VYSOKÉ UČENÍ TECHNICKÉ V BRNĚ
BRNO UNIVERSITY OF TECHNOLOGY

FAKULTA STROJNÍHO INŽENÝRSTVÍ
ÚSTAV MATEMATIKY
FACULTY OF MECHANICAL ENGINEERING
INSTITUTE OF MATHEMATICS

PERIODIC SOLUTIONS TO NONAUTONOMOUS DUFFING EQUATION

DIPLOMOVÁ PRÁCE
DIPLOMA THESIS

AUTOR PRÁCE
AUTHOR
QAZI HAMID ZAMIR

VEDOUCÍ PRÁCE
SUPERVISOR
doc. Ing. JIŘÍ ŠREMR, Ph.D.

BRNO 2020

BRNO UNIVERSITY OF TECHNOLOGY

VYSOKÉ UČENÍ TECHNICKÉ V BRNĚ

FACULTY OF MECHANICAL ENGINEERING

FAKULTA STROJNÍHO INŽENÝRSTVÍ

INSTITUTE OF MATHEMATICS

ÚSTAV MATEMATIKY

PERIODIC SOLUTIONS TO NONAUTONMOUS DUFFING EQUATION

PERIODICKÁ ŘEŠENÍ NEAUTONOMNÍ DUFFINGOVY ROVNICE

MASTER'S THESIS

DIPLOMOVÁ PRÁCE

AUTHOR Qazi Hamid Zamir

AUTOR PRÁCE

SUPERVISOR doc. Ing. Jiří Šremr, Ph.D.

VEDOUCÍ PRÁCE

BRNO 2020

Specification Master's Thesis

Department: Institute of Mathematics
Student: **Qazi Hamid Zamir**
Study programme: Applied Sciences in Engineering
Study branch: Mathematical Engineering
Supervisor: **doc. Ing. Jiří Šremr, Ph.D.**
Academic year: 2019/20

Pursuant to Act no. 111/1998 concerning universities and the BUT study and examination rules, you have been assigned the following topic by the institute director Master's Thesis:

Periodic solutions to nonautonmous Duffing equation

Concise characteristic of the task:

Ordinary differential equations of various types appear in the mathematical modelling in mechanics. Differential equations obtained are usually rather complicated nonlinear equations. However, using suitable approximations of nonlinearities, one can derive simple equations that are either well known or can be studied analytically. An example of such "approximative" equation is the so–called Duffing equation. Hence, the question on the existence of a periodic solution to the Duffing equation is closely related to the existence of periodic vibrations of the corresponding nonlinear oscillator.

Goals Master's Thesis:

Theoretical part:
1) To supplement knowledge in the theory of dynamical systems (in particular, sketch of a phase portrait).
2) To study fundamentals of the qualitative theory of boundary value problems (in particular, method of lower and upper functions for the periodic problem).

Practical part:
1) To analyse the existence of periodic solutions to the considered Duffing equation in the autonomous case.
2) To find conditions guaranteeing the existence of a periodic solution in the nonautonomous case.
3) To discuss the uniqueness of periodic solutions.

Recommended bibliography:

HABETS, P., De COSTER, C. Two-point boundary value problem: lower and upper solutions. Mathematics in Science and Engineering, 205, Elsevier B.V., Amsterdam, 2006, ISBN 978-0-4-4-52200-9.

HARTMAN, P. Ordinary differential equations, John Wiley & Sons, New York - London - Sydney, 1964.

KOVACIC, I. (ed.), BRENNAN, M. J. (ed.). The Duffing equation. Nonlinear oscillators and their behaviour, John Wiley & Sons, Ltd., Publication, Hoboken, NJ, 2011, ISBN 978-0-470-71549-9.

PERKO, L. Differential equations and dynamical systems, Text in Applied Mathematics, 7, Springer-Verlag, New York, 2001, ISBN 0-387-95116-4.

Deadline for submission Master's Thesis is given by the Schedule of the Academic year 2019/20

In Brno,

L. S.

prof. RNDr. Josef Šlapal, CSc.
Director of the Institute

doc. Ing. Jaroslav Katolický, Ph.D.
FME dean

Abstract

Ordinary differential equations of various types appear in the mathematical modelling in mechanics. Differential equations obtained are usually rather complicated nonlinear equations. However, using suitable approximations of nonlinearities, one can derive simple equations that are either well known or can be studied analytically. An example of such "approximative" equation is the so-called Duffing equation. Hence, the question on the existence of a periodic solution to the Duffing equation is closely related to the existence of periodic vibrations of the corresponding nonlinear oscillator.

Keywords

Differential equation, Duffing equation, periodic solution, existence, uniqueness.

Hamid, Z. Q.: *Periodic solutions to nonautonomous Duffing equation*, Brno University of Technology, Faculty of Mechanical Engineering, 2020. ?? pp. Supervisor: doc. Ing. Jiří Šremr, Ph.D.

I declare that I have worked on this thesis independently under a supervision of doc. Jiří Šremr, Ph.D. and using the sources listed in the bibliography.

Qazi Hamid Zamir

I am grateful to my supervisor doc. Jiří Šremr, Ph.D. without whose patience, guidance and support from the beginning to the final level, I could not able to complete this thesis. I would like to thank all instructors in department of mathematics for their support and willingness to share their knowledge which has been an integral part of my success, I would like to express my gratitude to my parents, my family, and especially my friends, who gave their guidance, support, ideas, views and shared some of their knowledge either directly or indirectly.

Qazi Hamid Zamir

Contents

Chapter 1

Introduction

The study of dynamical systems became a point of interest in recent mathematics and engineering researches. This is because of its essential attribute of being a time evolutionary procedure.

Mathematical modeling is a part of mathematics that mimics a certain result of reality by trying to describe and formulate the basic laws of natural sciences given phenomena by mathematical equations. In mathematical modeling in mechanics they often find differential equations of different types. These are mostly relatively complex linear equations, which, however, can be converted by a suitable approximation. One such equation is the Duffing equation. It is a second order differential equation with cubic non-linearity, which describes the chaotic behavior of some dynamic systems.

The Duffing equation in its numerous forms is used to describe many nonlinear systems. Although most physical systems can not be accurately described in this way for a wide range of operating conditions, such as frequency and amplitude of excitation, this equation can in many cases be used as an approximate definition so that their behavior can be qualitatively studied.In some cases, a comprehensive study may be conducted for low excitation amplitudes. It is in many cases the first step in switching from a linear system to a nonlinear one.

1.1 Motivation

The first objective is to interpretation of the Duffing equation in connection with the approximation of nonlinear oscillator. The second goal is the analysis of singular points and their stability in case of autonomous Duffing equation. We shall derive the level curves and corresponding phase portraits and analyze the existence of periodic solutions to the considered Duffing equation in the autonomous case, then we draw and interpret the phase portraits. We will find conditions guaranteeing the existence and uniqueness of a periodic solution in the non-autonomous case.

1.2 Structure of the thesis

This thesis work is organized as follows: Theoretical part and Practical part.

In second Chapter, We will first show the derivation of Duffing equation from a specific mathematical model. Using Newton's 2nd law of motion and the Taylor approximation of the nonlinear term gives the shape corresponding Duffing equation.

The third Chapter is devoted to the theoretical part which deals with the supplement knowledge in the theory of dynamical systems. We determine singular points and their type and stability.

The fourth Chapter is devoted to the practical part where we analyze the existence of periodic solutions to the considered Duffing equation in the autonomous case, we calculate level of Hamiltonian and sketch of corresponding phase portraits. We find conditions guaranteeing the existence of a periodic solution in the non-autonomous case and discuss the uniqueness of periodic solutions.

The fifth Chapter is devoted to main conclusion of the work.

Chapter 2

Duffing equation

In mathematical models of various oscillators, one can find the second order differential equation

$$y'' + \delta y' + \alpha y + \beta y^3 = \gamma \sin t, \tag{2.1}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. This equation is the central topics of Duffing's monograph [1] published in 1918 and still bears his name today (see also [4]). Equation (3.28) appears, for example, when approximating a nonlinearity in the equation of motion of a pendulum by Taylor's polynomial of the third order. It can be also interpreted as an equation of motion of a forced oscillator with a spring whose restoring force is given as a third order polynomial. The pendulum is the dynamical prototypical mechanism studied in nonlinear dynamics. The standard pendulum consists of a mass suspended from a string of length L and fixed at a pivot point P, see Fig. 2.1. The pendulum swings back and forth with occasional motion when moved to an initial angle and is released. For simplicity we assume that no external damping or driving force is acting on our physical system. We also assume negligible mass of the string. The equation of motion for the pendulum can be obtained by applying Newton's second law for rotational systems and has the form

$$-mg \sin \theta L = mL^2 \frac{d^2\theta}{dt^2}. \tag{2.2}$$

By rearranging the above equation, we will get a nonlinear equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L} \sin \theta = 0. \tag{2.3}$$

If the amplitude of angular displacement is small enough, then the approximation

$$\sin \theta \approx \theta - \frac{\theta^3}{3!} \tag{2.4}$$

is valid. Substituting the value of $\sin \theta$ in equation (2.3), we get

$$\theta'' + \frac{g}{L}\theta - \frac{g}{6L}\theta^3 = 0, \tag{2.5}$$

which is a particular case of the Duffing equation (3.28).
It is clear that equation (2.5) has three constant solutions $\theta_1(t) := 0$, $\theta_2(t) := \sqrt{6}$, and $\theta_3(t) := -\sqrt{6}$. A qualitative analysis namely the phase portrait, and the existence of non constant periodic solutions to (2.5) will be discussed in Section 4.1.

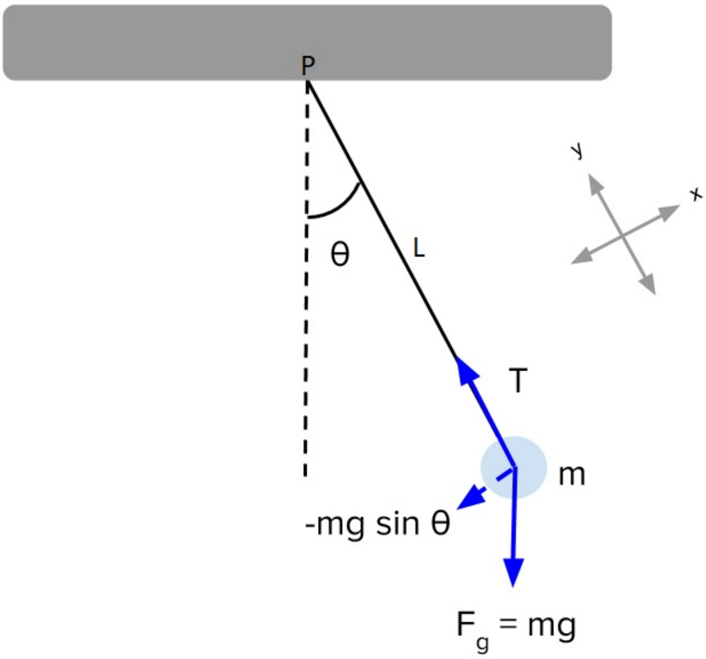

Figure 2.1: The simple pendulum

Now assume that the pivot point P oscillates vertically; oscillations are given by the T-periodic periodic function $d\colon \mathbb{R} \longrightarrow \mathbb{R}$. Then the equation of motion is of the form

$$\theta'' + \left(\frac{g}{L} + d''(t)\right)\sin\theta = 0$$

(see, e.g., [9, Section 11, Example 6.5]). Applying approximation (2.4), we get

$$\theta'' + \left(\frac{g}{L} + d''(t)\right)\theta - \frac{1}{6}\left(\frac{g}{L} + d''(t)\right)\theta^3 = 0. \tag{2.6}$$

Rewriting equation (2.6) in the form of variable u and substituting $p(t) := \frac{g}{L} + d''(t)$, $h(t) := \frac{1}{6}\left(\frac{g}{L} + d''(t)\right)$, we obtain the nonautonomous Duffing equation

$$u'' + p(t)u - h(t)u^3 = 0. \tag{2.7}$$

It is clear that equation (2.6) has three constant solutions $\theta_1(t) := 0$, $\theta_2(t) := \sqrt{6}$, $\theta_3(t) := -\sqrt{6}$ (similarly as equation (2.5)). In Section 4.2 (see Corollary 4.7), we show that, for T small enough, equation (2.6) has no non constant T-periodic solution.

