresponding strong 0-cuts.

Corollary 84. *If* **L** *has no zero-divisors then* $\rightharpoonup$ *is terminating iff* $\rightharpoonup$ *is strictly terminating iff* $\rightharpoonup$ *is well-founded iff* $\rightsquigarrow$ *is well-founded iff* $\rightsquigarrow$ *is terminating.*

Proof. The claim follows from the fact that if **L** has no zero-divisors then $\mathrm{re}(x_0, \ldots, x_n) \neq 0$ and $\mathrm{nt}(x_0, \ldots, x_n) = 0$ mean that, for each $y \in X$, $x_n \rightharpoonup y = 0$. Hence, $\rightharpoonup$ is terminating strictly. The rest follows from Theorem 82, Theorem 70, and properties of the classical termination. □

Finally, we present an assertion showing the equivalence of well-foundedness, termination, and Noetherian induction under the assumption of no zero divisors.

Theorem 85. *Let* **L** *has no zero-divisors. Then the following conditions are equivalent.*

(*i*) $\rightharpoonup$ *is well-founded;*

(*ii*) *for each inductive property* $\mathcal{P}$ *with respect to* $\rightharpoonup$ *we have* $||\mathcal{P}||_X = 1$*;*

(*iii*) $\rightharpoonup$ *is terminating.*

Proof. "$(i) \Rightarrow (ii)$": Apply Theorem 80.
"$(ii) \Rightarrow (iii)$": Consider property $\mathcal{P} \in L^X$ such that $\mathcal{P}(x) = 1$ if x does not have a nonterminating reduction; $\mathcal{P}(x) = 0$ otherwise. We first show that the property $\mathcal{P}$ is inductive, i.e. $\bigwedge_{y \in X}\big(x \rightharpoonup y \rightarrow P(y)\big) \leq P(x)$ is true for any $x \in X$. Since $P(x) \in \{0, 1\}$, it suffices to check the inequality for all $x \in X$ such that $\mathcal{P}(x) = 0$. Thus, take any $x \in X$ such that $\mathcal{P}(x) = 0$. By definition of $\mathcal{P}$, x has a nonterminating reduction $x, x_1, x_2, \ldots$ Obviously, x_1 has a nonterminating reduction as well, meaning $\mathcal{P}(x_1) = 0$. In addition to that, $x \rightharpoonup x_1 \neq 0$. As a consequence,

$$x \rightharpoonup x_1 \rightarrow \mathcal{P}(x_1) = x \rightharpoonup x_1 \rightarrow 0 = \bigvee\{c \in L \mid x \rightharpoonup x_1 \otimes c \leq 0\}.$$

Since **L** has no zero-divisors, $x \rightharpoonup x_1 \otimes c \leq 0$ iff $x \rightharpoonup x_1 \otimes c = 0$ iff $c = 0$ because $x \rightharpoonup x_1 \neq 0$. Therefore, $\bigvee\{c \in L \mid x \rightharpoonup x_1 \otimes c \leq 0\} = 0$, showing $x \rightharpoonup x_1 \rightarrow \mathcal{P}(x_1) = 0$. From this we further get $\bigwedge_{y \in X}\big(x \rightharpoonup y \rightarrow P(y)\big) = 0$, proving $\bigwedge_{y \in X}\big(x \rightharpoonup y \rightarrow P(y)\big) \leq P(x)$. Hence, we have shown that $\mathcal{P}$ is an inductive property. Therefore, using Theorem 80, we have $||\mathcal{P}||_X = 1$, i.e. no $x \in X$ has a nonterminating reduction, i.e. $\rightharpoonup$ is terminating.
"$(iii) \Rightarrow (i)$": Consequence of Corollary 84. □

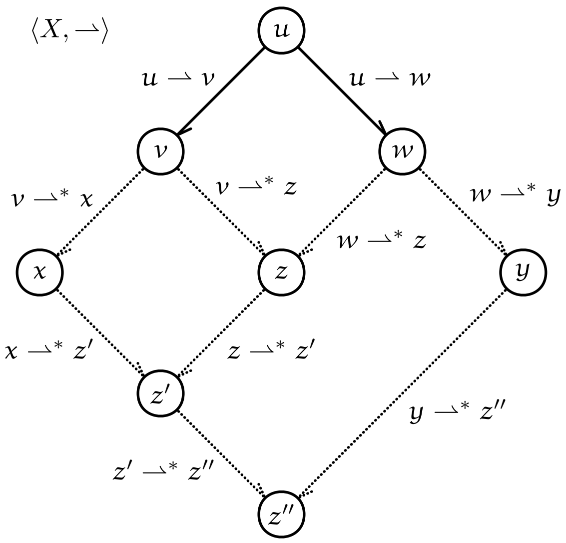

Figure 3.11: Illustration for the proof of Theorem 88.

We conclude this subsection with observations on confluent and terminating relations. The confluence of classical relations has a simpler characterization if the relation in question is terminating. Namely, there is a notion of a local confluence and each terminating relation is known to be confluent if and only if it is locally confluent. This characterization is beneficial since local confluence is much easier to check. We are going to show that in our setting there is also a notion of local confluence with similar properties.

Definition 86. We define a degree $\mathrm{LCFL}(\rightharpoonup)$ to which $\rightharpoonup$ is ***locally confluent*** by $\mathrm{LCFL}(\rightharpoonup) = S(\leftharpoondown \circ \rightharpoonup, \downharpoonleft)$. If $\mathrm{LCFL}(\rightharpoonup) = 1$ we say that $\rightharpoonup$ is locally confluent.

Remark 87. Observe the difference of confluence and local confluence. The only technical difference is that the definition of $\mathrm{LCFL}(\rightharpoonup)$ involves $\leftharpoondown \circ \rightharpoonup$ instead of $\upharpoonleft = \leftharpoondown^* \circ \rightharpoonup^*$. Clearly, $x \leftharpoondown \circ \rightharpoonup y$ is a degree to which there is z such that $z \rightharpoonup x$ and $z \rightharpoonup y$. Using graded subsethood, $\mathrm{LCFL}(\rightharpoonup)$ can be restated as follows:

$$\begin{aligned}\mathrm{LCFL}(\rightharpoonup) &= \textstyle\bigwedge_{x,y \in X}\big(\bigvee_{z \in X}(z \rightharpoonup x \otimes z \rightharpoonup y) \rightarrow x \downharpoonleft y\big) = \\ &= \textstyle\bigwedge_{x,y,z \in X}\big((z \rightharpoonup x \otimes z \rightharpoonup y) \rightarrow x \downharpoonleft y\big).\end{aligned}$$

Since $\leftharpoondown \circ \rightharpoonup \subseteq \leftharpoondown^* \circ \rightharpoonup^* = \upharpoonleft$, we get $\mathrm{CFL}(\rightharpoonup) \leq \mathrm{LCFL}(\rightharpoonup)$ because $\rightarrow$ is antitone in the first argument.

Theorem 88. *Let* $\rightharpoonup$ *be a well-founded* **L***-relation. Then* $\mathrm{CFL}(\rightharpoonup) = 1$ *iff* $\mathrm{LCFL}(\rightharpoonup) = 1$*, i.e.,* $\rightharpoonup$ *is confluent iff* $\rightharpoonup$ *is locally confluent.*

Proof. The "$\Rightarrow$" part of the claim follows from $\leftharpoondown \circ \rightharpoonup \subseteq \upharpoonleft$. In order to prove the "$\Leftarrow$" part, consider the following property:

$$\mathcal{P}(u) = \textstyle\bigwedge_{x,y \in X}\big((u \rightharpoonup^* x \otimes u \rightharpoonup^* y) \rightarrow x \downharpoonleft y\big).$$

We first prove that $\mathcal{P}$ is an inductive property. The proof is illustrated in Fig. 3.11. Take arbitrary $u \in X$. If u is irreducible (i.e., u is $\rightharpoonup$-minimal in X) then clearly $\mathcal{P}(u) = 1$. If u is not irreducible, we can proceed as follows. Assume that $\mathcal{P}(v) = 1$ for each v such that $u \rightharpoonup v \neq 0$. We are going to prove that the latter assumption implies $\mathcal{P}(u) = 1$.

First, notice that $u \rightharpoonup^* x \otimes u \rightharpoonup^* y \leq x \downharpoonright y$ is trivially true if x, y, and u are not distinct. Indeed, if $x = u$, we have

$$u \rightharpoonup^* x \otimes u \rightharpoonup^* y = 1 \otimes u \rightharpoonup^* y = u \rightharpoonup^* y \otimes 1 = u \rightharpoonup^* y \otimes y \rightharpoonup^* y \leq \\ \leq \bigvee_{z \in X}\big(u \rightharpoonup^* z \otimes y \rightharpoonup^* z\big) = u \downharpoonright y = x \downharpoonright y.$$

The other cases are analogous. Thus, consider $x \neq u \neq y$. Then,

$$u \rightharpoonup^* x \otimes u \rightharpoonup^* y = \bigvee_{v \in X}(u \rightharpoonup v \otimes v \rightharpoonup^* x) \otimes \bigvee_{w \in X}(u \rightharpoonup w \otimes w \rightharpoonup^* y) = \\ = \bigvee_{v,w \in X}(u \rightharpoonup v \otimes v \rightharpoonup^* x \otimes u \rightharpoonup w \otimes w \rightharpoonup^* y).$$

Notice that we can assume that the supremum "$\bigvee_{v,w \in X}$" in the last formula ranges over all $v, w \in X$ such that $u \rightharpoonup v \neq 0$ and $u \rightharpoonup w \neq 0$. Indeed, if either $u \rightharpoonup v = 0$ or $u \rightharpoonup w = 0$, we get $u \rightharpoonup v \otimes v \rightharpoonup^* x \otimes u \rightharpoonup w \otimes w \rightharpoonup^* y = 0$, i.e. such v and w are not essential.
Since $\rightharpoonup$ is locally confluent, $u \rightharpoonup v \otimes u \rightharpoonup w \leq v \downharpoonright w = \bigvee_{z \in X}(v \rightharpoonup^* z \otimes w \rightharpoonup^* z)$, i.e.

$$u \rightharpoonup^* x \otimes u \rightharpoonup^* y \leq \bigvee_{v,w \in X}\big(\bigvee_{z \in X}(v \rightharpoonup^* z \otimes w \rightharpoonup^* z) \otimes v \rightharpoonup^* x \otimes w \rightharpoonup^* y\big) = \\ = \bigvee_{v,w,z \in X}(v \rightharpoonup^* z \otimes w \rightharpoonup^* z \otimes v \rightharpoonup^* x \otimes w \rightharpoonup^* y).$$

Since $u \rightharpoonup v \neq 0$, $\mathcal{P}(v) = 1$ yields $v \rightharpoonup^* z \otimes v \rightharpoonup^* x \leq z \downharpoonright x = \bigvee_{z' \in X}(z \rightharpoonup^* z' \otimes x \rightharpoonup^* z')$. Therefore,

$$u \rightharpoonup^* x \otimes u \rightharpoonup^* y \leq \bigvee_{w,z \in X}\big(\bigvee_{z' \in X}(z \rightharpoonup^* z' \otimes x \rightharpoonup^* z') \otimes w \rightharpoonup^* z \otimes w \rightharpoonup^* y\big) = \\ = \bigvee_{w,z,z' \in X}(z \rightharpoonup^* z' \otimes x \rightharpoonup^* z' \otimes w \rightharpoonup^* z \otimes w \rightharpoonup^* y) \leq \\ \leq \bigvee_{w,z' \in X}(w \rightharpoonup^* z' \otimes x \rightharpoonup^* z' \otimes w \rightharpoonup^* y).$$

Moreover, $\mathcal{P}(w) = 1$, i.e. $w \rightharpoonup^* z' \otimes w \rightharpoonup^* y \leq z' \downharpoonright y = \bigvee_{z'' \in X}(z' \rightharpoonup^* z'' \otimes y \rightharpoonup^* z'')$, i.e.

