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Department of Optics

Bachelor thesis

The Cheeger constant and applications

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Abstract

This work deals with a geometric variation problem, which is called the Cheeger problem. The first part of the thesis containes mainly general results, the mathematical origin of the problem and its applications in various mathematical and physical structures. Within the general concepts of variational problems, the development and description of some specific topological structures that play an important role in the exact formulation of the Cheeger problem is also discussed. The backbone of the first part is the derivation of new results within the Cheeger constant for sets on curved Riemannian manifolds with arbitrary dimensions and tubular neighborhoods of geodesics in higher dimensions. The results are confirmed by exact proofs, of which those relating to the Cheeger constant of tubular neighborhoods on manifolds with constant sectional curvature are inherently different and thus provide independent confirmation of our hypothesis. The second part of this thesis contains the application of our result in the field of string theory. The general foundations and fundamental results of bosonic string theory, topology and *p*-brane action are presented. By connecting these concepts with our result, we create a set of conjectures. It provides a potential insight into the topological nature of Hamilton's variational principle and its validity, which could consist in the existence of a minimizer within the structure of the Cheeger constant.

Keywords: topology, manifold, Riemannian geometry, Riemannian manifold, isoperimetric problem, Laplace-Beltrami operator, Cheeger problem, variational problems, minimization problem, Sobolev space, p-Laplacian, curvature, Jacobi field equation, Riemannian metric, Fermi coordinates, smooth geodesic, Cheeger constant, Cheeger set. curved tubes, tubular neighbourhoods of curves, Spherical shells, curved strips, unbounded tubes, vector field, string theory, p-branes, bosonic strings, spacetime, world volume, string action, Fermi coordinates for spacetime, Hamilton's variational principle.

Affidavit

I declare that I have created this bachelor thesis independently under the guidance of doc. Mgr. David Krejčiřík, Ph.D. DSc. and that I have listed in the bibliography all sources used in the processing of the work.

In Olomouc on June 9, 2020.

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Contents

Chapter 1

1 The emergence of topology and the concept of manifold

Manifold is one of the most fundamental concept in topology and differential geometry. Before we grapple with a modern manifold concept, it may help us to know some of the history behind the idea. The main reason for doing this is to see that some of the more seemingly abstract parts of the definition of manifold didn't come out of nowhere.

Concept of manifold is, in my opinion one of the greatest achievements of mathematics, because it combines many important areas in pure mathematics and it also can be used to construct and describe many important concepts in multiple areas of theoretical physics. It is wrong to consider manifolds only in topology, because their origin historically lies in abstract geometry [1]. But if we want to get deeper understanding and dug deep to the beautiful idea of manifold, we must look back to history and mention some great mathematicians, who are more than connect with the very foundations of manifolds.

Geometry by virtue of its visualization capability is easy to understand and comprehend. Topology is on the other hand, more of pure abstract concept hence many find it difficult to understand. Geometry as a part of mathematics, is concerned with concepts of shape, relative position of figures and size. We can assume that geometry is the study of properties of space with the important relation to size of shapes [2]. Many great scientists have made contributions in this field. But lot has changed since the times of mathematical giants like Euler or Lobachevsky. Only the most beautiful and elegant ideas survived.

Major and historic contribution came with the remarkable genius of Leonhard Euler. By solving the problem known as the Seven Bridges of Königsberg, he gave birth to the interesting discipline of discrete mathematics, which is called graph theory [3]. As a part of his proof, he also discovered the formula relating the number of vertices, edges and faces of a convex polyhedron. This theorem is now called Euler equation and it also includes planar graphs [3]. The generalization of a Euler formula is located at the very beginning of topology. Another breaking contribution was from Giovanni G. Saccheri. In 1733, he wrote his last publication before his death, it was about Non- Euclidean geometry [1]. This type of geometry considers spaces where Euclid's postulate about parallel lines fails. His work fell into oblivion for the following century. Fortunately, Saccheri's idea was rediscovered and more developer by Nikolai I. Lobachevsky, Bernhard Riemann and János Bolyai. Riemann was very influential mathematician in his times, and he made a major paradigm shift in mathematics with Non-Euclidean geometry, as it freed scientists from the wrong believe that Euclid's axioms about flat geometry, were the only possible path to make inner-consistent postulate based theory. In his honor, the subject developed by his work is called Riemannian geometry [4]. Basic research among Riemannian geometry led to mathematical formulation of general relativity. Where spacetime continuum is described as non-Euclidean pseudo-Riemannian manifold. This is one of the most intuitive applications of Riemannian geometry in theoretical physics. In mathematical point of view, Riemann found the right path to extend the differential geometry of surfaces to n dimensions. The basic concept in Riemannian geometry is Riemann curvature tensor [5]. If we consider the trivial case, like surfaces, curvature tensor can be reduced to number, negative, zero or positive. The interesting situation appears, when the curvature tensor is constant. This implies the existence of models of the non-Euclidean geometries, as so the non-zero values of curvature tensor.