Chapter 3

Theoretical part

In this part, we provide some notions and results which we will need to to analyze the existence of periodic solutions to Duffing equation in both autonomous and nonautonomous cases.

3.1 Autonomous system of differential equations

Below we discuss fundamentals of the theory of dynamical systems, which we will need in Section 4.1 (see e.g., [7] for review).

Definition 3.1. Let $G \subseteq \mathbb{R}^n$ be an open set and $f_1, f_2, \ldots, f_n : G \to \mathbb{R}$ are continuous functions. System

$$\begin{aligned} x_1' &= f_1(x_1, x_2, \ldots, x_n) \\ x_2' &= f_2(x_1, x_2, \ldots, x_n) \\ &\vdots \\ x_n' &= f_n(x_1, x_2, \ldots, x_n) \end{aligned} \tag{3.1}$$

is called an autonomous system of n first order differential equations.
Differential equations that explicitly depend on time (i.e., $\dot{x} = f(t, x)$) are referred as nonautonomous differential equations or nonautonomous vector fields.

System (3.1) can be represented by vector equivalent functions form

$$x' = f(x). \tag{3.2}$$

Definition 3.2. Solution of equation (3.2) on the interval $J \subset \mathbb{R}$ we mean a vector function $x = (x_1, x_2, \ldots, x_n)$ whose components have continuous derivatives on J and x satisfies the system (3.2) for each $t \in J$.

Definition 3.3. Let $x_0 \in G$. Condition

$$x(0) = x^0 \tag{3.3}$$

is called the initial (Cauchy) condition. The problem of finding solution to the system (3.2) satisfying condition (3.3) is called the initial (Cauchy) problem. The solution of the initial problem (3.2),(3.3) we denote by $\varphi(\cdot, x^0)$.

Remark 1. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous on an open set G. Then the solution $x(t)$ of (3.2) can be extended over a maximal interval of existence (ϵ_-, ϵ_+). Additionally if (ϵ_-, ϵ_+) is a maximal interval of existence, then $x(t)$ tends to the boundary ∂G of G as $t \to -\epsilon$ and $t \to +\epsilon$.

Theorem 3.4 (Peano's)**.** *Let the function f defined in $G \subseteq \mathbb{R}^n$ be continuous around $x_0 \in G$. Then the initial value problem* (3.2), (3.3) *has at least one solution defined in the neighborhood of* 0.

Proof. There are many different proofs available for Peano's theorem. We can classify them into two fundamental types:
(A) Proofs based on the construction of a sequence of approximate solutions (mainly Euler Cauchy polygons or Tonelli sequences) which converges to some solution.
(B) Proofs based on fixed point theorems (mainly Schauder's Theorem) applied to the equivalent integral version of (3.2),(3.3) (see, e.g., [10] for review).

□

Theorem 3.5 (Picard's)**.** *Consider the initial value problem* (3.2),(3.3)*. Suppose that $f : G \to \mathbb{R}^n$ is continuous as well as Lipschitz continuous on some neighborhood of $(0, x^0)$, then for some $\epsilon > 0$, there exists a unique solution x to the initial value problem* (3.2),(3.3) *on the interval* $[t_0 - \epsilon, t_0 + \epsilon]$.

Proof. The proof is based on applying fixed-point theory and transforming the differential equation (3.2). By integrating both sides of (3.2) and applying the condition (3.3), any function satisfying the differential equation (3.2) must satisfy the integral equation

$$x(t) - x(t_0) = \int_{t_0}^{t} f(s, x(s))\mathrm{d}s. \tag{3.4}$$

To prove the existence of the solution is obtained by successive approximations(Picard iteration). Then by using the Banach fixed point theorem, it can be shown that the sequence of "Picard iterates" is convergent and that the limit is a solution to the problem. Then by an application of Gronwall's lemma, proving the uniqueness.(see, e.g., [11] for review). □

Remark 2. Picard's theorem proves both existence and uniqueness while Peano's theorem provides a very simple check able condition to ensure the existence of solutions for complicated systems of ordinary differential equations.

Theorem 3.6 ([7, Section 2.4, Theorem 1])**.** *Consider the initial value problem* (3.2),(3.3)*. Suppose that $f : G \to \mathbb{R}^n$ is continuous. Then there exists a maximal interval I_{x^0} for each point $x^0 \in G$, for which the initial value problem has a unique solution x, i.e., if initial value problem has a solution y on an interval J then $J \subset I_{x^0}$ and $y(t) = x(t)$ for all $t \in J$. Furthermore the maximal interval is open i.e., $I_{x^0} = (\alpha, \beta)$.*

Definition 3.7. The interval (α, β) in Theorem 3.6 is called the maximal interval of existence of the solution x of the initial value problem (3.2),(3.3).

Definition 3.8. Orbit of the solution $\varphi(\cdot, x^0)$ of the system (3.2) on the interval I is set of points $\varphi(t, x^0)$, where $t \in I$.

Proposition 3.9. *If x is a solution of* (3.2)*, then for any $\tau \in \mathbb{R}$, $x(t + \tau)$ is also a solution.*

Remark 3. Proposition 3.9 is valid only for *autonomous* case.

Definition 3.10. The graph of the solution $\varphi(\cdot, x^0)$ of the system (3.2) on the interval I is set of points $(t, \varphi(t, x^0))$, where $t \in I$.

Remark 4. Orbit of solution $\varphi(\cdot, x^0)$ of the system (3.2) is obtained by the projection of graph of this solution to $\mathbb{R}^n$.

Definition 3.11. Let $S \subset \mathbb{R}^n$, then a set of states S is called an invariant set of (3.2),(3.3) if for all $x^0 \in S$ and for all $t \in I_{x^0}$ and $t \geq 0$, the solution x of (3.2),(3.3) satifies $x(t) \in S$, where I_{x^0} is the maximal interval of existence.

Definition 3.12. (i) **Cycle**: An orbit is said to be a cycle if it corresponds to the periodic solution and has the shape of a closed curve.

(ii) **Homoclinical Orbit**: An orbit is said to be homoclinic to an invariant set if it approaches the invariant set in time evolution asymptotically as times goes up to $\pm\infty$.

(iii) **Heteroclinic Orbit**: Let Γ_1 and Γ_2 are two invariant sets. An orbit is said to be heteroclinic if it approaches Γ_1 asymptotically under time evolution as time goes to $-\infty$ and approaches Γ_2 asymptotically under time evolution as time goes to $+\infty$.

In Section 3.1 the definition for the dynamical system have been given for $\mathbb{R}^n$. Further we will limit ourselves to space $\mathbb{R}^2$.
Consider the autonomous two-dimensional system

$$\begin{aligned} x_1' &= f_1(x_1, x_2), \\ x_2' &= f_2(x_1, x_2), \end{aligned} \tag{3.5}$$

where $f_1,\ f_2$ are continuous functions along with the first order partial derivatives on a domain $G \subseteq \mathbb{R}^2$. According to Theorem 3.6, the initial problem (3.5),(3.3) has solution defined on the maximal interval of existence I_{x^0} and we denote this solution as $\varphi(\cdot, x^0)$.

Definition 3.13. Matrix

$$Df(\bar{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\bar{x_1}, \bar{x_2}) & \frac{\partial f_1}{\partial x_2}(\bar{x_1}, \bar{x_2}) \\ \\ \frac{\partial f_2}{\partial x_1}(\bar{x_1}, \bar{x_2}) & \frac{\partial f_2}{\partial x_2}(\bar{x_1}, \bar{x_2}) \end{pmatrix}$$

we call the Jacobi matrix of the function $f = (f_1, f_2)$ at the point $\bar{x} = (\bar{x_1}, \bar{x_2})$.

Definition 3.14. A point $\bar{x} = (\bar{x_1}, \bar{x_2})$ is referred to as fixed or critical or singular or equilibrium point of the system (3.5), if $f_1(\bar{x_1}, \bar{x_2}) = 0,\ f_2(\bar{x_1}, \bar{x_2}) = 0$.

Definition 3.15 ([7])**.** (i) If all the eigenvalues of Jacobi matrix $Df(\bar{x})$ have nonzero real part then the critical point $\bar{x}$ of (3.5) is called **hyperbolic** fix point. Otherwise, it is said to be **nonhyperbolic**.

(ii) If some of the eigenvalues of Jacobi matrix $Df(\bar{x})$ have positive real part, and others have negative real part then the hyperbolic fixed point is called a **saddle**.

(iii) If all the eigenvalues of Jacobi matrix $Df(\bar{x})$ have positive or negative real part then the hyperbolic fix point is called **source** or **sink** respectively.

(iv) If all the eigenvalues of jacobi matrix $Df(\bar{x})$ are non zero and purely imaginary then the fix point is called a **center**.

Definition 3.16. A critical point $\bar{x}$ of (3.5) is called stable if for all $\epsilon > 0$ there exists $\delta > 0$ such that $\|x^0 - \bar{x}\| < \delta$ implies that $\|\varphi(t, x^0) - \bar{x}\| < \epsilon$ for all $t \geq 0$. Otherwise the fixed point is said to be unstable.

Definition 3.17. A critical point $\bar{x}$ of (3.5) is called asymptotically stable if it is stable and there exists an $r > 0$ such that for every $x^0 \in G$ holds $\|x^0 - \bar{x}\| < r \Rightarrow \lim_{t\to\infty} \|\varphi(t, x^0) - \bar{x}\| = 0$.

Remark 5. Fixed points of the system (4.4),(4.5) can be classified by looking at the characteristics of equilibrium linearization i.e., the type of equilibria can be defined by measuring the Jacobian matrix at each of the system's equilibrium points, and then determining the resulting eigenvalues, so the behavior of the system can be calculated in the neighborhood of each equilibrium point(qualitatively determined, or even quantitatively).