$$u \rightharpoonup^* x \otimes u \rightharpoonup^* y \leq \bigvee_{z' \in X}\big(\bigvee_{z'' \in X}(z' \rightharpoonup^* z'' \otimes y \rightharpoonup^* z'') \otimes x \rightharpoonup^* z'\big) = \\ = \bigvee_{z',z'' \in X}(z' \rightharpoonup^* z'' \otimes y \rightharpoonup^* z'' \otimes x \rightharpoonup^* z') \leq \\ \leq \bigvee_{z'' \in X}(x \rightharpoonup^* z'' \otimes y \rightharpoonup^* z'') = x \downharpoonright y,$$

showing $(u \rightharpoonup^* x \otimes u \rightharpoonup^* y) \rightarrow x \downharpoonright y = 1$. Since $x, y \in X$ have been taken arbitrarily, we get $\mathcal{P}(u) = \bigwedge_{x,y \in X}\big((u \rightharpoonup^* x \otimes u \rightharpoonup^* y) \rightarrow x \downharpoonright y\big) = 1$. Hence, assuming $\mathcal{P}(v) = 1$ for each v such that $u \rightharpoonup v \neq 0$, we have shown $\mathcal{P}(u) = 1$. From the latter observation we directly obtain $\bigwedge_{v \in X}\big(u \rightharpoonup v \rightarrow \mathcal{P}(v)\big) = 1 = \mathcal{P}(u)$, meaning that $\mathcal{P}$ is an inductive property. Since $\rightharpoonup$ is well founded, we get $||\mathcal{P}||_X = 1$, i.e. $(u \rightharpoonup^* x \otimes u \rightharpoonup^* y) \leq x \downharpoonright y$ is true for all $u, x, y \in X$. Hence, $x \upharpoonright y = \bigvee_{u \in X}(u \rightharpoonup^* x \otimes u \rightharpoonup^* y) \leq x \downharpoonright y$ is true for all $x, y \in X$, proving that $\rightharpoonup$ is confluent. □

Corollary 89. *If* $\mathbf{L}$ *has no zero-divisors then a terminating* $\mathbf{L}$*-relation* $\rightharpoonup$ *is confluent iff* $\rightharpoonup$ *is locally confluent.*

Proof. Follows from Corollary 84 and Theorem 88. □

3.4.3 Normal Forms

Terminating and confluent relations play a crucial role in abstract rewriting systems because each element can be rewritten to a unique element in finitely many steps. The unique element is called a normal form. In this subsection, we present a preliminary study of normal forms and related issues.

Definition 90. Let $x \in X$. An element $y \in X$ is called a ***normal form*** of x if it satisfies the following property: if x has a terminating reduction $x_0, \ldots, x_n$ then $x_n = y$. The normal form of x (if it exists) is denoted by $\mathrm{nf}(x)$.

Obviously, the normal form $\mathrm{nf}(x)$ of x, if it exists, is determined uniquely. We now show that under the assumption of no zero-divisors each terminating confluent fuzzy relation has a normal form for any element.

Theorem 91. *Let* $\mathbf{L}$ *have no zero-divisors, let* $\rightharpoonup$ *be terminating and confluent. Then*

(*i*) *each* $x \in X$ *has the normal form;*
(*ii*) $x \rightharpoonup^* y \neq 0$ *implies* $\mathrm{nf}(x) = \mathrm{nf}(y)$.

Proof. (*i*): According to Lemma 74, any $x \in X$ has a terminating reduction. Suppose that x has terminating reductions $x = y_0, \ldots, y_n$ and $x = z_0, \ldots, z_m$. Since $\mathbf{L}$ has no zero-divisors, $\mathrm{re}(y_0, \ldots, y_n) \neq 0$ and $\mathrm{re}(z_0, \ldots, z_m) \neq 0$ yield $y_n \upharpoonleft z_m \neq 0$. In addition to that, y_n and z_m are irreducible (this also is a consequence of the fact that $\mathbf{L}$ has no zero-divisors). Using the fact that $\rightharpoonup$ is a confluence, $0 \neq y_n \upharpoonleft z_m \leq y_n \downharpoonleft z_m$. Hence, y_n and z_m are convergent to a nonzero degree $y_n \downharpoonleft z_m$ and both y_n and z_m are irreducible. This immediately gives $y_n = z_m = \mathrm{nf}(x)$, proving (*i*).
(*ii*): Let $x \rightharpoonup^* y \neq 0$. Then x has a terminating reduction

$$x = x_0, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n = \mathrm{nf}(x),$$

where $y = x_i$. Thus, y has a terminating reduction $y = x_i, \ldots, x_n = \mathrm{nf}(x)$. Applying (i), y has a normal form $\mathrm{nf}(y) = x_n$. Therefore, $\mathrm{nf}(x) = \mathrm{nf}(y)$. □

Remark 92. If $\mathbf{L}$ has no zero-divisors, nf can be seen as a map $\mathrm{nf}\colon X \to X$ which assigns to each element x from X its normal form $\mathrm{nf}(x)$. It is then convenient to consider an equivalence relation induced by such a map. Namely, we put $x \equiv_{\mathrm{nf}} y$ iff $\mathrm{nf}(x) = \mathrm{nf}(y)$. In the sequel, an equivalence class of $\equiv_{\mathrm{nf}}$ containing x will be denoted by $[x]_{\mathrm{nf}}$, i.e.

$$[x]_{\mathrm{nf}} = \{y \in X \mid \mathrm{nf}(x) = \mathrm{nf}(y)\}.$$

The following assertion shows how normal forms can be used to decide whether $x \rightleftharpoons^* y \neq 0$ and to estimate degrees $x \rightleftharpoons^* y$ to which x and y are convertible.

Theorem 93. *Let* $\mathbf{L}$ *have no zero-divisors, let* $\rightharpoonup$ *be terminating and confluent. Then*

(*i*) $x \rightleftharpoons^* y = \bigvee_{z \in [x]_{\mathrm{nf}}} (x \rightharpoonup^* z \otimes y \rightharpoonup^* z)$, *and*
(*ii*) $x \rightleftharpoons^* y \neq 0$ *iff* $\mathrm{nf}(x) = \mathrm{nf}(y)$.

Proof. (*i*): Due to the Theorem 13, $\rightharpoonup$ has the Church-Rosser property, i.e. $\mathrm{CR}(\rightharpoonup) = 1$. Theorem 7 yields $\rightleftharpoons^* \approx \downharpoonleft = 1$, i.e. $x \rightleftharpoons^* y = x \downharpoonleft y$ for each $x, y \in X$. By Definition 5 and using the fact that $[x]_{\mathrm{nf}} \subseteq X$,

$$x \rightleftharpoons^* y = x \downharpoonleft y = \bigvee_{z \in X} (x \rightharpoonup^* z \otimes y \rightharpoonup^* z) \geq \bigvee_{z \in [x]_{\mathrm{nf}}} (x \rightharpoonup^* z \otimes y \rightharpoonup^* z).$$

To prove the converse inequality, observe that for each $z \in X$ with $x \rightharpoonup^* z \otimes y \rightharpoonup^* z \neq 0$ we have $x \rightharpoonup^* z \neq 0$. Using the latter fact, Theorem 91 (ii) yields $\mathrm{nf}(x) = \mathrm{nf}(z)$, i.e. $z \in [x]_{\mathrm{nf}}$. Since $z \in X$ has been taken arbitrarily, we get

$$\bigvee_{z \in X} (x \rightharpoonup^* z \otimes y \rightharpoonup^* z) \leq \bigvee_{z \in [x]_{\mathrm{nf}}} (x \rightharpoonup^* z \otimes y \rightharpoonup^* z),$$

proving the equality in (i).
(ii): If $x \rightleftharpoons^* y \neq 0$ then there is $z \in X$ such that $x \rightharpoonup^* z \neq 0$, and $y \rightharpoonup^* z \neq 0$ by Theorem 13 and Theorem 7. Furthermore, Theorem 91 (ii) yields $\mathrm{nf}(x) = \mathrm{nf}(z) = \mathrm{nf}(y)$. Conversely, let $\mathrm{nf}(x) = \mathrm{nf}(y)$. Then there are reductions $x, \ldots, \mathrm{nf}(x)$ and $y, \ldots, \mathrm{nf}(y)$ such that $\mathrm{re}(x, \ldots, \mathrm{nf}(x)) \neq 0$ and $\mathrm{re}(y, \ldots, \mathrm{nf}(y)) \neq 0$, respectively. Thus, $x \rightharpoonup^* \mathrm{nf}(x) \neq 0$ and $y \rightharpoonup^* \mathrm{nf}(y) = y \rightharpoonup^* \mathrm{nf}(x) \neq 0$. Since $\mathbf{L}$ has no zero-divisors, $x \rightharpoonup^* \mathrm{nf}(x) \otimes y \rightharpoonup^* \mathrm{nf}(x) \neq 0$ from which we immediately get that $x \rightleftharpoons^* y = x \downharpoonleft y \neq 0$. □

We get the following

Corollary 94. *Let* $\mathbf{L}$ *have no zero-divisors. Then* $\leftrightsquigarrow^* = \equiv_{\mathrm{nf}}$*, where* $\leftrightsquigarrow^*$ *is the strong* 0*-cut of* $\rightleftharpoons^*$ *and* $\equiv_{\mathrm{nf}}$ *is defined as in Remark 92.*

Proof. Directly using Theorem 93 (ii). □

Remark 95. Theorem 93 and Corollary 94 allow us to decide whether $x \rightleftharpoons^* y = 0$. If x and y have the same normal form, $x \rightleftharpoons^* y \neq 0$ due to Theorem 93 (ii). In that case, if we wish to get the exact (nonzero) value of the convertibility $x \rightleftharpoons^* y$, we have to go through all elements $z \in [x]_{\mathrm{nf}} = [y]_{\mathrm{nf}}$, i.e. all elements that reduce to $\mathrm{nf}(x) = \mathrm{nf}(y)$ and compute the supremum of all $x \rightharpoonup^* z \otimes y \rightharpoonup^* z$. Since the equivalence class $[x]_{\mathrm{nf}}$ can be large (or even infinite), we can go over just a subset of $[x]_{\mathrm{nf}}$ to obtain a lower estimation of the convertibility degree $x \rightleftharpoons^* y$. Hence, normal forms can be used to compute estimates of degrees $x \rightleftharpoons^* y$ when computing the value directly is expensive.
Note that if $\mathbf{L} = \mathbf{2}$, Theorem 93 (ii) yields $x \rightleftharpoons^* y = 1$ iff $\mathrm{nf}(x) = \mathrm{nf}(y)$, which is the classical property of terminating and confluent relations. In case of $\mathbf{L} = \mathbf{2}$, Theorem 93 (i) collapses with Theorem 93 (ii) because $x \rightleftharpoons^* y = 1$ iff there is $z \in [x]_{\mathrm{nf}}$ such that $x \rightharpoonup^* z = 1$ and $y \rightharpoonup^* z = 1$, which is iff $\mathrm{nf}(x) = \mathrm{nf}(y)$.

Example 96. Consider the $\mathbf{L}$-relation $\rightharpoonup$ on the set X of colors from Fig. 3.12 and let $\mathbf{L}$ be the Goguen structure of truth degrees. This $\mathbf{L}$-relation describes a possible transformation of a picture color map provided one needs a new indexed color map containing 8 basic colors: x_1 (black), x_3 (blue), x_7 (green), x_9 (cyan), x_{19} (red), x_{21} (magenta), x_{25} (yellow), and x_{27} (white). The color x_i of each pixel in the picture will be substituted by its normal form $\mathrm{nf}(x_i)$. The similarity of an original color x_i and the substituted color $\mathrm{nf}(x_i)$ can be computed using Theorem 93 (i). Normal forms can be used to obtain a lower estimation of $x \rightleftharpoons^* y$. For instance, let $x = x_{11}$ and $y = x_{23}$. Since $\mathrm{nf}(x_{11}) = \mathrm{nf}(x_{23}) = x_{21}$ and $x_{21} \in [x_{11}]_{\mathrm{nf}}$, by Remark 95 we have $x_{11} \rightleftharpoons^* x_{23} \geq x_{11} \rightharpoonup^* x_{21} \otimes x_{23} \rightleftharpoons^* x_{21} = 0.1944$.