It was the first time, when the term manifold appears. Because Riemann made the major contributions to Riemannian geometry as a new mathematical discipline. This discipline was in particular a subset of differential geometry that studies Riemannian manifolds. This is the key to theories like general relativity, because Riemannian manifold has a beautiful property of being a smooth manifold with Riemannian metric [5]. In the particular case of general relativity, we have a construct that is called pseudo metric, because we have a pseudo-Riemannian manifold as a model for smooth spacetime continuum. Riemannian metric can be understood as an inner product on the tangent space at each point which varies smoothly from point to point. As we can see, an elegant idea of Riemannian metric provides in particular local notions of length of curves, angle, volume and surface area. From those some global quantities can be derived by integrating local contributions. In general, it implies the existence of local parametrization of curves and the existence of volume forms on Riemannian manifolds [5]. Volume forms may appear as basic property of Riemannian manifolds, but it is not appropriate to take it as a guaranteed fact. Without proper definition of Riemannian metric and the corresponding manifold, which have been improved over time, we would be stuck in many proofs of some elegant theorems. In my opinion, one of the most important example of using the volume forms occurs in the proof of Brouwer fixed-point theorem. There are, of course, many variants of the proof for this theorem. But none of them gives a direct generalization of the mentioned theorem to nonempty, smooth, orientable and compact manifolds. The essence of the evidence depends on the elementary theorem of analysis, the Stokes theorem. In particular, it is only the basic application of Stokes theorem on volume forms, which are the consequence of properties of the given manifold. As we can see, the properties of Riemannian manifolds with appropriate metric are essential and fundamental in many ways.

Riemannian geometry deals with a broad range of geometries, which can be categorized into two particular types of Non-Euclidean geometry, hyperbolical and spherical geometry. Also Euclidean geometry itself is included. In general, Non-Euclidean geometry describes elliptical and hyperbolic geometry. The main question for mathematicians who specialize in differential geometry was about the essential difference between these type of geometries. It shows up, that the main difference lies in the concept of parallel lines [4]. Again, the concept was more than two thousand years old, however it creates a crucial path for extending geometries. Euclid's fifth postulate about parallel lines which states, that on a two dimensional plane, for any given line and a point, which is not on the line, there exist exactly one line that intersects the mentioned point and does not intersect the original line [2]. For example, in hyperbolic geometry, there exist infinitely many lines that goes through the given point in plane, and not intersecting the original line. While in elliptic geometry, any line that goes through the original point intersects the line. Failure of Euclid's fifth postulate is the key for understanding, why the Euclidean geometry is only a particular subset of generalized Riemannian geometry.

However, it is not the only path, how to describe the fundamental differences of geometries. Another way to point out the basic deviations among various geometric images of Riemannian geometry is to describe curvature of lines in two dimensional plane. This is a typical Euclidean approach to understand how two straight lines behave in relation to the third straight line. Historically, it is an original Euclid's approach, which he used in ancient times to formulate his famous postulates of flat geometry [2]. Consider two straight lines extended in two dimensional plane that are both perpendicular to the third line. In Euclidean geometry, where the space is completely flat, the lines remain at constant distance from each other. In other words, we call them parallel. But when space is not flat and its behavior is described by hyperbolical geometry, these two lines curve away from each other. Distance between them is increasing as one moves further from the points of intersection with the common perpendicular line. In elliptic geometry the lines eventually intersect each other. Behavior of these lines intuitively implies existence of important quantity, which is called curvature. Riemann and Lobachevsky did not think in terms such as curvature. This changed when Carl F. Gauss got involved in the debate. Thanks to Gauss, the concept of curvature has its soul. Everything that was discovered and developer during the times of Riemann, Lobachevsky and Gauss gave a brilliant foundation for the formulation of topology.

Now we appreciate that we made a journey through historical development of basic ideas of Euclidean and Non-Euclidean geometry, because this will help us to understand the following topics. As we saw earlier, Euler was first one to publish a paper on topology and gave rise to graph theory as a byproduct. Euler's solution of seven bridge problem shows interesting properties. One of the most remarkable aspect of his result was independence on the lengths of the bridges. In general, the concept of distance played no role in the core of Euler's proof [3]. This is how a new intuition on geometry emerged and topology was born. It turns out, that the very nature of many geometric problems does not depend on the spatial information about the space itself, the essence is quite opposite. Everything that we need to know depends on the general characteristics of space. Specifically, on topology.