Theorem 3.18 ([7, Section 2.9])**.** *Let the function $f = (f_1, f_2)$ defined in $G \subseteq \mathbb{R}^2$ be continuous and $f(x_0) = 0$. Further, suppose that there is a real valued function $V \in C^1(G)$ satisfying $V(x_0) = 0$ and $V(x) > 0$ if $x \neq x_0$.*
Then

- *if $\dot{V}(x) \leq 0$ for all $x \in G$, x_0 is stable,*
- *if $\dot{V}(x) < 0$ for all $x \in G \setminus \{x_0\}$, x_0 is asymptotically stable,*
- *if $\dot{V}(x) > 0$ for all $x \in G \setminus \{x_0\}$, x_0 is unstable,*

where

$$\dot{V}(x) = V'_{x_1}(x_1, x_2)f_1(x_1, x_2) + V'_{x_2}(x_1, x_2)f_2(x_1, x_2) \quad \textit{for} \quad x = (x_1, x_2) \in G.$$

Theorem 3.19. *If $\bar{x}$ is a hyperbolic equilibrium point of (3.2) then it is stable if all of the eigenvalues of the matrix $Df(\bar{x})$ have negative real part and it is unstable if all of the eigenvalues of $Df(\bar{x})$ have positive real part.*

3.1.1 Hamiltonian system

Let $G \subseteq \mathbb{R}^2$ and let $H \in C^2(G)$. A system of the form

$$\begin{aligned} x_1' &= \frac{\partial H(x_1, x_2)}{\partial x_2}, \\ x_2' &= -\frac{\partial H(x_1, x_2)}{\partial x_1} \end{aligned} \tag{3.6}$$

is called the Hamiltonian system with 1 degree of freedom on G. We will consider the initial condition again together with the system (3.6),

$$x(0) = x^0.$$

The Hamiltonian system (3.6) is a special case of the system (3.5), where

$$f_1(x_1, x_2) = \frac{\partial H(x_1, x_2)}{\partial x_2}, \qquad f_2(x_1, x_2) = -\frac{\partial H(x_1, x_2)}{\partial x_1}.$$

Definition 3.20. Let $c \in \mathbb{R}$. The level $\mathcal{X}_c$ of the function H is the set of points $(x_1, x_2) \in G$ satisfying the equation

$$H(x_1, x_2) = c.$$

Theorem 3.21 ([7, Section 2.14, Theorem 1])**.** *Let $x^0 \in G$ and $\varphi(\cdot, x^0)$ be the solution of the system* (3.6) *along with the initial condition* (3.3) *at maximal interval $I_{x^0} \subseteq \mathbb{R}$. Then for every $t \in I_{x^0}$*

$$H(\varphi(t, x^0)) = H(x^0).$$

Remark 6. Hamiltonian has the meaning of total energy in physical models.

Theorem 3.21 tells that the values of *Hamiltonian* remains constant along the orbit of any solution of Hamiltonian system. Also, the system (3.6) conserves energy.

Corollary 3.22. *The levels $\mathcal{X}_c$ of Hamiltonian H consists of the orbits of the system* (3.6)*.*

It follows from the result of Corollary 3.22 that the level of Hamiltonian containing singular points $(\bar{x_1}, \bar{x_2})$ of the system (3.6) is a set of points $(x_1, x_2) \in G$ satisfying

$$H(x_1, x_2) = H(\bar{x_1}, \bar{x_2}).$$

In addition, each orbit of the system (3.6) is contained in some level of hamiltonian H. We can determine the orientation of of the orbits of the phase portraits of system (3.6) by Isocline method and the types of singular points by linearization method.

Remark 7. The ordinary differential equation can be solved by graphical method which is Isoclines. The isoclines for a differential equation of the form $x' = f(x)$, are lines in (x_1, x_2) plane obtained by setting right hand side i.e., $f(x)$ equal to a constant. By this method, for different constants we get a series of lines along which the solution curve have same gradient. The slope field can be visualized by calculating the gradient for each isocline.

The Jacobi matrix of the system (3.6) at the point (x_1, x_2) is of the shape

$$\mathbf{J}(x_1, x_2) = \begin{pmatrix} \frac{\partial^2 H}{\partial x_2 \partial x_1}(x_1, x_2) & \frac{\partial^2 H}{\partial x_2^2}(x_1, x_2) \\ & \\ -\frac{\partial^2 H}{\partial x_1^2}(x_1, x_2) & -\frac{\partial^2 H}{\partial x_2 \partial x_1}(x_1, x_2) \end{pmatrix}. \tag{3.7}$$

It is possible to decide the singular point $\bar{x}$ of the system (3.6) by using the determination of the jacobi matrix (3.7).

Theorem 3.23 ([7, Section 2.14, Theorem 2])**.** *Let $\bar{x} = (\bar{x_1}, \bar{x_2}) \in G$ be the singular point of* (3.6)*.*
Then

- *if det$(\boldsymbol{J}(\bar{x_1}, \bar{x_2})) < 0, \bar{x}$ is the saddle of the system* (3.6)*,*

- *if det$(\boldsymbol{J}(\bar{x_1}, \bar{x_2})) > 0, \bar{x}$ is the center of the system* (3.6)*.*

Second order differential equations of the form $x'' + f(x) = 0$ are possible to convert into so-called *conservative system*, which is a special type of Hamiltonian system. A system of shape

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -f(x_1). \end{aligned} \tag{3.8}$$

is conservative system, where $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable function. The Hamiltonian of system (3.8) has the form

$$H(x_1, x_2) = \frac{x_2^2}{2} + \int_0^{x_1} f(s) \mathrm{d}s \tag{3.9}$$

for $(x_1, x_2) \in \mathbb{R}$.

3.2 Periodic problem for second-order differential equation

Consider the periodic problem

$$u'' = f(t, u), \tag{3.10}$$

$$u(a) = u(b), \quad u'(a) = u'(b), \tag{3.11}$$

where $a < b$ and $f\colon [a, b] \to \mathbb{R}$ is a continuous function.

Definition 3.24. A function $u\colon [a, b] \to \mathbb{R}$ is said to be a *solution* to problem (3.10), (3.11) if it is continuous together with its first and second derivatives, satisfies

$$u''(t) = f(t, u(t)) \quad \text{for } t \in [a, b],$$

and meets periodic conditions (3.11).

Definition 3.25. A function $u : \mathbb{R} \longrightarrow \mathbb{R}$ of is called T-periodic if $u(t + T) = u(t)$ for all $t \in \mathbb{R}$.

To derive effective solvability conditions for problem (3.10), (3.11) various existence results can be applied. One of possible approaches is the so-called method of lower and upper functions.

Definition 3.26. A function $\alpha\colon [a, b] \to \mathbb{R}$, continuous together with its first and second derivatives, is said to be a *lower function* of problem (3.10), (3.11) if
(i) $\alpha''(t) \geq f(t, \alpha(t))$ for all $t \in [a, b]$.
(ii) $\alpha(a) = \alpha(b)$, $\alpha'(a) \geq \alpha'(b)$.
A function $\beta\colon [a, b] \to \mathbb{R}$, continuous together with its first and second derivatives, is said to be an *upper function* of problem (3.10), (3.11) if
(i) $\beta''(t) \leq f(t, \beta(t))$ for all $t \in [a, b]$.
(ii) $\beta(a) = \beta(b)$, $\beta'(a) \leq \beta'(b)$.

If the pair (α, β) of lower and upper functions is well-ordered in the sense of condition (3.12), then problem (3.10), (3.11) possesses a solution without any additional assumption on f. More precisely, the following theorem holds.

Theorem 3.27 ([2, Chapter I, Theorem 1.1])**.** *Assume that α and β are the lower and upper functions of problem* (3.10), (3.11), *respectively, such that*

$$\alpha(t) \leq \beta(t) \quad \textit{for } t \in [a, b]. \tag{3.12}$$

Then the problem (3.10), (3.11) *has at least one solution u such that*

$$\alpha(t) \leq u(t) \leq \beta(t), \quad \forall t \in [a, b].$$

For the linear second-order equation

$$u'' + q(t)u = h_0(t) \tag{3.13}$$

with continuous $q, h_0\colon [a, b] \to \mathbb{R}$ plenty of results are known. In Section 4.2, we will need Propositions 3.28 and 3.35 stated below. Recall that by a solution to (3.13) we understand a function $u\colon [a,b] \to \mathbb{R}$, which is continuous together with its first and second derivatives and satisfies equation (3.13) everywhere on $[a,b]$.

The first proposition concerns a number of zeros of solutions to equation (3.13). In Section 4.2, we will need the following statement, which is usually referred as Lyapunov's theorem.

Proposition 3.28 ([3, Chapter XI, Corollary 5.1])**.** *Assume that the homogeneous equation*

$$u'' + q(t)u = 0 \tag{3.14}$$

has a nontrivial solution possessing two zeros. Then

$$\int_a^b [q(t)]_+ \mathrm{d}t > \frac{4}{b-a}, \tag{3.15}$$

where

$$[q(t)]_+ = \frac{|q(t)| + q(t)}{2}.$$

Second proposition concerns the so-called third Fredholm's theorem for the problem (3.13), (3.11). It is a consequence of general results which are discussed in Chapter XII of [3].

Consider the nonhomogeneous system of n linear differential equations

$$y' = A(t)y + g(t), \tag{3.16}$$

where A is a continuous on $[a,b]$ matrix function (in general, complex-valued) and g is a continuous on $[a,b]$ vector function. Corresponding homogeneous system is of the form

$$y' = A(t)y. \tag{3.17}$$

In addition, consider the two-point boundary condition

$$My(a) - Ny(b) = 0, \tag{3.18}$$

where M, N are constant $n \times n$ matrices. The boundary condition (3.18) covers, in particular,

- periodic condition
$$y(a) = y(b),$$
if M, N are the identity matrices,
- initial condition in the point a
$$y(a) = 0,$$
if M is the identity matrix and N is the zero matrix,
- initial condition in the point b
$$y(b) = 0,$$
if N is the identity matrix and M is the zero matrix,
- the so-called Cauchy-Nicoletti conditions
$$y_1(a) = 0,\ \ldots,\ y_k(a) = 0,\ y_{k+1}(b) = 0,\ \ldots,\ y_n(b) = 0,$$
where $k \in \{1, \ldots, n-1\}$.

Proposition 3.29 ([3, Chapter XII, Theorem 1.1]). *Let M, N be such that the $n \times 2n$ matrix (M, N) is of rank n. Then* (3.16) *has a solution y satisfying* (3.18) *for every nonhomogeneous term g if and only if problem* (3.17), (3.18) *has no nontrivial solution; in which case y is unique and there exists a constant K, independent of g, such that*

$$\|y(t)\| \leq K \int_a^b \|g(s)\| \mathrm{d}s, \quad \forall t \in [a, b].$$

Now, a nature question is: What happens if problem (3.17), (3.18) has a nontrivial solution?

Consider the systems adjoint to (3.16), (3.17)

$$z' = -A^*(t)z - h(t), \tag{3.19}$$

$$z' = -A^*(t)z, \tag{3.20}$$

where A^* is the complex conjugate transpose of A, i.e., $A^* = \overline{A^T}$, and h is a continuous on $[a, b]$ vector function. Moreover, consider the boundary condition

$$Pz(a) - Qz(b) = 0, \tag{3.21}$$

where P, Q are constant $n \times n$ matrices. If y is a solution to (3.16) and z is a solution to (3.19), then Green's formula yields

$$\int_a^b \Big[g(s) \cdot z(s) - y(s) \cdot h(s)\Big] \mathrm{d}s = y(b) \cdot z(b) - y(a) \cdot z(a), \tag{3.22}$$

where the dot denotes scalar multiplication (see [3, Chapter IV, Lemma 7.2]). One can show that, if M, Q are nonsingular, then necessary and sufficient for (3.18), (3.21) to imply

$$y(b) \cdot z(b) - y(a) \cdot z(a) = 0 \tag{3.23}$$

in (3.22) is that

$$MP^* - NQ^* = 0 \tag{3.24}$$

holds. In general, we have

Lemma 3.30 ([3, Chapter XII, Lemma 1.2]). *Let M, N be such that* $\operatorname{rank}(M, N) = n$. *Then, there exist $n \times n$ matrices P, Q satisfying* $\operatorname{rank}(P, Q) = n$, (3.24), *and having property that the relations* (3.18), (3.21) *imply* (3.23) *(the pairs of vectors $z(a)$, $z(b)$ satisfying* (3.21) *are independent of the choice of P, Q).*

Boundary condition (3.21) satisfying the conditions of Lemma 3.30 is called the *adjoint boundary condition* to (3.18). Correspondingly, the problems (3.17), (3.18) and (3.20), (3.21) are called *adjoint problems*.