3.5 Conclusions and Open Problems

We have introduced and studied properties of fuzzy relations which are connected to the idea of rewriting and substituting. The research is motivated by the fact that in many cases, relations which appear in rewriting systems are fuzzy rather than crisp. For a given fuzzy reduction relation, we have defined reducibility, convergence, divergence, convertibility, Church-Rosser property, and confluence and investigated their graded properties. We have shown that such fuzzy relations have analogous properties and mutual relationship as in the ordinary case.
We have also introduced the notions mentioned above as properties of a fuzzy relation on a similarity space. We have presented a reductionist approach considering extensional closures which enables us to reduce considerations on rewriting over similarity spaces

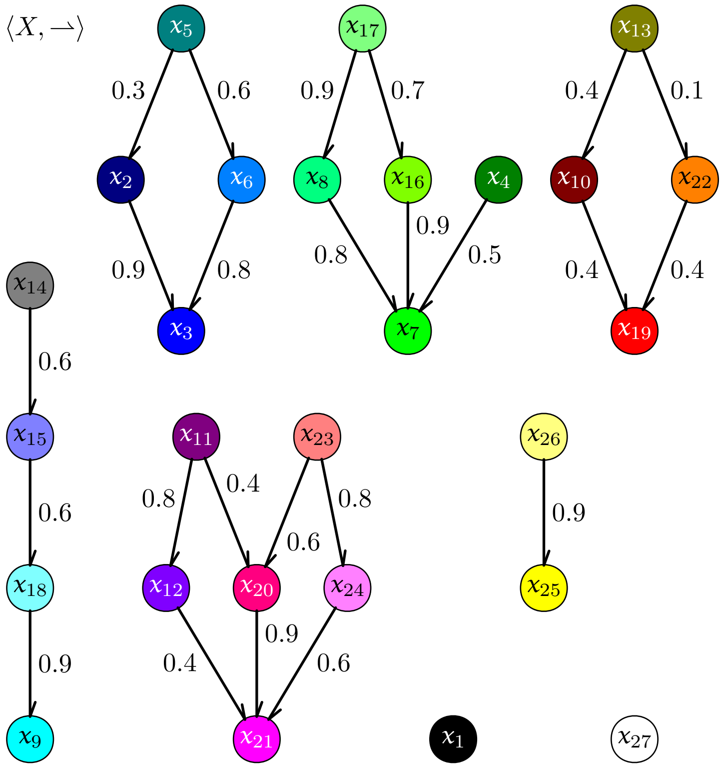

Figure 3.12: Color depth reduction.

to rewriting without taking similarities into account. Furthermore, the notions related to substitutability on generalized pseudometric spaces were introduced and a connection between these notions and notion on a similarity spaces was established.
Basic similarity issues of confluence and Church-Rosser property of fuzzy relations on similarity spaces were also studied. The main result is a collection of formulas providing estimations of convergence, divergence, convertibility, Church-Rosser property, and confluence degrees.
In addition to that, we have investigated termination, well-foundedness, and Noetherian induction from the point of view of fuzzy relations. We have outlined normal forms of fuzzy relations and their application in estimation of convertibility degrees.

Topics that have not been considered include:

- issues of termination of fuzzy relations on similarity and pseudometric spaces as well as further properties related to rewriting, see [2];
- preservation of properties like confluence by a-cuts;
- relationship to corresponding notions based on other types of compositions of fuzzy relations [35].

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Chapter 4

Representing Fuzzy Logic Programs by Graded Attribute Implications

This chapter shows that the fundamental notions of correct answers and semantic entailment, that appear in FLP and FAL respectively, are mutually reducible. This result may allow us to transport results from one theory to the other and vice versa. In addition to the reductions presented in Sections 4.1 and 4.3, we have also extended the existing Pavelka-style [48] Armstrong-like [1] axiomatization of FAL over infinite attribute sets and over arbitrary complete residuated lattices which is shown in Section 4.2.

The results summarized in this chapter have been published in [38].

4.1 Representing FAIs by Propositional FLPs

Let $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$ be a complete residuated lattice on the real unit interval. In this section, we consider FAIs of the form $A \Rightarrow B$, where both A and B are finite (i.e., there are finitely many attributes $y \in Y$ such that $A(y) > 0$ and $B(y) > 0$). In addition, we assume that all degrees $A(y)$ and $B(y)$ ($y \in Y$) are rational in order to satisfy the assumptions on definite programs from [53]. We call fuzzy attribute implications satisfying these two conditions *finitely presented* FAIs. In this section, we show that for each finite theory T of finitely presented FAIs (i.e., there are only finitely many FAIs which belong to T to a nonzero degree and all of them are finitely presented) we can find a corresponding definite program in which the correct answers can be used to describe degrees $||\cdots||_T$ of semantic entailment of finitely presented FAIs.

Remark 97. Although we are going to prove that for finite theories of finitely presented FAIs, there exist corresponding definite programs, it is important to understand that only a fragment of theories in sense of fuzzy attribute logic are covered this way. This is namely because we have made a restriction on structures of truth degrees. In fuzzy attribute logic, any complete residuated lattice can be taken for a structure of degrees, whereas in FLP, one works with (multi-adjoint) structures based on the real unit interval. Second, FAL admits general infinite theories whereas definite programs in FLP as in ordinary logic programming are considered finite for computational reasons.

In order to simplify considerations about semantic entailment of FAIs, we utilize the observation that for each theory T which is considered as an $\mathbf{L}$-set of FAIs, we may find an equivalent theory T' (i.e., a theory with the same models and thus the same semantic entailment) which is crisp. According to [12], for T, we may take

$$T' = \{A \Rightarrow T(A \Rightarrow B)\otimes B \mid T(A \Rightarrow B)\otimes B \nsubseteq A\}. \tag{4.1}$$

The fact that $\mathrm{Mod}(T)$ coincides with $\mathrm{Mod}(T')$ can be easily shown because $c \leq ||A \Rightarrow B||_M$ iff $||A \Rightarrow c\otimes B||_M = 1$ holds for any $A, B \in L^Y$ and $c \in L$. Recall that in (4.1), $T(A \Rightarrow B)\otimes B$ denotes an **L**-set which results from **L**-set B by an a-multiple for a being $T(A \Rightarrow B)$. Also note that the condition $T(A \Rightarrow B)\otimes B \nsubseteq A$ ensures that we do not put in T' inessential formulas which are true in all models to degree 1. The consequent of formulas in T' can further be simplified but we do not discuss the issue here. Anyway, instead of considering theories as **L**-sets, we can restrict ourselves only to crisp theories without any loss of expressive power.
In the subsequent characterization, we use the following construction of a language of definite programs. We consider a language $\mathcal{L}$ with only nullary relation symbols $R = \{top, y_1, y_2, \ldots, y_k\}$ (i.e., $\mathrm{ar}(top) = 0$ and $\mathrm{ar}(y_i) = 0$ for all $i = 1, \ldots, k$). The symbol top serves a technical role and its purpose is to represent the truth degree 1. The remaining relation symbols correspond to attributes which appear in the antecedents or consequents of finitely presented FAIs from T to a nonzero degree. Clearly, R is a finite set since T is supposed to be finite and there are only finitely many pairwise different attributes in all finitely presented FAIs in T which belong to antecedents and/or consequents of the FAIs to nonzero degrees. Notice that the Herbrand base $\mathcal{B}_P$ of any program P written in $\mathcal{L}$ is equal to R. In addition, we assume that $\mathcal{L}$ contains the following logical connectives and aggregations:

(i) $\Rightarrow$ (interpreted by the residuum $\rightarrow$ in **L**),

(ii) $\curlywedge$ (interpreted by the infimum $\wedge$ in **L**),

(iii) a unary aggregation $\mathbb{ts}$ (interpreted by an idempotent truth-stressing hedge *, i.e. $M^\sharp(\mathbb{ts}(\varphi)) = (M^\sharp(\varphi))^*$ for each ground formula φ),

(iv) for each rational $a \in (0, 1]$ a binary aggregation $\mathbb{sh}_a$ called an a-shift aggregation (interpreted by $M^\sharp(\mathbb{sh}_a(\varphi, \psi)) = M^\sharp(\varphi) \wedge (a \rightarrow M^\sharp(\psi))$ for all ground formulas φ, ψ).

Remark 98. Our particular selection of the language will become clear in the next theorem. Let us note here that the choice is not the only possible. For instance, one may introduce a language with unary relation symbols corresponding to attributes and a single constant or a language with a single relation symbol and constant denoting attributes. Our choice of the language is mainly to show that counterparts of finite theories of finitely presented FAIs can be constructed as *propositional* fuzzy logic programs.

The first observation on the relationship of FAL and FLP is the following:

Theorem 99. *For each finite theory T of finitely presented FAIs and a finitely presented $A \Rightarrow B$ there is a definite program P such that $||A \Rightarrow B||_T \geq a$ iff for each attribute $y \in Y$ such that $a \otimes B(y) > 0$, the pair $\langle a \otimes B(y), \emptyset\rangle$ is a correct answer for the program P and atomic formula y.*

Proof. We can assume that T is crisp. If it is not, we can take a corresponding crisp theory given by (4.1). Since all FAIs in T are finitely presented, for any $A \Rightarrow B \in T$ and arbitrary attribute $y \in Y$, we can consider a rule of FLP

$$y \Leftarrow \mathbb{ts}\big(\mathbb{sh}_{A(z_1)}(top, z_1) \curlywedge \cdots \curlywedge \mathbb{sh}_{A(z_n)}(top, z_n)\big), \tag{4.2}$$

where $z_1, \ldots, z_n$ are exactly the attributes from Y which belong to A to a nonzero degree provided that $A \neq \emptyset$. In the special case of $A = \emptyset$, we can let (4.2) be just the fact y. Notice that (4.2) is a properly defined rule of a definite program written in the language $\mathcal{L}$ as described before. We denote the rule (4.2) by $y \Leftarrow A$. Moreover, for any finite crisp T of finitely presented FAIs, we can consider an **L**-set of rules P_T defined by

$$P_T(\varphi) = \begin{cases} 1, & \text{if } \varphi \text{ is } top, \\ \bigvee\{B(y) \,|\, B \in L^Y \text{ such that } A \Rightarrow B \in T\}, & \text{if } \varphi \text{ is } y \Leftarrow A, \\ \bigvee\{B(y) \,|\, B \in L^Y \text{ such that } \emptyset \Rightarrow B \in T\}, & \text{if } \varphi \text{ is } y, \\ 0, & \text{otherwise.} \end{cases} \tag{4.3}$$