Lemma 3.31 ([3, Chapter XII, Lemma 1.3]). *Let M, N be such that* $\operatorname{rank}(M, N) = n$ *and P, Q be constant matrices such that boundary condition* (3.21) *is adjoint to* (3.18). *Then, problems* (3.17), (3.18) *and* (3.20), (3.21) *have the same number of linearly independent solutions.*

We are in a position to formulate the existence results for the problem (3.16), (3.18) provided that the homogeneous problem (3.17), (3.18) has a nontrivial solution.

Proposition 3.32 ([3, Chapter XII, Theorem 1.2]). *Let M, N be such that* $\operatorname{rank}(M,N) = n$ *and let* (3.17), (3.18) *and* (3.20), (3.21) *be adjoint problems. Suppose that* (3.17), (3.18) *has exactly k linearly independent solutions $y_1, \dots, y_k$ and let $z_1, \dots, z_k$ be linearly independent solutions to problem* (3.20), (3.21). *Then, system* (3.16) *has a solution y_0 satisfying* (3.18) *if and only if*

$$\int_a^b g(s) \cdot z_j(s) \mathrm{d}s = 0, \quad \forall j = 1, \dots, k. \tag{3.25}$$

In this case, solutions of problem (3.16), (3.18) *are given*

$$y_0 + \alpha_1 y_1 + \cdots \alpha_k y_k,$$

where $\alpha_1, \dots, \alpha_k$ are arbitrary constants.

Now we apply Propositions 3.29 and 3.32 to the nonhomogeneous periodic problem (3.13), (3.11). It is clear that problem (3.13), (3.11) can be rewritten as problem (3.16), (3.18), in which

$$A(t) = \begin{pmatrix} 0 & 1 \\ -q(t) & 0 \end{pmatrix}, \qquad g(t) = \begin{pmatrix} 0 \\ h_0(t) \end{pmatrix} \quad \text{for } t \in [a,b] \tag{3.26}$$

and

$$M = N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.27}$$

and that the following lemma holds.

Lemma 3.33. *Let A and g be defined by* (3.26) *and let* (3.27) *hold.*

If u is a solution to problem (3.13), (3.11), *then the vector function* (u, u') *is a solution to problem* (3.16), (3.18).

Conversely, if (y_1, y_2) *is a solution to problem* (3.16), (3.18), *then $y_1' = y_2$ and y_1 is a solution to problem* (3.13), (3.11).

Therefore, Proposition 3.29 yields the so-called Fredholm's alternative for periodic problem (3.13), (3.11).

Proposition 3.34. *Problem* (3.13), (3.11) *has a unique solution for an arbitrary nonhomogeneous term h_0 if and only if the corresponding homogeneous problem* (3.14), (3.11) *has only the trivial solution.*

The complex conjugate transpose of the matrix function A given by (3.26) is of the form

$$A^*(t) = \begin{pmatrix} 0 & -q(t) \\ 1 & 0 \end{pmatrix} \quad \text{for } t \in [a,b].$$

Moreover, the boundary condition (3.18) with M, N given by (3.27) is, in fact, the periodic condition $y(a) = y(b)$. One can show that the adjoint of the periodic condition $y(a) = y(b)$ is equivalent to the periodic condition $z(a) = z(b)$ and, thus, we put

$$P = Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Observe that

$$\begin{gathered}(y_1, y_2) \text{ is a solution to problem (3.17), (3.18)} \\ \Updownarrow \\ (y_2, -y_1) \text{ is a solution to problem (3.20), (3.21).}\end{gathered} \tag{3.28}$$

Since g is given by (3.26), for any solution $z = (z_1, z_2)$ to system (3.20), we have

$$\int_a^b g(s) \cdot z(s) \mathrm{d}s = \int_a^b \begin{pmatrix} 0 \\ h_0(s) \end{pmatrix} \cdot \begin{pmatrix} z_1(s) \\ z_2(s) \end{pmatrix} \mathrm{d}s = \int_a^b h_0(s) z_2(s) \mathrm{d}s. \tag{3.29}$$

Therefore, on account of (3.28), (3.29), and Lemma 3.33, Proposition 3.32 implies

Proposition 3.35. *Problem* (3.13), (3.11) *has a solution if and only if the condition*

$$\int_a^b u_0(s) h_0(s) \mathrm{d}s = 0$$

holds for every solution u_0 *to homogeneous problem* (3.14), (3.11).

Chapter 4

Periodic solutions to Duffing equation

In this part, we study the existence as well as uniqueness of T-periodic solutions to the equation

$$u'' = -p(t)u + h(t)u^3, \tag{4.1}$$

where $p, h\colon \mathbb{R} \to \mathbb{R}$ are continuous T-periodic functions. By a solution to equation (4.1) on the interval $I \subseteq \mathbb{R}$, as usual, we understand a function $u\colon I \to \mathbb{R}$, which is continuous together with its first and second derivatives and satisfies equation (4.1) everywhere on I. A solution to (4.1), which is defined and T-periodic on $\mathbb{R}$, is referred to as a T-periodic solution to (4.1).

4.1 Autonomous equation

We first consider equation (4.1) in which both p and h are constant functions. In other words, we consider the Duffing equation (2.5) introduced in Section 2, i.e., the equation

$$y'' = -ay + by^3, \tag{4.2}$$

where a and b are constants. The signs of coefficients a and b determines the type of oscillation.
The general solution of (4.2) for $a > 0$ and $b > 0$, associated with closed phase trajectories, is expressed in terms of the Jacobi elliptic sine as

$$y(t) = \bar{A}sn(\omega t, k), \tag{4.3}$$

where $\bar{A}$ is the amplitude of the oscillations, $\omega = \sqrt{a - \frac{|b|\bar{A}^2}{2}}$, and $k = \sqrt{\frac{|b|}{2}}\frac{\bar{A}}{\omega}$ is the modulus of the elliptic sine. The expression (4.3) is valid for $\bar{A} \leq \sqrt{\frac{a}{|b|}}$, i.e., $k \leq 1$. It describes periodic oscillations with period $\frac{4K(k)}{\omega}$, where $K(k)$ is the full elliptic integral of the first kind. For $k \to 1$ the oscillation period tends to infinity and $y(t)$ tends to the expression of the motion along separatrix between two points is described by the following equation

$$y(t) = \pm\sqrt{\frac{a}{|b|}}\tanh\sqrt{\frac{a}{2}}t$$

(see, e.g.,[4] or [5] for review).

We will provide the analytic description of all orbits in the phase portraits in Fig. 4.7 on p. 36.
Equation (4.2) can be rewritten as Hamiltonian system in the following way. Let

$$\begin{aligned} x_1 &= y, \\ x_2 &= y'. \end{aligned}$$

Then

$$x_1' = x_2, \tag{4.4}$$

$$x_2' = -ax_1 + bx_1^3. \tag{4.5}$$

Thus, in view of (3.9), we have the Hamiltonian as following

$$H(x_1, x_2) = \frac{1}{2}x_2^2 + \frac{a}{2}x_1^2 - \frac{b}{4}x_1^4 \quad \text{for } x_1, x_2 \in \mathbb{R}. \tag{4.6}$$

In Section 4.1.1-4.1.3, we will assume that $a > 0$, $b > 0$.

4.1.1 Equilibrium points

The equilibrium points of the system (4.4),(4.5) can be obtained by solving the following algebraic equations

$$\begin{aligned} x_2 &= 0, \\ -ax_1 + bx_1^3 &= 0. \end{aligned}$$

Clearly, $\bar{x} = (0,0)$, $\bar{x} = (+\sqrt{\frac{a}{b}}, 0)$ and $\bar{x} = (-\sqrt{\frac{a}{b}}, 0)$ are the only equilibrium points of (4.4),(4.5). Since we have $f(x_1, x_2) = (x_2, -ax_1 + bx_1^3)$, the derivatives are

$$Df(\bar{x}) = \begin{bmatrix} 0 & 1 \\ -a + 3bx_1^2 & 0 \end{bmatrix}, \quad Df(0,0) = \begin{bmatrix} 0 & 1 \\ -a & 0 \end{bmatrix}, \quad Df\left(+\sqrt{\frac{a}{b}}, 0\right) = \begin{bmatrix} 0 & 1 \\ 2a & 0 \end{bmatrix},$$

and

$$Df\left(-\sqrt{\frac{a}{b}}, 0\right) = \begin{bmatrix} 0 & 1 \\ 2a & 0 \end{bmatrix}.$$

Thus, according to Definition 3.15 in Section 3.1 the equilibrium point $(0,0)$ is a **center** and $\left(+\sqrt{\frac{a}{b}}, 0\right)$, $\left(-\sqrt{\frac{a}{b}}, 0\right)$ are **saddles**. The stability of any hyperbolic equilibrium point $\bar{x}$ of (4.4),(4.5) is determined by the signs of the real parts of the eigenvalues λ_k of the matrix $Df(\bar{x})$. Therefore, according to Theorem 3.19 the hyperbolic equilibrium points $\left(+\sqrt{\frac{a}{b}}, 0\right)$ and $\left(-\sqrt{\frac{a}{b}}, 0\right)$ are **unstable**. Since, the equilibrium point $x_0 = (0,0)$ is nonhyperbolic so we will apply Theorem 3.18.
Consider the system (4.4),(4.5) where the continuous function $q(x_1) = ax_1 - bx_1^3$ satisfies $x_1 q(x_1) > 0$ for $x_1 \neq 0$, $|x_1| < \sqrt{\frac{a}{b}}$. The total energy of the system (3.9)

$$H(x_1, x_2) = \frac{x_2^2}{2} + \int_0^{x_1} q(s)ds$$

serves as a *Liapunov function* for this system.
Since

$$H(x_1,x_2)=\frac{1}{2}x_2^2+x_1^2\left(\frac{a}{2}-\frac{b}{4}x_1^2\right)$$

$\exists U_\delta(0,0)$ such that $H(x_1,x_2)>0$ for all $(x_1,x_2)\in U_\delta$, $|x_1|<\sqrt{\frac{a}{b}}$,
hence,

$$\dot{H}(x_1,x_2)=q(x_1)x_2+x_2[(-q(x_1)]=0 \quad \forall\ [x_1,x_2]\in\mathbb{R}^2.$$

The energy is constant on the solution curves and the origin $(0,0)$ is a **stable** equilibrium point.