Clearly, P_T is a definite program in $\mathcal{L}$ in sense of [53]. Indeed, there are only finitely many rules $y \Leftarrow A$ (and facts) such that $P_T(y \Leftarrow A) > 0$ and all degrees $P_T(y \Leftarrow A)$ are rational since in (4.3), the supremum goes over a finite set of rational degrees.
Our proof continues by observing that $||A \Rightarrow B||_T \geq a$ iff $||A \Rightarrow a{\otimes}B||_T = 1$ which is indeed true (cf. [12] and see the comments after Remark 97). Moreover, the latter identity holds true iff $||\emptyset \Rightarrow a{\otimes}B||_{T\cup\{\emptyset\Rightarrow A\}} = 1$. Indeed, if $||A \Rightarrow a{\otimes}B||_T = 1$, then due to the monotony of the semantic entailment, we can conclude that $||A \Rightarrow a{\otimes}B||_{T\cup\{\emptyset\Rightarrow A\}} = 1$, meaning that $S(A, M)^* \leq S(a{\otimes}B, M)$ for each $M \in \mathrm{Mod}(T \cup \{\emptyset \Rightarrow A\})$ and, in addition, $M \supseteq A$, i.e., $S(A, M)^* = 1$ which further gives that $S(a{\otimes}B, M) = 1$ for each $M \in \mathrm{Mod}(T\cup \{\emptyset \Rightarrow A\})$. This immediately yields $S(a{\otimes}B, M) = ||\emptyset \Rightarrow a{\otimes}B||_{T\cup\{\emptyset\Rightarrow A\}} = 1$. Conversely, if $||\emptyset \Rightarrow a{\otimes}B||_{T\cup\{\emptyset\Rightarrow A\}} = 1$, we exploit the observation that $\mathrm{Mod}(T)$ is an $\mathbf{L}^*$-closure system [7], i.e., $\mathrm{Mod}(T)$ is closed under arbitrary intersections and a-shifts by fixed points of the idempotent truth-stressing hedge *. In particular case, for any $M \in \mathrm{Mod}(T)$, we get that $S(A, M)^* {\rightarrow} M \in \mathrm{Mod}(T)$. Recall that $S(A, M)^* {\rightarrow} M$ denotes an **L**-set which results from $M \in L^Y$ by an a-shift for a being $S(A, M)^*$. Since $A \subseteq S(A, M)^* {\rightarrow} M$, we can conclude that $S(A, M)^* {\rightarrow} M$ is a model of $T \cup \{\emptyset \Rightarrow A\}$. Thus, $||\emptyset \Rightarrow a{\otimes}B||_{T\cup\{\emptyset\Rightarrow A\}} = 1$ yields that $||\emptyset \Rightarrow a{\otimes}B||_{S(A,M)^* \rightarrow M} = 1$, i.e., $S(a{\otimes}B, S(A, M)^* {\rightarrow} M) = 1$ which is true iff $a{\otimes}B \subseteq S(A, M)^* {\rightarrow} M$ iff $S(A, M)^* \leq S(a{\otimes}B, M)$ which is true iff $||A \Rightarrow a{\otimes}B||_M = 1$. Since we have taken $M \in \mathrm{Mod}(T)$ arbitrarily, it follows that $||A \Rightarrow a{\otimes}B||_T = 1$. Altogether, we have shown that

$$||A \Rightarrow B||_T \geq a \text{ iff } ||\emptyset \Rightarrow a{\otimes}B||_{T\cup\{\emptyset\Rightarrow A\}} = 1. \tag{4.4}$$

We further prove that $||\emptyset \Rightarrow a{\otimes}B||_{T\cup\{\emptyset\Rightarrow A\}} = 1$ iff

$$a \otimes B(y) \leq ||\emptyset \Rightarrow \{{}^1\!/y\}||_{T\cup\{\emptyset\Rightarrow A\}} \tag{4.5}$$

holds for all $y \in Y$ such that $a{\otimes}B(y) > 0$. But this is indeed true since $||\emptyset \Rightarrow a{\otimes}B||_{T\cup\{\emptyset\Rightarrow A\}} = 1$ iff $S(a{\otimes}B, M) = 1$ for all $M \in \mathrm{Mod}(T \cup \{\emptyset \Rightarrow A\})$ which is true iff $a \otimes B(y) \leq M(y)$ for all $y \in Y$ and all such M. Since, $M(y)$ can be written as $S(\{{}^1\!/y\}, M)$, the latter is equivalent to $a \otimes B(y) \leq S(\{{}^1\!/y\}, M)$ iff $a \otimes B(y) \leq ||\emptyset \Rightarrow \{{}^1\!/y\}||_M$ for all $M \in \mathrm{Mod}(T \cup \{\emptyset \Rightarrow A\})$ which is equivalent to (4.5). At this point, we have shown that $||A \Rightarrow B||_T \geq a$ iff (4.5) holds.
Now it suffices to show that (4.5) is equivalent to $M^\sharp_\forall(y) \geq a \otimes B(y)$ for all $y \in Y$ (such that $a \otimes B(y) > 0$) and all $M \in \mathrm{Mod}\big(P_{T\cup\{\emptyset\Rightarrow A\}}\big)$. Since y is an atomic ground formula, it suffices to show that (4.5) is true iff $M(y) \geq a \otimes B(y)$ for all for all $y \in Y$ and all $M \in \mathrm{Mod}\big(P_{T\cup\{\emptyset\Rightarrow A\}}\big)$. We prove this claim by showing $\mathrm{Mod}(T) = \mathrm{Mod}(P_T)$ for any crisp finite theory T consisting of finitely presented FAIs and its counterpart P_T given by (4.3). Observe that $M \in \mathrm{Mod}(P_T)$ iff for each $y \Leftarrow A$, we have $M^\sharp_\forall(y \Leftarrow A) \geq P_T(y \Leftarrow A)$. If $A \neq \emptyset$, we have by definition $M^\sharp_\forall(y \Leftarrow A) = S(A, M)^* \rightarrow M(y)$. If $A = \emptyset$, we have

$$M^\sharp_\forall(y) = M(y) = S(\emptyset, M)^* \rightarrow M(y) = S(A, M)^* \rightarrow M(y). \tag{4.6}$$

Hence, in both the cases, $M_\forall^\sharp(y \Leftarrow A) \geq P_T(y \Leftarrow A)$ holds true iff $S(A,M)^* \to M(y) \geq P_T(y \Leftarrow A)$. From (4.3), we have $B(y) \leq S(A,M)^* \to M(y)$ for all $A \Rightarrow B \in T$, i.e., by adjointness $S(A,M)^* \leq B(y) \to M(y)$ for all $A \Rightarrow B \in T$ and all $y \in Y$. The latter is true iff $S(A,M)^* \leq S(B,M)$ for all $A \Rightarrow B \in T$, i.e. iff $M \in \mathrm{Mod}(T)$.
We can now conclude the proof as follows. We have observed that for given T and $A \Rightarrow B$, we have $||A \Rightarrow B||_T \geq a$ iff $a \otimes B(y) \leq ||\emptyset \Rightarrow \{{}^1/y\}||_{T\cup\{\emptyset\Rightarrow A\}}$ for all $y \in Y$ which is true iff $a \otimes B(y) \leq M(y) = M_\forall^\sharp(y)$ for all $y \in Y$ and all $M \in \mathrm{Mod}\big(P_{T\cup\{\emptyset\Rightarrow A\}}\big)$. The latter is true iff for each $y \in Y$ such that $a \otimes B(y) > 0$, the pair $\langle a \otimes B(y), \emptyset\rangle$ is a correct answer for the program $P_{T\cup\{\emptyset\Rightarrow A\}}$ and atomic formula y. □

Considering characterization of degrees $||\cdots||_T$ of semantic entailment of FAIs, we have the following consequence of Theorem 99:

Theorem 100. *For each finite theory T of finitely presented FAIs and a finitely presented $A \Rightarrow B$ there is a definite program P such that $||A \Rightarrow B||_T$ is the greatest degree $a \in L$ for which the following condition holds: for any $y \in Y$, $\langle a \otimes B(y), \emptyset\rangle$ is a correct answer for P and any $y \in Y$ provided that $a \otimes B(y) > 0$.*

Proof. For T and $A \Rightarrow B$, consider $P_{T\cup\{\emptyset\Rightarrow A\}}$ as in (4.3) and apply the fact that $||A \Rightarrow B||_T$ is the greatest degree $a \in L$ such that $||A \Rightarrow B||_T \geq a$. □

Therefore, we have shown that for T and A, we can find a *propositional* fuzzy logic program from which we can express degrees of semantic entailment of FAIs of the form $A \Rightarrow B$. Due to the limitations of FLP, the result is restricted to finite theories consisting of finitely presented FAIs, and structures of degrees defined on the real unit interval.

Remark 101. Note that regarding [53, Theorem 3], our aggregations $\mathbb{ts}$ and $\mathbb{sh}_a$ are not lower semi-continuous in general. That is, in general one cannot directly apply [53, Theorem 3] and Theorem 100 to obtain a characterization of $||A \Rightarrow B||_T$ using computed answers in FLP. On the other hand, if $\mathbf{L}$ is the standard Łukasiewicz algebra and * is the identity, then both $\mathbb{ts}$ and $\mathbb{sh}_a$ will be interpreted by continuous truth functions, i.e., one may use the machinery of computed answers in FLP to deduce the degrees to which finitely presented FAIs follow from a finite theory of finitely presented FAIs.

4.2 Completeness for FAIs over Infinite Attribute Sets

Before we show the reduction in the opposite direction, we provide a syntactic characterization of $||\cdots||_T$ for FAIs over infinite sets of attributes and over arbitrary $\mathbf{L}$. An analogous result has been shown in [12], where we have considered arbitrary $\mathbf{L}$ and finite Y. The limitation to finite Y in [12] was mainly for historical reasons because originally FAIs were developed as rules extracted from object-attribute data tables, i.e., the sets of attributes were considered finite. Nevertheless, we show here that the main results from [12] hold for any infinite Y. In addition to that, we present here the completeness results for a simplified set of deduction rules.
Let us consider the following deduction rules:

$$\text{(Ax)}:\frac{}{A\cup B \Rightarrow A},\quad \text{(Mul)}:\frac{A \Rightarrow B}{c^*\otimes A \Rightarrow c^*\otimes B},\quad (\text{Cut}_\omega):\frac{A \Rightarrow B,\, \{B\cup C \Rightarrow D_i \,|\, i \in I\}}{A\cup C \Rightarrow \bigcup_{i\in I} D_i},$$

where $A, B, C, D_i \in L^Y$ $(i \in I)$, and $c \in L$.

Remark 102. The first two rules come from [12], the rule (Cut_ω) is an *infinitary rule* saying that "from $A \Rightarrow B$ and (in general infinitely many) FAIs $B \cup C \Rightarrow D_i$, infer $A \cup C \Rightarrow \bigcup_{i\in I} D_i$". For $|I| = 1$, the infinitary (Cut_ω) becomes the ordinary (Cut) from [12].

A proof from a set T of FAIs (a theory) is defined as a labeled infinitely branching rooted (directed) tree with finite depth [54]. Denoting by $\mathbf{T} = \langle l, Z\rangle$ the rooted tree with the root label l (a FAI) and subtrees from Z, we introduce the following notion:

(i) for each $A \Rightarrow B \in T$, tuple $\mathbf{T} = \langle A \Rightarrow B, \emptyset\rangle$ is a proof of $A \Rightarrow B$ from T,

(ii) if $\mathbf{T}_i = \langle \varphi_i, \dots\rangle$ $(i \in I)$ are proofs from T and if φ results from φ_i $(i \in I)$ by any of the deduction rules (Ax), (Mul), or (Cut_ω), then $\mathbf{T} = \langle \varphi, \{\mathbf{T}_i \,|\, i \in I\}\rangle$ is a proof of φ from T.