4.1.2 Level curves and orbits of system (4.4), (4.5)

Following ideas described in Section 3.1.1, we get the Hamiltonian system (4.4),(4.5) and the Hamiltonian (4.6).
Let us briefly describe another approach, how to get the form of the Hamiltonian.
The equation (4.2), equivalent to the conservation system having an energy function which is constant on orbits.
Consider

$$\ddot{x}+ax-bx^3=0,$$

without loss of generality, multiply it by $\dot{x}$

$$\dot{x}\left(\ddot{x}+ax-bx^3\right)=\dot{x}.0,$$

or

$$\dot{x}\ddot{x}+a\dot{x}x-b\dot{x}x^3=0,$$

which can be written as

$$\frac{d}{dt}\left(\frac{1}{2}\dot{x}^2+\frac{a}{2}x^2-\frac{b}{4}x^4\right)=0,$$

hence,

$$\frac{1}{2}\dot{x}^2+\frac{a}{2}x^2-\frac{b}{4}x^4=c=\quad\text{constant}$$

or

$$\frac{1}{2}x_2^2+\frac{a}{2}x_1^2-\frac{b}{4}x_1^4=c. \tag{4.7}$$

Equation (4.7) shows that the unforced, undamped Duffing oscillator has a first integral or more precisely a function c of dependent variables whose level curve give the orbits equation (4.2), or if we think of kinetic energy as $\frac{1}{2}x_2^2$, where the mass m has been scaled to be 1, the potential energy as $V(x_1)\equiv\frac{a}{2}x_1^2-\frac{b}{4}x_1^4$ and the total energy c of the system. The Duffing unforced, undamped oscillator is a *Hamiltonian System*.
It follows from Corollary 3.22 in Section 3.1.1 that in order to describe the orbits of system (4.4), (4.5), we need set $\mathcal{X}_c=\{(x_1,x_2)\in\mathbb{R}^2 : H(x_1,x_2)=c\}$, where c is admissible constant. The magnitude of c is very important because the phase portrates qualitively depends on it. The level of Hamiltonian corresponding equilibria (or fixed points) $y_1=0$ of (4.2) is $H(y_1,0)=0$ and to equilibria $y_{2,3}=\pm\sqrt{\frac{a}{b}}$ of (4.2) is $H(y_{2,3},0)=\frac{a^2}{4b}$.

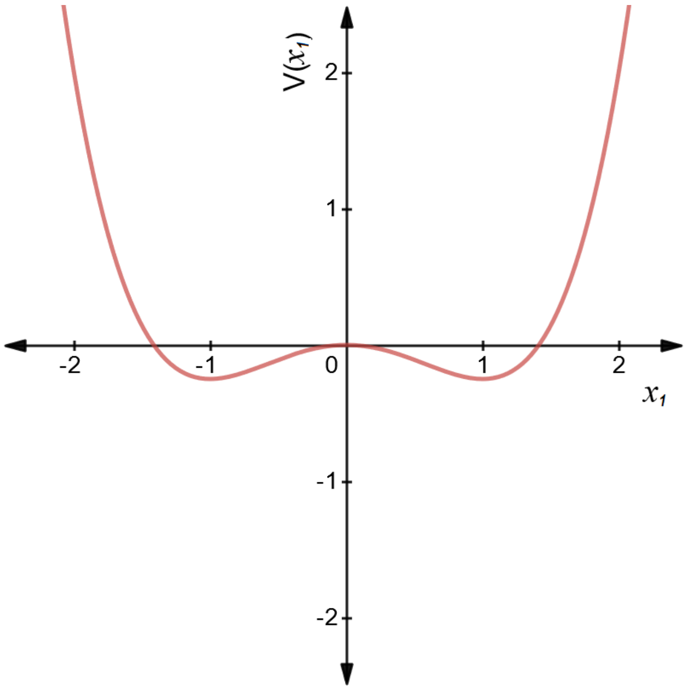

Figure 4.1: Level curves of potential Energy Function $V(x_1)$.

Case 1: $c = 0$. Now we will analytically describe the set $\mathcal{X}_0$. From (4.6) we obtain,

$$\frac{1}{2}x_2^2 + \frac{a}{2}x_1^2 - \frac{b}{4}x_1^4 = 0, \tag{4.8}$$

which yields

$$x_2^2 = -ax_1^2 + \frac{b}{2}x_1^4. \tag{4.9}$$

For the values of x_1, we have the following condition,

$$-ax_1^2 + \frac{b}{2}x_1^4 \geq 0$$

thus,

$$x_1^2(-a + \frac{b}{2}x_1^2) \geq 0.$$

This inequality is satisfied if either $x_1 = 0$ or $-a + \frac{b}{2}x_1^2 \geq 0$. If $x_1 = 0$, then (4.8) yields $x_2 = 0$, which corresponds to equilibrium $(0, 0)$ of system (4.4), (4.5), i.e., to equilibrium $y_1 = 0$ of equation (4.2). If

$$-a + \frac{b}{2}x_1^2 \geq 0,$$

i.e.,

$$|x_1| \geq \sqrt{\frac{2a}{b}},$$

it follows from (4.9) that the level curve $\mathcal{X}_0$ contains the following two orbits

$$x_2 \;=\; \pm\sqrt{-ax_1^2 + \frac{b}{2}x_1^4} \;\;,\;\; x_1 \geq \sqrt{\frac{2a}{b}} \;,$$

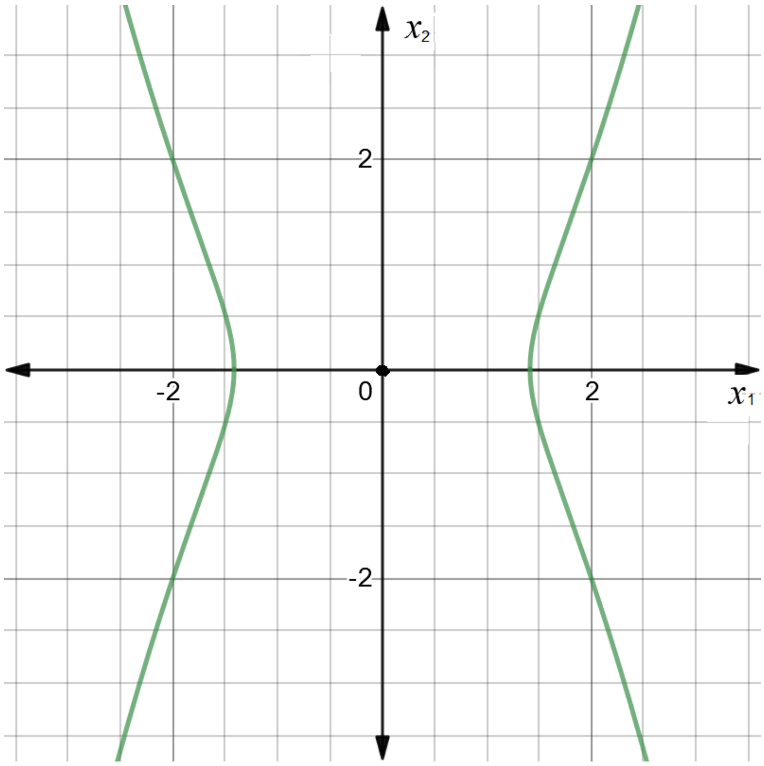

Figure 4.2: Level curve $\mathcal{X}_0$ of (4.6) for $a = 1 = b$.

$$x_2 = \pm\sqrt{-ax_1^2 + \frac{b}{2}x_1^4} \ , \ \ x_1 \leq -\sqrt{\frac{2a}{b}} \ .$$

The lever curve $\mathcal{X}_0$ is shown in Fig. 4.2.

Case 2: $c = \frac{a^2}{4b}$**.** Now we will analytically describe the set $\mathcal{X}_{\frac{a^2}{4b}}$. From equation (4.6), we can write

$$\begin{aligned}
\frac{1}{2}x_2^2 + \frac{a}{2}x_1^2 - \frac{b}{4}x_1^4 &= \frac{a^2}{4b}, \\
x_2^2 &= -ax_1^2 + \frac{b}{2}x_1^4 + \frac{a^2}{2b}, \\
x_2^2 &= \frac{b}{2}\left(x_1^4 - \frac{2a}{b}x_1^2 + \frac{a^2}{b^2}\right), \\
x_2^2 &= \frac{b}{2}\left(x_1^2 - \frac{a}{b}\right)^2, \\
|x_2| &= \sqrt{\frac{b}{2}}\left|x_1^2 - \frac{a}{b}\right|.
\end{aligned}$$

Hence, the level curve $\mathcal{X}_{\frac{a^2}{4b}}$ contains equilibria $\left(\sqrt{\frac{a}{b}}, 0\right)$, $\left(-\sqrt{\frac{a}{b}}, 0\right)$ and six orbits as following,

$$x_2 = \sqrt{\frac{b}{2}}\left(x_1^2 - \frac{a}{b}\right) \quad , \ \ x_1 < -\sqrt{\frac{a}{b}} \ ,$$

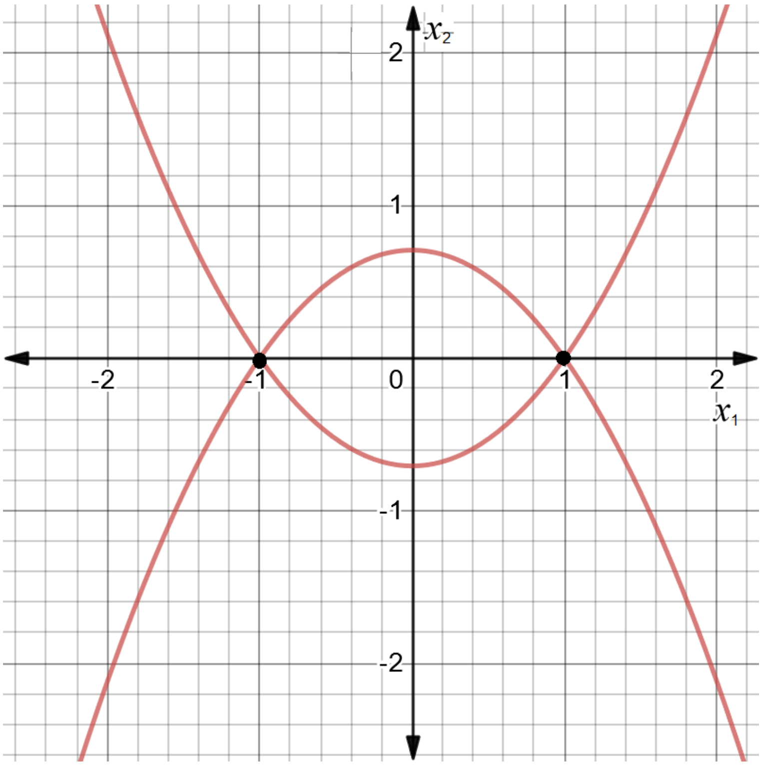

Figure 4.3: Level curves $\mathcal{X}_{\frac{a^2}{4b}}$ of (4.6) for $a = b = 1$.

$$
\begin{aligned}
x_2 &= -\sqrt{\frac{b}{2}}\left(x_1^2 - \frac{a}{b}\right) \quad , \quad x_1 < -\sqrt{\frac{a}{b}} \, , \\
x_2 &= \sqrt{\frac{b}{2}}\left(x_1^2 - \frac{a}{b}\right) \quad , \quad -\sqrt{\frac{a}{b}} < x_1 < \sqrt{\frac{a}{b}} \, , \\
x_2 &= -\sqrt{\frac{b}{2}}\left(x_1^2 - \frac{a}{b}\right) \quad , \quad -\sqrt{\frac{a}{b}} < x_1 < \sqrt{\frac{a}{b}} \, , \\
x_2 &= \sqrt{\frac{b}{2}}\left(x_1^2 - \frac{a}{b}\right) \quad , \quad x_1 > \sqrt{\frac{a}{b}} \, , \\
x_2 &= -\sqrt{\frac{b}{2}}\left(x_1^2 - \frac{a}{b}\right) \quad , \quad x_1 > \sqrt{\frac{a}{b}} \, .
\end{aligned}
$$

The corresponding level curve is shown in Fig. 4.3.