Furthermore, $A \Rightarrow B$ is provable from T, written $T \vdash A \Rightarrow B$, if there is a proof $A \Rightarrow B$ from T. The degree $|A \Rightarrow B|_T$ to which $A \Rightarrow B$ is provable from T is defined by $|A \Rightarrow B|_T = \bigvee\{c \in L \,|\, T \vdash A \Rightarrow c{\otimes}B\}$. We can now prove the following characterization:

Theorem 103 (ordinary-style completeness)**.** *For any crisp theory T and $A \Rightarrow B$,*

$$T \vdash A \Rightarrow B \quad \textit{iff} \quad ||A \Rightarrow B||_T = 1. \tag{4.7}$$

Proof. Soundness is obvious since both (Ax) and (Mul) are known to be sound (see [12] for details) and the soundness of (Cut_ω) can be shown by a similar argument as for the ordinary (Cut). Namely, if $||A \Rightarrow B||_T = 1$ and $||B \cup C \Rightarrow D_i||_T = 1$ for all $i \in I$, then for each $M \in \mathrm{Mod}(T)$, we have $S(A, M)^* \leq S(B, M)$ and $S(B \cup C, M)^* \leq S(D_i, M)$ for all $i \in I$. Using the monotony of *,

$$\begin{aligned} S(A \cup C, M)^* &= \big(S(A, M) \wedge S(C, M)\big)^* \leq \big(S(B, M) \wedge S(C, M)\big)^* \\ &= S(B \cup C, M)^* \leq S(D_i, M) \end{aligned}$$

for all $i \in I$. Hence, $S(A \cup C, M)^* \leq \bigwedge_{i\in I} S(D_i, M) = S(\bigcup_{i\in I} D_i, M)$. We therefore have $||A \cup C \Rightarrow \bigcup_{i\in I} D_i||_M = 1$ for each $M \in \mathrm{Mod}(T)$, showing $||A \cup C \Rightarrow \bigcup_{i\in I} D_i||_T = 1$. Thus, the soundness can be proved by induction on the depth of a proof.
The converse implication of (4.7) is shown indirectly. Assume that $T \nvdash A \Rightarrow B$. We show there is $M \in \mathrm{Mod}(T)$ such that $||A \Rightarrow B||_M < 1$. We let $\mathcal{S}_T^A \subseteq L^Y$ be a system of $\mathbf{L}$-sets defined by $\mathcal{S}_T^A = \{C \in L^Y \,|\, T \vdash A \Rightarrow C\}$. Due to (Ax), $\mathcal{S}_T^A$ is nonempty and due to (Cut_ω), $\mathcal{S}_T^A$ has a greatest element. Indeed, for $M = \bigcup \mathcal{S}_T^A$ observe that from $A \Rightarrow A$ and $A \Rightarrow C$ (for all $C \in \mathcal{S}_T^A$), one infers $A \Rightarrow M$ using (Cut_ω). Therefore, $T \vdash A \Rightarrow M$.
Furthermore, $M \in \mathrm{Mod}(T)$ and $||A \Rightarrow B||_M < 1$ can be shown using $T \vdash A \Rightarrow M$ by the same arguments as in the the proof of [12, Lemma 3.5] taking into account the fact that the ordinary (Cut) is a particular case of (Cut_ω). We therefore omit the rest of the proof and refer to [12] for details. □

Remark 104. Note that the approach from [12] involves an ordinary (Cut) and an infinitary rule (Add_ω) saying that from $A \Rightarrow B_i$ (for all $i \in I$), one infers $A \Rightarrow \bigcup_{i\in I} B_i$. As a consequence of Theorem 103, we get that (Cut_ω) is derivable from (Cut) and (Add_ω) and vice versa. The observation on mutual substitutability of the rules does not answer the question whether the approach from [12] can be extended to infinite Y, however. We have answered this in Theorem 103 by checking the critical part of the completeness result which depends on (infinite) Y and $\mathbf{L}$.

The following is a consequence of the ordinary-style completeness:

Corollary 105 (graded-style completeness). *For any theory T and $A \Rightarrow B$,*

$$|A \Rightarrow B|_T = ||A \Rightarrow B||_T.$$

Proof. Consequence of Theorem 103 utilizing the fact $||A \Rightarrow B||_T$ is the supremum of all $c \in L$ such that $||A \Rightarrow c{\otimes}B||_T = 1$ and that T can be considered crisp. □

Completeness results like Corollary 105 are usually called Pavelka-style completeness results. We have obtained the result over arbitrary **L** at the cost of introducing an infinitary rule, cf. [25, 29, 48].
In addition to the graded completeness of fuzzy attribute logic we have established before, there is an alternative characterization of entailment degrees using least models. In our previous papers, we have described a construction of least models and we established the characterization of degrees of semantic entailment for finite **L** and Y which was motivated by solving issue in concept lattices constrained by linguistic hedges [17] where infinite **L** and Y are not considered because of computational issues. Again, the results can be generalized for both **L** and Y being infinite as we show in the rest of this section.
For any crisp theory T, consider an operator $[\cdots]_T$ defined by

$$[M]_T = M \cup \bigcup\{S(A,M)^{*}{\otimes}B \mid A \Rightarrow B \in T\} \tag{4.8}$$

for all $M \in L^Y$. Clearly, $M \in \mathrm{Mod}(T)$ iff M is a fixed point of $[\cdots]_T$. Indeed, we have $M = [M]_T$ iff $[M]_T \subseteq M$ which is true iff for all $A \Rightarrow B \in T$, we have $S(A,M)^{*}{\otimes}B \subseteq M$, meaning $S(A,M)^{*} \leq S(B,M)$, i.e., $||A \Rightarrow B||_M = 1$ for all $A \Rightarrow B \in T$ which is true iff $M \in \mathrm{Mod}(T)$.
Since (4.8) is extensive and monotone, we may apply Tarski fixpoint theorem [52] in its constructive version [20] to get a closure operator whose fixed points are the fixed points of $[\cdots]_T$. That is, for arbitrary ordinal number κ, we let

$$M_T^{\kappa} = \begin{cases} M, & \text{if } \kappa = 0,\\ \left[M_T^{\kappa-1}\right]_T, & \text{if } \kappa \text{ is a successor ordinal,}\\ \bigcup_{\mu<\kappa} M_T^{\mu}, & \text{if } \kappa \text{ is a limit ordinal,} \end{cases} \tag{4.9}$$

and let $\mathrm{lfp}_T(M) = M_T^{\kappa-1}$, where κ is a successor ordinal such that $M_T^{\kappa-1} = M_T^{\kappa}$. As a consequence of the results from [20], lfp_T is a closure operator whose fixed points are the fixed points of (4.8), i.e., the fixed points of lfp_T are exactly the models of T. Therefore, $\mathrm{lfp}_T(M)$ is the *least model of T containing M*.

Remark 106. Furthermore, it follows that for each **L** and Y there is an ordinal μ such that $\mathrm{lfp}_T(M) = M_T^{\mu}$ for all $M \in L^Y$. In general, such μ can be taken as $\mu = \mathrm{card}(Y \times L)$ which if both Y and **L** are finite means $\mu = \mathrm{card}(Y) \cdot \mathrm{card}(L)$ and if either of Y and L is infinite, it means $\mu = \max(\mathrm{card}(Y), \mathrm{card}(L))$.

Using least models, we can characterize degrees of semantic entailment:

Theorem 107. *For any T and $A \Rightarrow B$, we have $||A \Rightarrow B||_T = S\big(B, \mathrm{lfp}_T(A)\big)$.*

Proof. By analogous arguments as in [10, Theorem 1] considering a general closure operator lfp_T for arbitrary **L** and Y. See also [50] for an analogous weaker result. □

4.3 Representing FLPs by FAIs over Herbrand Bases

We now explore the opposite direction of the transformation between definite programs and theories consisting of fuzzy attribute implications. In this section, we consider **L** to be a complete residuated lattice on the real unit interval equipped with * defined by (2.25). For a definite program P, we consider a theory consisting of FAIs where the set of attributes is represented by the Herbrand base $\mathcal{B}_P$. In general, $\mathcal{B}_P$ is infinite and therefore the FAIs are formulas with infinite antecedents and consequents. Note that in the important case when F consists solely of constants, $\mathcal{B}_P$ is finite and thus we work with FAIs that can be understood as formulas in the usual sense.
For any definite program P with Herbrand base $\mathcal{B}_P$, we introduce the following notation which also appears in [53, Definition 4]. For each $A \in L^{\mathcal{B}_P}$ such that $A \neq \emptyset$, we put

$$A^\circ(\chi) = \bigvee\{P(\psi \Leftarrow \xi) \otimes A^\sharp(\xi\eta) \,|\, \xi\eta \text{ is ground and } \psi\eta \text{ equals } \chi\}, \tag{4.10}$$

for all $\chi \in \mathcal{B}_P$. Note that $A^\circ(\chi)$ should be read the following way: "$A^\circ(\chi)$ is a degree to which P contains a rule $\psi \Leftarrow \xi$ such that a ground instance $\xi\eta$ of its tail is true under A and χ is a ground instance of its head ψ which results by applying η." Note that the multiplication $\otimes$ which appears in (4.10) is the multiplication which is adjoint to the residuum $\rightarrow$ interpreting $\Rightarrow$. In addition, we put

$$\emptyset^\circ(\chi) = \bigvee\{P(\psi) \,|\, \psi\eta \text{ equals } \chi\}. \tag{4.11}$$

for all $\chi \in \mathcal{B}_P$. Technically, (4.11) can be seen as a special case of (4.10) since facts can be seen as rules $\psi \Leftarrow$, we keep the distinction here to emphasize that facts are atomic formulas whereas rules are compound. Nevertheless, $A^\circ \in L^{\mathcal{B}_P}$ for all $A \in L^{\mathcal{B}_P}$ and we may let T_P be the set

$$T_P = \{A \Rightarrow A^\circ \,|\, A \in L^{\mathcal{B}_P}\} \tag{4.12}$$

of fuzzy attribute implications over $\mathcal{B}_P$. The construction of T_P ensures that it has the same models as P:

Lemma 108. *Let P be a definite program. Then* $\mathrm{Mod}(P) = \mathrm{Mod}(T_P)$.

Proof. First, we may use an argument that $M \in \mathrm{Mod}(P)$ iff $M^\circ \subseteq M$ and $\emptyset^\circ \subseteq M$. This is almost immediate and it has already been observed in [53, Theorem 2], we just distinguish the cases of (4.10) and (4.11). We now proceed by showing both the inclusions of $\mathrm{Mod}(P) = \mathrm{Mod}(T_P)$. Let $M \in \mathrm{Mod}(P)$ and take $A \Rightarrow A^\circ \in T_P$. If $A \subseteq M$ and $A \neq \emptyset$, then by realizing that the operator $^\circ$ is monotone (this is indeed true since all connectives appearing in a head of a rule are interpreted by monotone functions), we get that $A^\circ \subseteq M^\circ \subseteq M$, i.e., $||A \Rightarrow A^\circ||_M = 1$ because * is globalization. In addition, if $A = \emptyset$, then we can conclude $||\emptyset \Rightarrow \emptyset^\circ||_M = 1$ due to $\emptyset^\circ \subseteq M$. Since A has been taken arbitrarily, we get $M \in \mathrm{Mod}(T_P)$. Now it suffices to show the converse inclusion. Let $M \in \mathrm{Mod}(T_P)$. Since M is a model of T_P, we have $||\emptyset \Rightarrow \emptyset^\circ||_M = 1$, meaning $\emptyset^\circ \subseteq M$ and $||M \Rightarrow M^\circ||_M = 1$, meaning $M^\circ \subseteq M$ which are together equivalent with $M \in \mathrm{Mod}(P)$. □

The following theorem exploits Theorem 103 and Lemma 108 and establishes the opposite reduction to that from Section 4.1.

Theorem 109. *For every definite program P there is a set T of FAIs such that for each atomic formula φ and substitution θ there is a crisp $B_\varphi \in L^{\mathcal{B}_P}$ so that $\langle a, \theta\rangle$ is a correct answer for P and φ iff $T \vdash \emptyset \Rightarrow a{\otimes}B_\varphi$ and $a > 0$.*

Proof. Let T be defined as in (4.12). For the atomic formula φ, we introduce a crisp $\mathbf{L}$-set B_φ so that for each $\psi \in \mathcal{B}_P$,

$$B_\varphi(\psi) = \begin{cases} 1, & \text{if } \psi \text{ is a ground instance of } \varphi\theta, \\ 0, & \text{otherwise.} \end{cases} \tag{4.13}$$

By definition, $\langle a, \theta\rangle$ is a correct answer for P and φ iff $M_\forall^\sharp(\varphi\theta) \geq a > 0$ for all $M \in \mathrm{Mod}(P)$. Using Lemma 108 the condition is true iff $M(\varphi\theta\eta) \geq a > 0$ for all $M \in \mathrm{Mod}(T)$ and for all substitutions η such that $\varphi\theta\eta$ is ground. The latter condition holds true iff $||\emptyset \Rightarrow \{^1/\varphi\theta\eta\}||_M \geq a > 0$ for all η such that $\varphi\theta\eta \in \mathcal{B}_P$ and all $M \in \mathrm{Mod}(T)$. Taking into account (4.13), we get $||\emptyset \Rightarrow B_\varphi||_M \geq a > 0$ for all $M \in \mathrm{Mod}(T)$ which is equivalent to $||\emptyset \Rightarrow a\otimes B_\varphi||_T = 1$ and $a > 0$. Now, using Theorem 103, the latter is true iff $T \vdash \emptyset \Rightarrow a\otimes B_\varphi$ and $a > 0$. □

Let us comment on the previous result.

Remark 110. (1) In Theorem 109, we have used multiplication $\otimes$ and logical connective $\Rightarrow$ without any specification. In fact, since B_φ is crisp, all $\otimes$ yield the same $a\otimes B_\varphi$, i.e., the choice of $\otimes$ is not essential. The same applies to $\Rightarrow$, it can be any residuated implication, this is a consequence of having the truth-stressing hedge * as the globalization which suppresses the role of $\Rightarrow$ and its truth function $\rightarrow$, see [14, Theorem 15] for details.
(2) The previous assertion can be seen as an alternative syntactic characterization of correct answers in fuzzy logic programming. The utilized formulas in A° are constructed from models of definite programs. A problem we consider interesting is to describe more concise representations of T°, i.e., to find a theory which is equivalent to T° from (4.12) and is not redundant. Classic results related to this issue can be found in [28, 41].
(3) In [53], the author uses various connectives together with the aggregations but, in fact, the aggregations are more universal and the connectives (conjunctions and disjunctions) used therein can be seen as binary aggregations. From the proof of Lemma 108, we can see that the key property of such aggregations is monotony. It is not the case of the residua (which are antitone in the first argument) but their role is different from the other connectives since the (symbols of) implications cannot be used in tails of the rules.

The paper [53] does not introduce semantic entailment from definite programs which is quite usual in the logic programming since its agenda is different from that of general logic aiming at exploring notions of entailment. At least in case of facts and particular conjunctions of facts, we can introduce graded semantic entailment and provide its Pavelka-style characterization based on Theorem 109. For each definite program P and fact φ, we let

$$||\varphi||_P = \textstyle\bigwedge_{M\in\mathrm{Mod}(P)} M_\forall^\sharp(\varphi). \tag{4.14}$$

As a consequence of Theorem 109, we establish the following characterization.

Corollary 111. *For every definite program P there is a set T of FAIs such that for each fact φ there is a crisp $B_\varphi \in L^{\mathcal{B}_P}$ such that*

$$||\varphi||_P = ||\emptyset \Rightarrow B_\varphi||_T = |\emptyset \Rightarrow B_\varphi|_T = S\big(B_\varphi, \mathrm{lfp}_T(\emptyset)\big). \tag{4.15}$$

Proof. Let T and B_φ be defined as in (4.12) and (4.13), respectively. Observe that

$$\begin{aligned} ||\varphi||_P &= \textstyle\bigwedge_{M\in\mathrm{Mod}(P)} M_\forall^\sharp(\varphi) = \bigwedge_{M\in\mathrm{Mod}(T)} \bigwedge_{\chi\in B_\varphi} M(\chi) \\ &= \textstyle\bigwedge_{M\in\mathrm{Mod}(T)} \bigwedge_{\chi\in\mathcal{B}_P} (B_\varphi(\chi) \rightarrow M(\chi)) = ||\emptyset \Rightarrow B_\varphi||_T. \end{aligned}$$

Again, the particular choice of $\rightarrow$ is not essential since B_φ is crisp. This follows from Theorem 103, Theorem 107, and Theorem 109. □

Remark 112. (1) Formulas in Corollary 111 can be extended to arbitrary min-conjunctions $\varphi_1 \curlywedge \cdots \curlywedge \varphi_n$ of n facts in which case it suffices to take $B_{\varphi_1 \curlywedge \cdots \curlywedge \varphi_n} = \bigcup_{i=1}^{n} B_\varphi$ and one can establish an analogous characterization as in the corollary. For other compound formulas, the situation does not seem to be straightforward and we consider it as an open problem.
(2) Since B_φ is crisp, the expressions involved in (4.15) simplify, e.g.

$$S\big(B_\varphi, \mathrm{lfp}_T(\emptyset)\big) = \textstyle\bigwedge_{\chi \in B_\varphi} (\mathrm{lfp}_T(\emptyset))(\chi),$$

which for φ being a ground atomic formula yields $||\varphi||_T = (\mathrm{lfp}_T(\emptyset))(\varphi)$. From the point of view of FAL, we can view this fact, which is mentioned in the proof of [53, Theorem 3], as a consequence of the least model characterization of semantic entailment in FAL and the existing reduction.

4.4 Boolean Case Reduction

It has been observed that entailment of FAIs is reducible to entailment of ordinary attribute implications using a transformation based on (**L**-set)-representative [5] subsets of $L \times Y$. In what follows we assume that * is globalization. Following [5, 9], we can use the fact that

$$||A \Rightarrow B||_T = 1 \text{ iff } \lfloor T \rfloor \models \lfloor A \rfloor \Rightarrow \lfloor B \rfloor, \tag{4.16}$$

where

$$\lfloor M \rfloor = \{\langle y, a\rangle \in Y \times L \,|\, a \leq M(y)\}, \tag{4.17}$$

$$\lfloor T \rfloor = \{\lfloor A \rfloor \Rightarrow \lfloor B \rfloor \,|\, A \Rightarrow B \in T\} \tag{4.18}$$

for any $M \in L^Y$ and any crisp theory T and $\lfloor T \rfloor \models \lfloor A \rfloor \Rightarrow \lfloor B \rfloor$ denotes the ordinary semantic entailment of attribute implications. Notice that since all $\lfloor A \rfloor$, $\lfloor B \rfloor$, and the formulas in $\lfloor T \rfloor$ are crisp, we can view them as their ordinary counterparts. Hence, (4.16) shows that under globalization the semantic entailment of FAIs to degree 1 can be characterized by an ordinary semantic entailment of ordinary attribute implications [24]. Therefore, by a combination of our observations in Section 4.3 with (4.16), we may find a counterpart of a fuzzy logic program expressed by ordinary attribute implications:

Corollary 113. *For every definite program P there is a set T of attribute implications such that for each atomic formula φ and substitution θ there is $C_\varphi \subseteq \mathcal{B}_P \times L$ so that $\langle a, \theta\rangle$ is a correct answer for P and φ iff $T \models \emptyset \Rightarrow C_\varphi$ and $a > 0$.*

Proof. Take $T = \lfloor T_P \rfloor$, where T_P is given by (4.12) and let $C_\varphi = \lfloor a \otimes B_\varphi \rfloor$, where B_φ is given by (4.13). Now, apply Theorem 109, Theorem 103 (for **L** being a two-element Boolean algebra) together with (4.16). □

Further characterizations derived from Corollary 113 and Corollary 111 are possible. For instance:

$$\begin{aligned} ||\varphi||_P &= \textstyle\bigvee\big\{a \in L \,|\, \lfloor T_P \rfloor \vdash \emptyset \Rightarrow \lfloor a \otimes B_\varphi \rfloor\big\} \\ &= \textstyle\bigvee\big\{a \in L \,|\, \lfloor a \otimes B_\varphi \rfloor \subseteq \mathrm{lfp}_{\lfloor T_P \rfloor}(\emptyset)\big\}, \end{aligned}$$

where $\mathrm{lfp}_{\lfloor T_P \rfloor}(\emptyset)$ is the least model of $\lfloor T_P \rfloor$ in the usual sense and can be seen as a special case of the operator from Section 4.2 for **L** being the two-element Boolean algebra. Analogously, $\vdash$ is the ordinary provability based on (Ax) and (Cut_ω) with **L**-sets replaced by ordinary sets in which case the rules become the ordinary (but infinitary) Armstrong-rules.

4.5 Illustrative Examples

This section shows some illustrative examples of the reductions introduced in Sections 4.1, Section 4.3, and Section 4.4.

Example 114. Let **L** be the standard Łukasiewicz structure of truth degrees, i.e., a complete residuated lattice on the unit interval with its genuine ordering $\leq$ and adjoint operations $\otimes$, $\rightarrow$ defined by $a \otimes b = \max(0, a+b-1)$ and $a \rightarrow b = \min(1, 1-a+b)$. Let * be the identity.
Furthermore, consider a set of attributes of cars $Y = \{lA, lM, hAT, hFE, hP\}$ which mean: "low age", "low mileage", "automatic transmission", "high fuel economy", and "high price" respectively. Let T be a set containing the following FAIs over Y:

$$\{{}^{0.7}\!/lA, {}^{0.9}\!/lM, {}^{0.4}\!/hAT\} \Rightarrow \{{}^{0.6}\!/hFE, {}^{0.9}\!/hP\},$$
$$\{{}^{0.8}\!/lA\} \Rightarrow \{{}^{0.7}\!/lM\}.$$

Using Theorem 99, we can find a FLP P_T that corresponds to FAIs from T. The program P_T will contain the following rules:

$$\begin{aligned}
hFE &\overset{0.6}{\Leftarrow} \mathbb{ts}\big(\mathbb{sh}_{0.7}(top, lA) \mathbin{\&} \mathbb{sh}_{0.9}(top, lM) \mathbin{\&} \mathbb{sh}_{0.4}(top, hAT)\big),\\
hP &\overset{0.9}{\Leftarrow} \mathbb{ts}\big(\mathbb{sh}_{0.7}(top, lA) \mathbin{\&} \mathbb{sh}_{0.9}(top, lM) \mathbin{\&} \mathbb{sh}_{0.4}(top, hAT)\big),\\
lM &\overset{0.7}{\Leftarrow} \mathbb{ts}\big(\mathbb{sh}_{0.8}(top, lA)\big),\\
top &\overset{1}{\Leftarrow} .
\end{aligned}$$

Obviously, the aggregator $\mathbb{ts}$ interpreted by identity can be omitted. Furthermore, all aggregations interpreting $\mathbb{sh}_a(y,z)$ as well as the function $\wedge$ interpreting conjunctor $\mathbin{\&}$ are continuous in this case. Thus, we can use [53, Theorem 3] and Theorem 100 to characterize $||A \Rightarrow B||_T$ using computed answers for program $P_{T\cup\{\emptyset\Rightarrow A\}}$ and queries $y \in Y$ with $B(y) > 0$.
For example, a user asks a question "How much expensive are quite new cars with automatic transmission?", i.e., more precisely "To what degree $a \in L$, is $\{{}^{0.6}\!/lA, {}^{1}\!/hAT\} \Rightarrow \{{}^{1}\!/hP\}$ true in T?". To get the answer, we first extend P_T to $P_{T\cup\{\emptyset\Rightarrow A\}}$ by adding facts $lA \overset{0.6}{\Leftarrow}$ and $hAT \overset{1}{\Leftarrow}$ to the program. Then, we can easily compute an answer to query hP using the usual admissible rules of FLPs [53] (all substitutions are $\emptyset$):

$$\begin{aligned}
&hP,\\
&0.9 \otimes \big(\mathbb{sh}_{0.7}(top, lA) \mathbin{\&} \mathbb{sh}_{0.9}(top, lM) \mathbin{\&} \mathbb{sh}_{0.4}(top, hAT)\big),\\
&0.9 \otimes \big(\mathbb{sh}_{0.7}(top, lA) \mathbin{\&} \mathbb{sh}_{0.9}(top, 0.7 \otimes \mathbb{sh}_{0.8}(top, lA)) \mathbin{\&} \mathbb{sh}_{0.4}(top, hAT)\big),\\
&0.9 \otimes \big(\mathbb{sh}_{0.7}(top, 0.6) \mathbin{\&} \mathbb{sh}_{0.9}(top, 0.7 \otimes \mathbb{sh}_{0.8}(top, 0.6)) \mathbin{\&} \mathbb{sh}_{0.4}(top, 1)\big),\\
&0.9 \otimes \big(0.7 \rightarrow 0.6 \wedge 0.9 \rightarrow (0.7 \otimes (0.8 \rightarrow 0.6)) \wedge 0.4 \rightarrow 1\big),\\
&0.5.
\end{aligned}$$

Using [53, Theorem 3], Theorem 100 and the computed answer $\langle 0.5, \emptyset\rangle$, we immediately get $||\{{}^{0.6}\!/lA, {}^{1}\!/hAT\} \Rightarrow \{{}^{1}\!/hP\}||_T = 0.5$.