Case 3: $c > \frac{a^2}{4b}$. Consider $H(x_1, x_2) = c$, we obtain

$$\frac{1}{2}x_2^2 + \frac{a}{2}x_1^2 - \frac{b}{4}x_1^4 = c,$$

$$x_2^2 = -ax_1^2 + \frac{b}{2}x_1^4 + 2c. \tag{4.10}$$

From this we obtain the condition

$$-ax_1^2 + \frac{b}{2}x_1^4 + 2c \geq 0,$$

$$\begin{aligned} -ax_1^2 + \frac{b}{2}x_1^4 + \frac{a^2}{2b} - \frac{a^2}{2b} + 2c &\geq 0, \\ \frac{b}{2}\left(x_1^2 - \frac{a}{b}\right)^2 &\geq \frac{a^2}{2b} - 2c, \end{aligned}$$

$$\left(x_1^2 - \frac{a}{b}\right)^2 \geq \frac{a^2 - 4bc}{b^2}. \tag{4.11}$$

Since $c > \frac{a^2}{4b}$, clearly,

$$\frac{a^2 - 4bc}{b^2} < 0$$

and thus, inequality (4.11) is satisfied for any $x_1 \in \mathbb{R}$. Therefore the corresponding orbits are

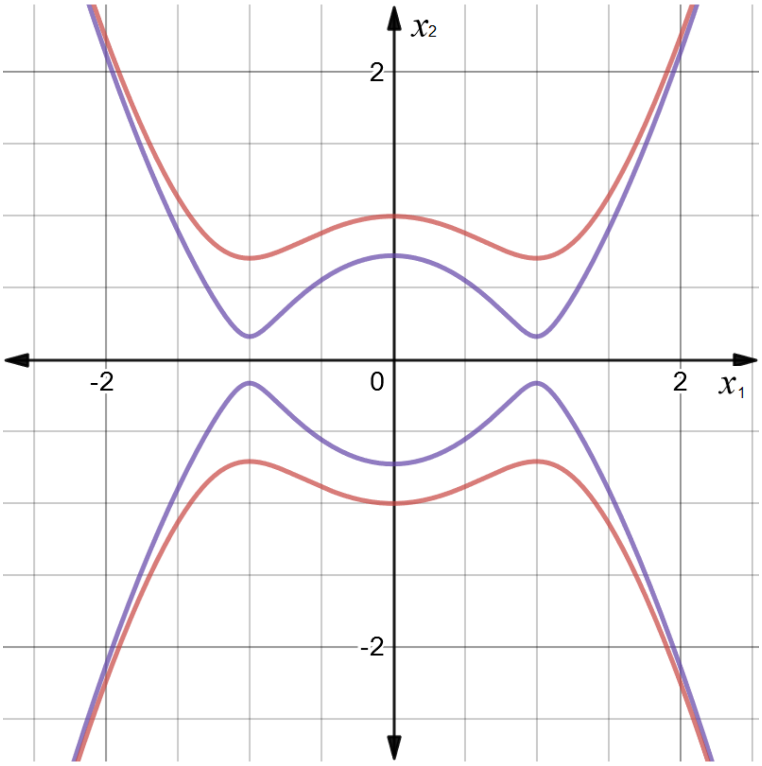

Figure 4.4: Level curves $\mathcal{X}_c$ of (4.6) for $a = b = 1$ and $c = \frac{10}{19}, c = 1$.

$$\begin{aligned} x_2 &= \sqrt{\frac{b}{2}x_1^4 - ax_1^2 + 2c} \quad , \quad x_1 \in \mathbb{R}, \\ x_2 &= -\sqrt{\frac{b}{2}x_1^4 - ax_1^2 + 2c} \quad , \quad x_1 \in \mathbb{R}. \end{aligned}$$

The corresponding level curve is shown in Fig. 4.4.

Case 4: $c < 0$**.** Just as in case 3, we get (4.10) and the condition (4.11). Since $c < 0$, condition (4.11) is satisfied iff

$$x_1^2 - \frac{a}{b} \geq \sqrt{\frac{a^2 - 4bc}{b^2}},$$

$$
\begin{aligned}
x_1^2 &\geq \frac{a}{b} + \sqrt{\frac{a^2 - 4bc}{b^2}}, \\
|x_1| &\geq \sqrt{\frac{a}{b} + \sqrt{\frac{a^2 - 4bc}{b^2}}}.
\end{aligned}
$$

So the corresponding orbits are as following,

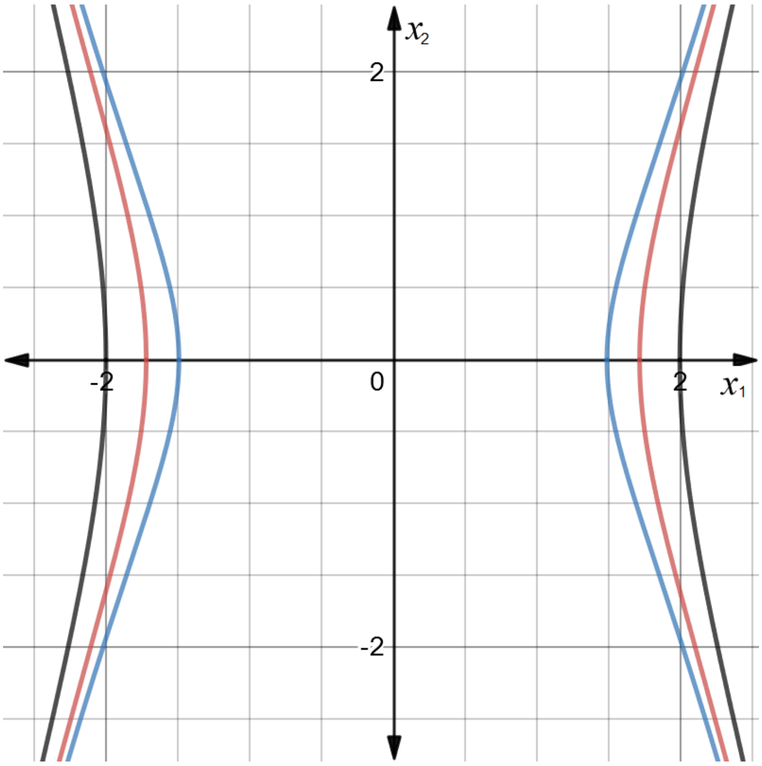

Figure 4.5: Level curves $\mathcal{X}_c$ of (4.6) for $a = b = 1$ and $c = -1$, $c = -\sqrt{2}$, $c = -4$.

$$
\begin{aligned}
x_2 &= \pm\sqrt{\frac{b}{2}x_1^4 - ax_1^2 + 2c} \;\;, \;\; x_1 \geq \sqrt{\frac{a}{b} + \sqrt{\frac{a^2 - 4bc}{b^2}}}, \\
x_2 &= \pm\sqrt{\frac{b}{2}x_1^4 - ax_1^2 + 2c} \;\;, \;\; x_1 \leq -\sqrt{\frac{a}{b} + \sqrt{\frac{a^2 - 4bc}{b^2}}}.
\end{aligned}
$$

The corresponding level curve is shown in Fig. 4.5.

Case 5: $0 < c < \frac{a^2}{4b}$**.** Just as in case 3, we get (4.10) and the condition (4.11). Since $0 < c < \frac{a^2}{4b}$, condition (4.11) is satisfied iff either

$$
x_1^2 - \frac{a}{b} \geq \sqrt{\frac{a^2 - 4bc}{b^2}},
$$

or

$$
x_1^2 - \frac{a}{b} \leq -\sqrt{\frac{a^2 - 4bc}{b^2}}.
$$

Hence, we have the following three orbits

$$
\begin{aligned}
x_2 &= \pm\sqrt{\frac{b}{2}x_1^4 - ax_1^2 + 2c} \ , \quad x_1 \geq \sqrt{\frac{a}{b} + \sqrt{\frac{a^2-4bc}{b^2}}}, \\
x_2 &= \pm\sqrt{\frac{b}{2}x_1^4 - ax_1^2 + 2c} \ , \quad x_1 \leq -\sqrt{\frac{a}{b} + \sqrt{\frac{a^2-4bc}{b^2}}}, \\
x_2 &= \pm\sqrt{\frac{b}{2}x_1^4 - ax_1^2 + 2c} \ , \quad -\sqrt{\frac{a}{b} + \sqrt{\frac{a^2-4bc}{b^2}}} \leq x_1 \leq \sqrt{\frac{a}{b} + \sqrt{\frac{a^2-4bc}{b^2}}}.
\end{aligned}
$$

and the corresponding level curves are shown in Fig. 4.6.

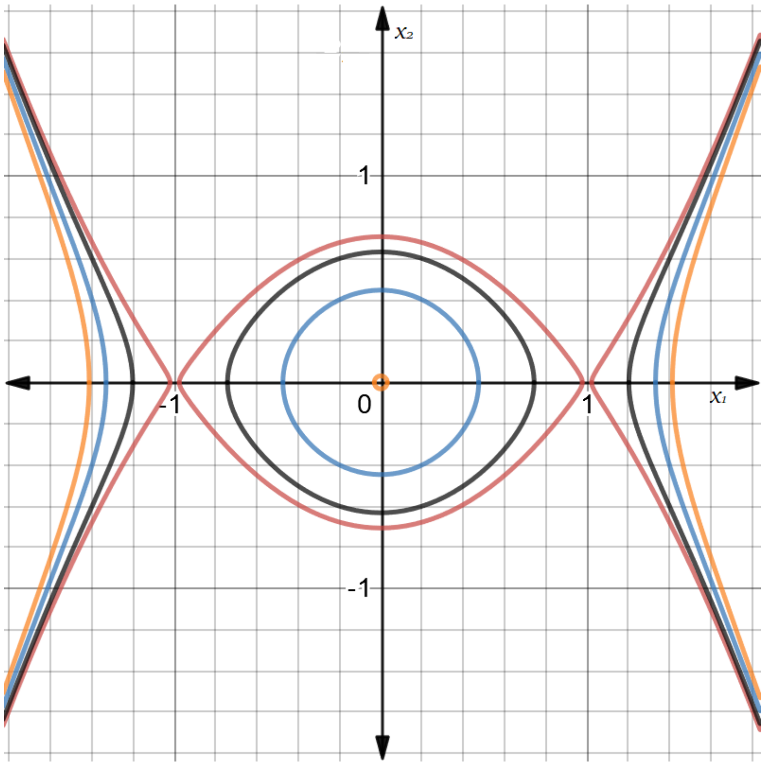

Figure 4.6: Level curves $\mathcal{X}_c$ of (4.6) for $a = b = 1$ and $c = 0.001$, $c = 0.15$, $c = 0.25$, $c = 0.40$, $c = 0.499$.

4.1.3 Phase portrait for the Duffing equation, solutions of (4.2)

In the previous section we provided analytical description of all orbits of the hamiltonian system (4.4), (4.5) corresponding to the Duffing equation (4.2). Moreover, in Figs. 4.2–4.6, level curves of hamiltonian (4.6) are shown.