Example 115. Let $L = \{0, 0.1, 0.2, \ldots, 0.9, 1\}$ be a chain of 11 elements (truth degrees) equipped with Łukasiewicz $\langle \otimes_L, \leftarrow_L \rangle$ and Gödel $\langle \otimes_G, \leftarrow_G \rangle$ pair of truth functions.

We define a language for a simple fuzzy logic program describing properties of hotels and their suitability for a sport fan as follows. Let $R = \{near, cost, suitable\}$ with $\mathrm{ar}(near) = 2$, $\mathrm{ar}(cost) = 1$ and $\mathrm{ar}(suitable) = 1$. The meanings of these predicates are "locations are near", "(accommodation) cost is low" and "(accommodation) is suitable". Further, let $F = \{\mathsf{hotel}, \mathsf{center}, \mathsf{stadium}\}$ where $\mathrm{ar}(\mathsf{hotel}) = \mathrm{ar}(\mathsf{center}) = \mathrm{ar}(\mathsf{stadium}) = 0$. These constants represent a particular hotel, the stadium and the city center. For simplicity, we consider only one hotel here. In order to make the example concise, we also use sorts (or types) of constants and variables in this example, which is quite usual in fuzzy logic programming [53]. This way, we can specify that all constants in F are locations, but only hotel can be used for accommodation. Thus, atomic formulas $cost(\mathsf{stadium})$, $cost(\mathsf{center})$, $suitable(\mathsf{stadium})$ and $suitable(\mathsf{center})$ does not exist in our language. The Herbrand base of this language is obviously

$$\begin{aligned}\mathcal{B}_P = \{&near(\mathsf{hotel}, \mathsf{hotel}), near(\mathsf{hotel}, \mathsf{center}), near(\mathsf{hotel}, \mathsf{stadium}),\\ &near(\mathsf{center}, \mathsf{hotel}), near(\mathsf{center}, \mathsf{center}), near(\mathsf{center}, \mathsf{stadium}),\\ &near(\mathsf{stadium}, \mathsf{hotel}), near(\mathsf{stadium}, \mathsf{center}), near(\mathsf{stadium}, \mathsf{stadium}),\\ &cost(\mathsf{hotel}), suitable(\mathsf{hotel})\}\,.\end{aligned}$$

Now, we consider the following fuzzy logic program P using the given language and structure of truth degrees.

$$\begin{aligned}near(\mathsf{hotel}, \mathsf{center}) &\overset{0.8}{\Leftarrow}\\ near(\mathsf{stadium}, \mathsf{center}) &\overset{0.6}{\Leftarrow}\\ cost(\mathsf{hotel}) &\overset{0.7}{\Leftarrow}\\ near(\mathbb{X}, \mathbb{X}) &\overset{1}{\Leftarrow}\\ near(\mathbb{X}, \mathbb{Y}) &\overset{1}{\Leftarrow} near(\mathbb{Y}, \mathbb{X})\\ near(\mathbb{X}, \mathbb{Z}) &\overset{0.7}{\Leftarrow}_L near(\mathbb{X}, \mathbb{Y}) \mathbin{\&}_L near(\mathbb{Y}, \mathbb{Z})\\ suitable(\mathbb{X}) &\overset{0.8}{\Leftarrow}_L \mathbb{avg}(near(\mathbb{X}, \mathsf{stadium}) \mathbin{\&}_G near(\mathbb{X}, \mathsf{center}), cost(\mathbb{X}))\end{aligned}$$

where $\Leftarrow_L$ is interpreted by $\leftarrow_L$, $\mathbin{\&}_L$ by $\otimes_L$, $\mathbin{\&}_G$ by $\otimes_G$ and $\mathbb{avg}$ by rounded arithmetic average.

Using the construction from Section 4.3, we get a theory T_P of FAIs whose models are exactly models of P. First, we construct $\emptyset^\circ$ by (4.11):

$$\begin{aligned}\emptyset^\circ = \bigl\{&{}^{1}\!/near(\mathsf{hotel}, \mathsf{hotel}), {}^{0.8}\!/near(\mathsf{hotel}, \mathsf{center}), {}^{1}\!/near(\mathsf{center}, \mathsf{center}),\\ &{}^{0.6}\!/near(\mathsf{stadium}, \mathsf{center}), {}^{1}\!/near(\mathsf{stadium}, \mathsf{stadium}), {}^{0.7}\!/cost(\mathsf{hotel})\bigr\}\,.\end{aligned}$$

Then, for an arbitrary **L**-set A on $\mathcal{B}_P$, we get the corresponding A° by (4.10).

$$\begin{aligned}A = \{&{}^{a_1}/near(\mathsf{hotel},\mathsf{hotel}), {}^{a_2}/near(\mathsf{hotel},\mathsf{center}), {}^{a_3}/near(\mathsf{hotel},\mathsf{stadium}),\\ &{}^{a_4}/near(\mathsf{center},\mathsf{hotel}), {}^{a_5}/near(\mathsf{center},\mathsf{center}), {}^{a_6}/near(\mathsf{center},\mathsf{stadium}),\\ &{}^{a_7}/near(\mathsf{stadium},\mathsf{hotel}), {}^{a_8}/near(\mathsf{stadium},\mathsf{center}), {}^{a_9}/near(\mathsf{stadium},\mathsf{stadium}),\\ &{}^{a_{10}}/cost(\mathsf{hotel}), {}^{a_{11}}/suitable(\mathsf{hotel})\}\\ A^\circ = \Big\{&{}^{b_1}/near(\mathsf{hotel},\mathsf{hotel}), {}^{b_2}/near(\mathsf{hotel},\mathsf{center}), {}^{b_3}/near(\mathsf{hotel},\mathsf{stadium}),\\ &{}^{b_4}/near(\mathsf{center},\mathsf{hotel}), {}^{b_5}/near(\mathsf{center},\mathsf{center}), {}^{b_6}/near(\mathsf{center},\mathsf{stadium}),\\ &{}^{b_7}/near(\mathsf{stadium},\mathsf{hotel}), {}^{b_8}/near(\mathsf{stadium},\mathsf{center}), {}^{b_9}/near(\mathsf{stadium},\mathsf{stadium}),\\ &{}^{b_{10}}/suitable(\mathsf{hotel})\Big\}\end{aligned}$$

where $a_1, \dots, a_{11} \in L$ are arbitrary elements such that there is some $a_i > 0$ and $b_1, \dots, b_{10} \in L$ can be computed as follows:

$$\begin{aligned}b_1 &= \bigvee\{a_2 \otimes_L a_4 \otimes_L 0.7, a_3 \otimes_L a_7 \otimes_L 0.7, a_1\},\\ b_2 &= \bigvee\{a_1 \otimes_L a_2 \otimes_L 0.7, a_2 \otimes_L a_5 \otimes_L 0.7, a_3 \otimes_L a_8 \otimes_L 0.7, a_4\},\\ b_3 &= \bigvee\{a_1 \otimes_L a_3 \otimes_L 0.7, a_2 \otimes_L a_6 \otimes_L 0.7, a_3 \otimes_L a_9 \otimes_L 0.7, a_7\},\\ b_4 &= \bigvee\{a_4 \otimes_L a_1 \otimes_L 0.7, a_5 \otimes_L a_4 \otimes_L 0.7, a_6 \otimes_L a_7 \otimes_L 0.7, a_2\},\\ b_5 &= \bigvee\{a_4 \otimes_L a_2 \otimes_L 0.7, a_6 \otimes_L a_8 \otimes_L 0.7, a_5\},\\ b_6 &= \bigvee\{a_4 \otimes_L a_3 \otimes_L 0.7, a_5 \otimes_L a_6 \otimes_L 0.7, a_6 \otimes_L a_1 \otimes_L 0.7, a_8\},\\ b_7 &= \bigvee\{a_7 \otimes_L a_1 \otimes_L 0.7, a_8 \otimes_L a_4 \otimes_L 0.7, a_9 \otimes_L a_7 \otimes_L 0.7, a_3\},\\ b_8 &= \bigvee\{a_7 \otimes_L a_2 \otimes_L 0.7, a_8 \otimes_L a_5 \otimes_L 0.7, a_9 \otimes_L a_8 \otimes_L 0.7, a_6\},\\ b_9 &= \bigvee\{a_7 \otimes_L a_3 \otimes_L 0.7, a_8 \otimes_L a_6 \otimes_L 0.7, a_9\},\\ b_{10} &= \left\lfloor \frac{(a_3 \otimes_G a_2) + a_{10}}{2} \right\rfloor.\end{aligned}$$

The theory T_P consist of 121 ($|\mathcal{B}_P| \cdot |L|$) FAIs: $\emptyset \Rightarrow \emptyset^\circ$ and all possible $A \Rightarrow A^\circ$. By the technique described in Section 4.4, we can also compose a crisp theory of AIs $\lfloor T_P \rfloor$. From $\emptyset \Rightarrow \emptyset^\circ$ we get $\lfloor \emptyset \rfloor \Rightarrow \lfloor \emptyset^\circ \rfloor$ where

$$\begin{aligned}\lfloor \emptyset \rfloor = \{&\langle near(\mathsf{hotel},\mathsf{hotel}), 0\rangle, \langle near(\mathsf{hotel},\mathsf{center}), 0\rangle, \langle near(\mathsf{hotel},\mathsf{stadium}), 0\rangle,\\ &\langle near(\mathsf{center},\mathsf{hotel}), 0\rangle, \langle near(\mathsf{center},\mathsf{center}), 0\rangle, \langle near(\mathsf{center},\mathsf{stadium}), 0\rangle,\\ &\langle near(\mathsf{stadium},\mathsf{hotel}), 0\rangle, \langle near(\mathsf{stadium},\mathsf{center}), 0\rangle,\\ &\langle near(\mathsf{stadium},\mathsf{stadium}), 0\rangle, \langle cost(\mathsf{hotel}), 0\rangle, \langle suitable(\mathsf{hotel}), 0\rangle\},\\ \lfloor \emptyset^\circ \rfloor = \{&\langle near(\mathsf{hotel},\mathsf{hotel}), 0\rangle, \langle near(\mathsf{hotel},\mathsf{hotel}), 0.1\rangle, \dots, \langle near(\mathsf{hotel},\mathsf{hotel}), 1\rangle,\\ &\langle near(\mathsf{hotel},\mathsf{center}), 0\rangle, \dots, \langle near(\mathsf{hotel},\mathsf{center}), 0.8\rangle,\\ &\langle near(\mathsf{hotel},\mathsf{stadium}), 0\rangle, \langle near(\mathsf{center},\mathsf{hotel}), 0\rangle,\\ &\langle near(\mathsf{center},\mathsf{center}), 0\rangle, \dots, \langle near(\mathsf{center},\mathsf{center}), 1\rangle,\\ &\langle near(\mathsf{center},\mathsf{stadium}), 0\rangle, \langle near(\mathsf{stadium},\mathsf{hotel}), 0\rangle,\\ &\langle near(\mathsf{stadium},\mathsf{center}), 0\rangle, \dots, \langle near(\mathsf{stadium},\mathsf{center}), 0.6\rangle,\\ &\langle near(\mathsf{stadium},\mathsf{stadium}), 0\rangle, \dots, \langle near(\mathsf{stadium},\mathsf{stadium}), 1\rangle,\\ &\langle cost(\mathsf{hotel}), 0\rangle, \dots, \langle cost(\mathsf{hotel}), 0.7\rangle\}.\end{aligned}$$

In the same way, we can construct $\lfloor A \rfloor \Rightarrow \lfloor A^\circ \rfloor$ form each $A \Rightarrow A^\circ$.

$$
\begin{aligned}
\lfloor A \rfloor = \{ & \langle near(\mathsf{hotel}, \mathsf{hotel}), 0\rangle, \dots, \langle near(\mathsf{hotel}, \mathsf{hotel}), a_1\rangle, \\
& \langle near(\mathsf{hotel}, \mathsf{center}), 0\rangle, \dots, \langle near(\mathsf{hotel}, \mathsf{center}), a_2\rangle, \\
& \dots, \\
& \langle suitable(\mathsf{hotel}), 0\rangle, \dots, \langle suitable(\mathsf{hotel}), a_{11}\rangle, \} \\
\lfloor A^\circ \rfloor = \{ & \langle near(\mathsf{hotel}, \mathsf{hotel}), 0\rangle, \dots, \langle near(\mathsf{hotel}, \mathsf{hotel}), b_1\rangle, \\
& \dots, \\
& \langle near(\mathsf{stadium}, \mathsf{stadium}), 0\rangle, \dots, \langle near(\mathsf{stadium}, \mathsf{stadium}), b_9\rangle, \\
& \langle cost(\mathsf{hotel}), 0\rangle, \\
& \langle suitable(\mathsf{hotel}), 0\rangle, \dots, \langle suitable(\mathsf{hotel}), b_{10}\rangle, \}
\end{aligned}
$$

The theory $\lfloor T_P \rfloor$ of boolean AIs also consist of $\lfloor \emptyset \rfloor \Rightarrow \lfloor \emptyset^\circ \rfloor$ and all $\lfloor A \rfloor \Rightarrow \lfloor A^\circ \rfloor$.

Finally, we use our results from Section 4.1 to create a crisp logic program $P_{\lfloor T_P \rfloor}$ corresponding to $\lfloor T_P \rfloor$ as well as to T_P and to the original fuzzy logic program P. From $\lfloor \emptyset \rfloor \Rightarrow \lfloor \emptyset^\circ \rfloor$ we get the following formulas of $P_{\lfloor T_P \rfloor}$:

$$
\begin{aligned}
top & \overset{1}{\Leftarrow} \\
\langle near(\mathsf{hotel}, \mathsf{hotel}), 0\rangle & \overset{1}{\Leftarrow} \mathbb{ts}(\mathbb{sh}_1(top, \langle near(\mathsf{hotel}, \mathsf{hotel}), 0\rangle) \wedge \dots \\
& \qquad \dots \wedge \mathbb{sh}_1(top, \langle suitable(\mathsf{hotel}), 0\rangle)), \\
& \dots \\
\langle near(\mathsf{hotel}, \mathsf{hotel}), 1\rangle & \overset{1}{\Leftarrow} \mathbb{ts}(\mathbb{sh}_1(top, \langle near(\mathsf{hotel}, \mathsf{hotel}), 0\rangle) \wedge \dots \\
& \qquad \dots \wedge \mathbb{sh}_1(top, \langle suitable(\mathsf{hotel}), 0\rangle)), \\
\langle near(\mathsf{hotel}, \mathsf{center}), 0\rangle & \overset{1}{\Leftarrow} \mathbb{ts}(\mathbb{sh}_1(top, \langle near(\mathsf{hotel}, \mathsf{hotel}), 0\rangle) \wedge \dots \\
& \qquad \dots \wedge \mathbb{sh}_1(top, \langle suitable(\mathsf{hotel}), 0\rangle)), \\
& \dots \\
\langle near(\mathsf{hotel}, \mathsf{center}), 0.8\rangle & \overset{1}{\Leftarrow} \mathbb{ts}(\mathbb{sh}_1(top, \langle near(\mathsf{hotel}, \mathsf{hotel}), 0\rangle) \wedge \dots \\
& \qquad \dots \wedge \mathbb{sh}_1(top, \langle suitable(\mathsf{hotel}), 0\rangle)), \\
& \dots \\
\langle cost(\mathsf{hotel}), 0.7\rangle & \overset{1}{\Leftarrow} \mathbb{ts}(\mathbb{sh}_1(top, \langle near(\mathsf{hotel}, \mathsf{hotel}), 0\rangle) \wedge \dots \\
& \qquad \dots \wedge \mathbb{sh}_1(top, \langle suitable(\mathsf{hotel}), 0\rangle)).
\end{aligned}
$$

Similarly, we get other rules of $P_{\lfloor T_P \rfloor}$ from each $\lfloor A \rfloor \Rightarrow \lfloor A^\circ \rfloor$:

$$\begin{aligned}
\langle near(\mathsf{hotel}, \mathsf{hotel}), 0\rangle &\stackrel{1}{\Leftarrow} \mathbb{ts}(\mathbb{sh}_1(top, \langle near(\mathsf{hotel}, \mathsf{hotel}), a_1\rangle) \mathbin{\&} \dots \\
&\quad \cdots \mathbin{\&} \mathbb{sh}_1(top, \langle suitable(\mathsf{hotel}), a_{11}\rangle)), \\
&\dots \\
\langle near(\mathsf{hotel}, \mathsf{hotel}), b_1\rangle &\stackrel{1}{\Leftarrow} \mathbb{ts}(\mathbb{sh}_1(top, \langle near(\mathsf{hotel}, \mathsf{hotel}), a_1\rangle) \mathbin{\&} \dots \\
&\quad \cdots \mathbin{\&} \mathbb{sh}_1(top, \langle suitable(\mathsf{hotel}), a_{11}\rangle)), \\
\langle near(\mathsf{hotel}, \mathsf{center}), 0\rangle &\stackrel{1}{\Leftarrow} \mathbb{ts}(\mathbb{sh}_1(top, \langle near(\mathsf{hotel}, \mathsf{hotel}), 0\rangle) \mathbin{\&} \dots \\
&\quad \cdots \mathbin{\&} \mathbb{sh}_1(top, \langle suitable(\mathsf{hotel}), 0\rangle)), \\
&\dots \\
\langle near(\mathsf{hotel}, \mathsf{center}), b_2\rangle &\stackrel{1}{\Leftarrow} \mathbb{ts}(\mathbb{sh}_1(top, \langle near(\mathsf{hotel}, \mathsf{hotel}), a_1\rangle) \mathbin{\&} \dots \\
&\quad \cdots \mathbin{\&} \mathbb{sh}_1(top, \langle suitable(\mathsf{hotel}), a_{11}\rangle)), \\
&\dots \\
\langle cost(\mathsf{hotel}), b_{10}\rangle &\stackrel{1}{\Leftarrow} \mathbb{ts}(\mathbb{sh}_1(top, \langle near(\mathsf{hotel}, \mathsf{hotel}), a_1\rangle) \mathbin{\&} \dots \\
&\quad \cdots \mathbin{\&} \mathbb{sh}_1(top, \langle suitable(\mathsf{hotel}), a_{11}\rangle)).
\end{aligned}$$

All these rules can be further simplified. Note that the aggregator $\mathbb{sh}_1$ (interpreted by 1-shift, i.e. identity) can be obviously omitted. Moreover, the conjunctor $\mathbin{\&}$ performs just as a boolean conjunction for crisp values and $\mathbb{ts}$ (interpreted by globalization here) is also indistinguishable from identity when applied on crisp values. Altogether, $P_{\lfloor T_P \rfloor}$ can be seen as a boolean logic program containing the following rules for all $a_1, \dots, a_{11} \in L$ such that there is some $a_i > 0$.

$$\begin{aligned}
\langle near(\mathsf{hotel}, \mathsf{hotel}), 0\rangle &\Leftarrow \langle near(\mathsf{hotel}, \mathsf{hotel}), 0\rangle, \dots, \langle suitable(\mathsf{hotel}), 0\rangle \\
&\dots \\
\langle near(\mathsf{hotel}, \mathsf{hotel}), 1\rangle &\Leftarrow \langle near(\mathsf{hotel}, \mathsf{hotel}), 0\rangle, \dots, \langle suitable(\mathsf{hotel}), 0\rangle \\
\langle near(\mathsf{hotel}, \mathsf{center}), 0\rangle &\Leftarrow \langle near(\mathsf{hotel}, \mathsf{hotel}), 0\rangle, \dots, \langle suitable(\mathsf{hotel}), 0\rangle \\
&\dots \\
\langle near(\mathsf{hotel}, \mathsf{center}), 0.8\rangle &\Leftarrow \langle near(\mathsf{hotel}, \mathsf{hotel}), 0\rangle, \dots, \langle suitable(\mathsf{hotel}), 0\rangle \\
&\dots \\
\langle cost(\mathsf{hotel}), 0.7\rangle &\Leftarrow \langle near(\mathsf{hotel}, \mathsf{hotel}), 0\rangle, \dots, \langle suitable(\mathsf{hotel}), 0\rangle \\
\langle near(\mathsf{hotel}, \mathsf{hotel}), 0\rangle &\Leftarrow \langle near(\mathsf{hotel}, \mathsf{hotel}), a_1\rangle, \dots, \langle suitable(\mathsf{hotel}), a_{11}\rangle \\
&\dots \\
\langle near(\mathsf{hotel}, \mathsf{hotel}), b_1\rangle &\Leftarrow \langle near(\mathsf{hotel}, \mathsf{hotel}), a_1\rangle, \dots, \langle suitable(\mathsf{hotel}), a_{11}\rangle \\
\langle near(\mathsf{hotel}, \mathsf{center}), 0\rangle &\Leftarrow \langle near(\mathsf{hotel}, \mathsf{hotel}), a_1\rangle, \dots, \langle suitable(\mathsf{hotel}), a_{11}\rangle \\
&\dots \\
\langle near(\mathsf{hotel}, \mathsf{center}), b_2\rangle &\Leftarrow \langle near(\mathsf{hotel}, \mathsf{hotel}), a_1\rangle, \dots, \langle suitable(\mathsf{hotel}), a_{11}\rangle \\
&\dots \\
\langle cost(\mathsf{hotel}), b_{10}\rangle &\Leftarrow \langle near(\mathsf{hotel}, \mathsf{hotel}), a_1\rangle, \dots, \langle suitable(\mathsf{hotel}), a_{11}\rangle
\end{aligned}$$

Further, we add the "basic" facts to $P_{\lfloor T_P \rfloor}$ to get a boolean logic program which is equivalent to a given fuzzy logic program P. For each $y \in \mathcal{B}_P$, we add a fact $\langle y, 0\rangle$ to $P_{\lfloor T_P \rfloor}$. In this particular case, the "basic" facts are $\langle near(\mathsf{hotel}, \mathsf{hotel}), 0\rangle$, $\langle near(\mathsf{hotel}, \mathsf{center}), 0\rangle$,..., $\langle suitable(\mathsf{hotel}), 0\rangle$.

4.6 Conclusions

We have shown that fuzzy attribute implications (in sense of Bělohlávek and Vychodil) and fuzzy logic programs (in sense of Vojtáš) are mutually reducible (with some limitations to structures of degrees) and correct answers for fuzzy logic programs and queries can be described via semantic entailment of fuzzy attribute implications and *vice versa.* Furthermore, we have shown a complete Pavelka-style axiomatization for fuzzy attribute logic (FAL) over arbitrary **L** and infinite sets of attributes using a new deduction system containing an infinitary cut. Together with the reduction we have shown in the paper, this gives us a new syntactic characterization of correct answers in fuzzy logic programming (FLP). The results have shown a new theoretical insight and a link of two branches of rule-based reasoning methods. Future research will focus on various other issues interrelating FLP and FAL.

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