It allows us to illustrate the phase portrait for the Duffing equation (4.2), see Fig. 4.7. Therefore, without knowledge of exact form of solutions, we can claim that the Duffing

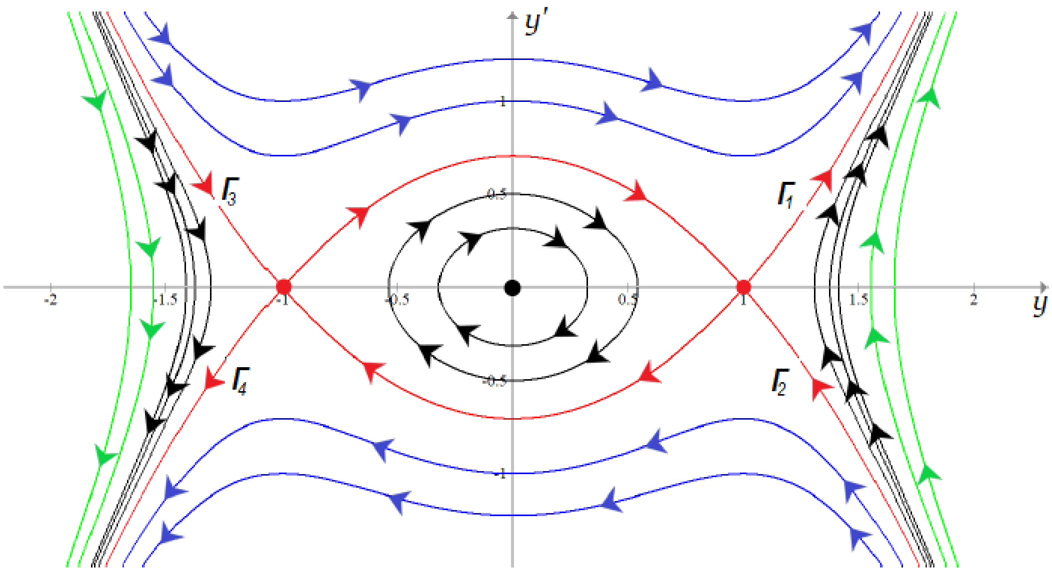

Figure 4.7: Phase portrait for Duffing equation (4.2) for $a = 1, b = 1$.

equation (4.2) has the following types of solutions:

- three constant solutions

$$y_1(t) = 0, \qquad y_2(t) = \sqrt{\frac{a}{b}}, \qquad y_3(t) = -\sqrt{\frac{a}{b}}$$

  corresponding to the critical points $(0,0)$, $(\sqrt{\frac{a}{b}}, 0)$ and $(-\sqrt{\frac{a}{b}}, 0)$ of the hamiltonian system (4.4), (4.5). The constant solution shown in Fig. 4.7 by a black dot is stable and the solutions shown by red dots are unstable.
- periodic sign-changing solutions, they correspond to *black* cyclic curves in the phase portrait Fig. 4.7.
- heteroclinic solutions y_4, resp. y_5, corresponding to *red* curve, resp. *red* curve, in the phase portrait Fig. 4.7, which satisfies

$$\lim_{t\to-\infty} y_4(t) = -\sqrt{\frac{a}{b}}, \qquad \lim_{t\to+\infty} y_4(t) = \sqrt{\frac{a}{b}}$$

  and

$$\lim_{t\to-\infty} y_5(t) = \sqrt{\frac{a}{b}}, \qquad \lim_{t\to+\infty} y_5(t) = -\sqrt{\frac{a}{b}}.$$

- positive unbounded from above solutions corresponding to *blue* color curve in the phase portrait Fig. 4.7.
- negative unbounded from below solutions corresponding to *blue* color curve in the phase portrait Fig. 4.7.

- sign-changing unbounded increasing solutions corresponding to *green* color curve in the phase portrait Fig. 4.7.
- sign-changing unbounded decreasing solutions corresponding to *green* color curve in the phase portrait Fig. 4.7.
- positive, increasing and unbounded solution y_6 corresponding to Γ_1 curve in the phase portrait Fig. 4.7 such that

$$\lim_{t \to -\infty} y_6(t) = \sqrt{\frac{a}{b}}.$$

- positive, decreasing and unbounded solution y_7 corresponding to Γ_2 curve in the phase portrait Fig. 4.7 such that

$$\lim_{t \to +\infty} y_7(t) = \sqrt{\frac{a}{b}}.$$

- negative, increasing and unbounded solution y_8 corresponding to Γ_3 curve in the phase portrait Fig. 4.7 such that

$$\lim_{t \to +\infty} y_8(t) = -\sqrt{\frac{a}{b}}.$$

- negative, decreasing and unbounded solution y_9 corresponding to Γ_4 curve in the phase portrait Fig. 4.7 such that

$$\lim_{t \to -\infty} y_9(t) = -\sqrt{\frac{a}{b}}.$$

4.2 Nonautonomous equation

It follows from the results of the previous section that, for any $T > 0$, autonomous equation (4.2) with $a, b > 0$ has a positive T-periodic solution. Indeed, equation (4.2) with $a, b > 0$ has a positive equilibrium which is obviously a T-periodic solution to (4.2) for any $T > 0$. The first question discussed in this section is the existence of a positive T-periodic solution to nonautonomous equation (4.1).
Let us start with a simple lemma.

Lemma 4.1. *If u is a T-periodic solution to equation* (4.1)*, then the restriction $u|_{[0,T]}$ is a solution to the periodic problem*

$$\begin{aligned} u'' &= -p(t)u + h(t)u^3, \\ u(0) = u(T), &\quad u'(0) = u'(T). \end{aligned} \tag{4.12}$$

Conversely, if u is a solution to problem (4.12)*, then the T-periodic extension of u to the whole real axis is a T-periodic solution to equation* (4.1)*.*

This lemma implies that there is one-to-one correspondence between T-periodic solutions of the equation (4.1) on the real axis and solutions of the corresponding boundary value problem on the interval of the length T. Now we want to find some conditions on the functions p and h guaranteeing that the equation (4.1) has at least one positive T-periodic solution.
By virtue of Lemma 4.1, it follows from [8, Corollary 4.1] the following proposition.

Proposition 4.2. *Let*

$$h(t) \geq 0, \quad p(t) \geq 0 \quad \text{for } t \in \mathbb{R}\ , \quad h(t) \not\equiv 0, \tag{4.13}$$

and

$$0 < \int_0^T p(t)dt \leq \frac{4}{T}.$$

Then, equation (4.1) *has at least one positive T-periodic solution.*

We will show that if the functions p, h are positive, then the hypothesis $\int_0^T p(t)dt \leq \frac{4}{T}$ in Proposition 4.2 can be omitted. To prove the existence of a positive solution to equation (4.1), we apply Lemma 4.1 and Theorem 3.27 with $a := 0$, $b := T$, and

$$f(t, x) := -p(t)x + h(t)x^3.$$

If we construct a pair of well-ordered lower and upper functions for the periodic problem (4.12) and if the lower function satisfies $\alpha(t) > 0$, then by Theorem 3.27 it follows that there exists at least one positive T-periodic solution to the periodic problem (4.12).

Proposition 4.3. *Let $0 < m \leq M$ be such that*

$$mh(t) \leq p(t) \leq Mh(t) \qquad \text{for all } t \in \mathbb{R}. \tag{4.14}$$

Then, equation (4.1) *has at least one positive T-periodic solution.*

Proof. Let

$$\alpha(t) := \sqrt{m}, \quad \beta(t) := \sqrt{M} \quad \text{for } t \in [0,T].$$

It follows from (4.14) that

$$mh(t) \leq p(t) \quad \text{for } t \in [0,T].$$

We can rewrite the above equation as following

$$\begin{aligned} -p(t) + mh(t) &\leq 0, \\ -p(t)\sqrt{m} + h(t)(\sqrt{m})^3 &\leq 0, \\ -p(t)\sqrt{m} + h(t)(\sqrt{m})^3 &\leq (\sqrt{m})'', \end{aligned}$$

and, thus,

$$-p(t)\alpha(t) + h(t)\alpha^3(t) \leq \alpha''(t) \quad \text{for all } t \in [0,T]. \tag{4.15}$$

Similarly, for the function β

$$Mh(t) \geq p(t) \quad \text{for } t \in [0,T].$$

We can rewrite the above equation as following

$$\begin{aligned} -p(t) + Mh(t) &\geq 0, \\ -p(t)\sqrt{M} + h(t)(\sqrt{M})^3 &\geq 0, \\ -p(t)\sqrt{M} + h(t)(\sqrt{M})^3 &\geq (\sqrt{M})'', \end{aligned}$$

and, thus,

$$-p(t)\beta(t) + h(t)\beta^3(t) \geq \beta''(t) \quad \text{for all } t \in [0,T]. \tag{4.16}$$

Combining (4.15) and (4.16) and using the hypothesis $0 < m \leq M$, we conclude that α and β are positive lower and upper functions of problem (4.12) satisfying

$$\alpha(t) \leq \beta(t) \quad \text{for } t \in [0,T].$$

Therefore, it follows from Theorem 3.27 that problem (4.12) has a solution u such that $\alpha(t) \leq u(t) \leq \beta(t)$ for $t \in [0,T]$.
Consequently, Lemma 4.1 yields that equation (4.1) possesses at least one positive T-periodic solution. □

Now we provide an effective condition guaranteeing the existence of a positive T-periodic solution to (4.1).

Theorem 4.4. *Let $p(t) > 0$, $h(t) > 0$ for $t \in \mathbb{R}$. Then, equation* (4.1) *has at least one positive T-periodic solution.*

Proof. Since we assume that the functions p and h are strictly positive and continuous, we can put

$$m := \min\left\{\frac{p(t)}{h(t)} : \quad t \in [0,T]\right\},$$

and

$$M := \max\left\{\frac{p(t)}{h(t)} : \quad t \in [0,T]\right\}.$$

Then $0 < m \leq M$ and the condition

$$mh(t) \leq p(t) \leq Mh(t) \quad \text{for all } t \in \mathbb{R}$$

holds. Therefore, the conclusion of the theorem follows from Proposition 4.3. □

It follows from Theorem 4.4 that if the functions p, h are positive, then the hypothesis

$$\int_0^T p(t)dt \leq \frac{4}{T} \tag{4.17}$$

in Proposition 4.2 can be omitted.
On the other hand, under the assumption (4.17), we can provide some further information about solutions to equation (4.1). Following the idea provided in [6] we arrive at the next theorem.

Theorem 4.5. *Let condition* (4.13) *hold and*

$$\int_0^T p(t)\mathrm{d}t \leq \frac{4}{T}. \tag{4.18}$$

Then, equation (4.1) *has at most one positive T-periodic solution. Moreover, any non-trivial T-periodic solution to equation* (4.1) *is either positive or negative.*

Proof. Suppose on contrary that u_1, u_2 are positive T-periodic solutions to equation (4.1) such that $u_2(t_0) > u_1(t_0)$ (without loss of generality) for some $t_0 \in \mathbb{R}$. Then either

$$u_1(t) < u_2(t) \qquad \forall\, t \in \mathbb{R},$$

or there exists $\gamma \in \mathbb{R}$ such that

$$u_1(\gamma) = u_2(\gamma). \tag{4.19}$$

Case 1: $u_1(t) < u_2(t)$, $\forall\, t \in \mathbb{R}$.
Since u_1 and u_2 are the solutions so they satisfy equation (4.1), i.e.,

$$\begin{aligned} u_1''(t) + [p(t) - h(t)u_1^2(t)]u_1(t) &= 0, \qquad \forall t \in [0, T], \\ u_2''(t) + [p(t) - h(t)u_2^2(t)]u_2(t) &= 0, \qquad \forall t \in [0, T]. \end{aligned}$$

Therefore,

$$u_2''(t) + [p(t) - h(t)u_1^2(t)]u_2(t) = h(t)u_2^3(t) - h(t)u_1^2(t)u_2(t) \qquad \forall t \in [0, T] \tag{4.20}$$

which gives us,

$$u_2''(t) + [p(t) - h(t)u_1^2(t)]u_2(t) = h(t)[u_2^2(t) - u_1^2(t)]u_2(t) \qquad \forall\, t \in [0, T]. \tag{4.21}$$

It is clear that u_1 is a positive solution to the homogeneous problem (3.11), (3.14) with

$$q(t) := p(t) - h(t)u_1^2(t), \quad a = 0, b = T,$$

and u_2 is a solution to the non-homogeneous problem (3.11),(3.13) with

$$q(t) := p(t) - h(t)u_1^2(t), \quad h_0(t) := h(t)[u_2^2(t) - u_1^2(t)]u_2(t), \ a = 0, \ b = T.$$

Therefore, Proposition 3.35 yields

$$\int_0^T u_1(t)h(t)[u_2^2(t) - u_1^2(t)]u_2(t)\mathrm{d}t = 0. \tag{4.22}$$

Since,$h(t) \geq 0$, $u_1(t) > 0$, $u_2^2(t) - u_1^2(t) > 0$ and $u_2(t) > 0$, it follows from (4.22) that $h(t) \equiv 0$, which is a contradiction to our hypothesis.

Case 2: $\exists\gamma \ \in \mathbb{R}$ such that (4.19) holds. Define,

$$w(t) := u_2(t) - u_1(t), \quad q(t) := p(t) - h(t)[(u_2^2(t) + u_1(t)u_2(t) + u_1^2(t))] \ \text{ for } t \in \mathbb{R}. \quad (4.23)$$

We have

$$\begin{aligned} w(\gamma) &= 0, \\ w''(t) &= -p(t)u_2(t) + h(t)u_2^3(t) + p(t)u_1(t) - h(t)u_1^3(t) \\ &= -p(t)w(t) + h(t)[u_2^3(t) - u_1^3(t)] \\ &= -p(t)w(t) + h(t)[(u_2(t) - u_1(t))(u_2^2(t) + u_1(t)u_2(t) + u_1^2(t))] \\ &= -[p(t) - h(t)(u_2^2(t) + u_1(t)u_2(t) + u_1^2(t))]w(t) \quad \text{for } t \in \mathbb{R}. \end{aligned}$$

Since

$$0 \leq h(t)(u_2^2(t) + u_1(t)u_2(t) + u_1^2(t)) \ \text{ for } t \in \mathbb{R},$$

we get

$$q(t) \leq p(t),$$

which implies

$$[q(t)]_+ \leq p(t), \qquad \forall\, t \in \mathbb{R},$$

because $p(t) \geq 0$ for $t \in \mathbb{R}$. Now

$$\begin{aligned} w''(t) + q(t)w(t) &= 0, \qquad \forall t \in [\gamma, \gamma + T] \\ w(\gamma) = 0, \quad w(\gamma + T) &= 0, \\ w(t) &\not\equiv 0 \quad \text{on} \quad [\gamma, \gamma + T]. \end{aligned}$$

Hence, Proposition 3.28 with $a = \gamma, b = \gamma + T$ yields

$$\int_\gamma^{\gamma+T} [q(t)]_+ \mathrm{d}t > \frac{4}{T} \ .$$

As we have, $p(t) \geq \ [q(t)]_+$ for $t \in \mathbb{R}$, so

$$\begin{aligned} \int_\gamma^{\gamma+T} p(t)\mathrm{d}t &\geq \int_\gamma^{\gamma+T} [q(t)]_+ \mathrm{d}t \\ &> \frac{4}{T}. \end{aligned}$$

The integral have same value for each interval of length T since p is T-periodic, and, thus,

$$\int_0^T p(t)\mathrm{d}t \ = \ \int_\gamma^{\gamma+T} p(t)\mathrm{d}t > \frac{4}{T}, \qquad (4.24)$$

which contradicts (4.18). Hence, equation (4.1) has at most one positive T-periodic solution.

Now, we will prove that every nontrivial T-periodic solution to (4.1) is either positive or negative. Suppose on the contrary that equation (4.1) has a nontrivial T-periodic solution u, which has zero value at some point t_1, i.e. $u(t_1) = 0$.

Then

$$u''(t) = -[p(t) - h(t)u^2(t)]u(t) \qquad \text{for } t \in \mathbb{R}. \qquad (4.25)$$

Define

$$q(t) := p(t) - h(t)u^2(t) \quad \text{for } t \in \mathbb{R}.$$

Similarly as above we get

$$[q(t)]_+ \leq p(t) \quad \text{for } t \in \mathbb{R}$$

and rewriting equation (4.25), we obtain

$$u''(t) + q(t)u(t) = 0, \qquad \forall\, t \in [t_1, t_1 + T],$$

$$u(t_1) = 0,\ u(t_1 + T) = 0, \quad u(t) \not\equiv 0, \quad \text{on} \quad [t_1, t_1 + T].$$

By Proposition 3.28,

$$\int_{t_1}^{t_1+T} [q(t)]_+ \mathrm{d}t > \frac{4}{T}.$$

Therefore,

$$\begin{aligned} \int_0^T p(t)\mathrm{d}t &= \int_{t_1}^{t_1+T} p(t)\mathrm{d}t \\ &\geq \int_{t_1}^{t_1+T} [q(t)]_+ \mathrm{d}t \\ &> \frac{4}{T}, \end{aligned}$$

which contradicts (4.18). So every nontrivial T-periodic solution to equation $u'' = -p(t)u + h(t)u^3$ is either positive or negative. □

Combining Theorem 4.4 and 4.5, we get

Theorem 4.6. *Let $p(t) > 0$, $h(t) > 0$, $\forall\, t \in \mathbb{R}$ and the condition* (4.18) *hold. Then equation* (4.1) *has exactly three T-periodic solutions (positive, negative, and trivial).*

Now we provide a corollary for equation (2.6) derived in Section 2.

Corollary 4.7. *Let $0 < T \leq 2\sqrt{\frac{L}{g}}$ and $d : \mathbb{R} \longrightarrow \mathbb{R}$ be a T-periodic function, which is continuous together with its first and second derivatives and satisfies*

$$\frac{g}{L} + d''(t) \geq 0 \ \ \textit{for } t \in \mathbb{R}, \quad d''(t) \not\equiv -\frac{g}{L}. \tag{4.26}$$

Then, equation (2.6) *has exactly three T-periodic solutions*

$$\theta_1(t) := 0,\ \theta_2(t) := \sqrt{6},\ \theta_3(t) := -\sqrt{6}. \tag{4.27}$$

Proof. It is clear that the functions θ_1, θ_2, and θ_3 given by (4.27) are solutions to equation (2.6).
Put

$$p(t) := \frac{g}{L} + d''(t), \quad h(t) := \frac{1}{6}\left(\frac{g}{L} + d''(t)\right) \quad \text{for } t \in \mathbb{R}.$$

Then hypothesis (4.26) yields (4.13) and, moreover,

$$\int_0^T p(t)\mathrm{d}t = \int_0^T \left(\frac{g}{L} + d''(t)\right) \mathrm{d}t = \frac{g}{L}T \leq \frac{4}{T}\ .$$

Consequently, the conclusion of the corollary follows from Theorem 4.5. □

We conclude this section by the following considerations. It follows from Section 4.1.3 that equation (2.5), i.e.,

$$\theta'' + \frac{g}{L}\theta - \frac{g}{6L}\theta^3 = 0,$$

has periodic non-constant solutions with unknown periods. By Theorem 4.5, these periods can be estimated from below.

Corollary 4.8. *Let θ be a non-constant T-periodic solution to equation* (2.5)*. Then θ changes its sign and*

$$T > 2\sqrt{\frac{L}{g}}. \tag{4.28}$$

Proof. Put

$$p(t) := \frac{g}{L}, \quad h(t) := \frac{g}{6L} \quad \text{for } t \in \mathbb{R}.$$

Then (4.13) holds and

$$\int_0^T p(t)\mathrm{d}t = \frac{g}{L}T. \tag{4.29}$$

It follows from Section 4.1.3 that the solution θ changes its sign. Moreover, Theorem 4.5 yields

$$\frac{g}{L}T > \frac{4}{T}$$

and, thus, estimate (4.28) is satisfied. □

Chapter 5

Conclusion

In this work, we discussed stability of singular points, existence and uniqueness of periodic solutions to nonautonomous Duffing equation, we drew phase portraits and level curves for different intervals and interpreted them. As for the analysis itself, it was first necessary to derive the equation of motion of a certain oscillator using Newton's second law. We then replaced nonlinear term using the Taylor polynomial and we thus obtained the required differential equation. The third chapter of this work was devoted to the theoretical part of the thesis which deals with the definitions of basic concepts from the theory of autonomous and nonautonomous systems differential equations. The concept of hamiltonian and its levels. The last chapter was devoted to the analysis of the periodic solutions to Duffing equation. The work discussed all possible cases that for the Duffing equation could occur. For each case, singular points were determined. Subsequently, the sets of orbits for each case were calculated and corresponding level curves were drawn and taken using Desmos Graphing calculator.

List of Figures

Bibliography

[1] DUFFING, G. Erzwungen Schwingungen bei veränderlicher Eigenfrequenz und ihre technisch Bedeutung, Vieweg Heft 41/42, Vieweg, Braunschweig, 1918.

[2] HABETS, P. and C. De COSTER. Two-point boundary value problem: lower and upper solutions. Mathematics in Science and Engineering, 205, Elsevier B.V., Amsterdam, 2006, ISBN 978-0-444-52200-9.

[3] HARTMAN, P. Ordinary differential equations, John Wiley & Sons, New York-London-Sydney, 1964.

[4] KOVACIC, I. (ed.) and M. J. BRENNAN (ed.). The Duffing equation. Nonlinear oscillators and their behaviour, John Wiley & Sons, Ltd., Publication, Hoboken, NJ, 2011, ISBN 978-0-470-71549-9.

[5] KOVACIC, I., L. CVETICANIN, M. ZUKOVIC and Z. RAKARIC. Jacobi eliptic functions: A review of nonlinear oscillatory application problems, Journal of Sound and Vibratiion 380 (2016), 1–36.

[6] LOMTATIDZE, A. and J. ŠREMR. On periodic solutions to second-order Duffing type equations, Nonlinear Anal. Real World Appl. 40 (2018), 215–242.

[7] PERKO, L. Differential equations and dynamical systems, Text in Applied Mathematics, 7, Springer-Verlag, New York, 2001, ISBN 0-387-95116-4.

[8] TORRES, P.J. Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselsbii fixed point theorem, J.Differential Equations 190 (2003), 643-662.

[9] MORIN, D. Introduction to classical Mechanics: With Problems and Solutions, Cambridge University Press, Cambridge, 2008, ISBN 978-0-521-87622-3.

[10] DOW, M. A. and R. VÝBORNÝ. Elementary proofs of Peanos Existence Theorem, J. Aust. Math. Soc. 15 (1973), 366–372.

[11] NAGLE, R., B. SAFF and SNIDER. Fundamentals of Differential Equations and Boundary Value Problems, University of South Florida, 2003, ISBN 978-0-321-97710-6.