Finally, we have overcome the necessary obstacle in the form of different geometries and got to the desired topology. I am convinced that another great scientist in the history of mathematics experienced similar feelings. Of course we are talking about the great Jules H. Poincaré. He was one the few mathematicians who had a complex interdisciplinary overview in pure mathematics, theoretical physics and philosophy of science. He was responsible for formulating many of the fundamental discoveries that covered theories of deterministic systems as well as topology [1]. But most important to us is the Poincaré conjecture. Poincaré was first who focused only on the intrinsic properties of space. We can also give him primacy in a use of the term manifold to describe properties of topological spaces. The essence of his hypothesis concerns the classification of manifolds. One of the most trivial cases are two dimensional manifolds. These types of manifolds were well understood. But Poincaré went further and proposed that every closed, simply connected 3-manifold is homeomorphic to the 3-sphere [6]. In other words, 3-manifolds are the same objects as 3-spheres. The term 3-sphere describes the boundary of unit ball in four dimensional space. In language of theoretical physics, Poincaré hypothesis has gained in importance due to the development of cosmology. Within the cosmological models, 3-manifolds can be considered as a topological description of universe. This description of universe as a 3-manifold, can form the background of a consistent cosmological model, which was also one of the reasons, why the proof of the Poincaré hypothesis was becoming more important.

The nature of Poincaré hypothesis is greatly reflected in its proof. While the best topologists were wondering about proof of three dimensional case, the generalized Poincaré conjecture had already been proven [7]. So for some reason, the formulation of proof of three dimensional case, was diametrically more difficult compared to the generalized version. Which is very interesting, because our natural intuition says, that the classification of arbitrary dimensional manifolds will be much more complicated. However, not everything is as obvious as it appears. Especially in topology or in any part of mathematics and theoretical physics that is somehow connected with topology. After nearly a century of effort by top mathematicians, the proof emerged. Solution of Poincaré hypothesis came from the hands of Grigori Perelman [8]. In my opinion, this brilliant mathematician is the embodiment of the essence of topology itself and can just be considered as one of the best mathematician of our millennium. Grigori Perelman posted the first of a series of three eprints on the internet platform outlining a solution of the Poincaré conjecture. Now let's try to outline the main principle of Perelman's proof. The core of the proof is based on the knowledge Riemann has come up with. In the most primitive context, the main strategy is based on the precise definition of Riemannian manifold. As we mentioned previously in the historical overview, Riemannian manifold has a beautiful property of being a smooth manifold with metric. This metric is called Riemannian metric. We know that the metric varies from point to point smoothly. At the simplest approximation, it is precisely this property of Riemannian metric that the proof depends on. We are actually looking for homeomorphism from closed, simply connected 3-manifold to the 3-sphere. In other words, we are trying to find out to what extent the 3-manifold is similar to 3-sphere [6]. Now, the Riemannian metric comes in the front line. When we put the metric on the unknown closed, simply connected 3-manifold, the desired homeomorphism can be replaced by particular improvement of the metric. The essence of homeomorphism reformulation lies in the basics of Ricci flow on manifolds [8]. Ricci flow was first introduced by the great topologist Richard Hamilton. Grigori Perelman based his solution on the surgeries that can be carried out on manifolds by Ricci flow. The beautiful mechanism of the flow tells us how the metric can be improved as the time of performed surgeries on manifold increases. The problem is that at the beginning we know almost nothing about the manifold. But as time of performed surgeries increases the manifold becomes easier to understand. So now we have relatively straightforward mechanism to improve the Riemannian metric of the underlying 3-manifold. However, the result that we must achieve is also clear. The Riemannian metric must be improved until the curvature of the 3-manifold is constant. If it is possible, then we have the direct homeomorphism between the 3-manifold and 3-sphere. This is a radically simplified insight to the highly elegant structure of the Perelman's creation [8].

From the historical point of view, the question about Poincaré hypothesis turned out to be extraordinarily difficult. But when we get more into the Perelman's proof, we find that the proof he published does not only verify Poincaré's hypothesis, but opens the door to a completely new geometry. To sum up, it can be stated that Perelman only shook hand with

Poincaré and went on. Proof of Poincaré hypothesis was the culmination of a long era and an entirely new part of modern topology and geometry was opened. The elegance of the solution lies not only in abstract proof but also in its applications, which can be found, for example in the formulation of particular cosmological models, but also in string theory [9]. It is the flourishing of all scientific endeavors in the field of modern topology. Thus, we have completely summarized the brief history of topology. In my point of view, it was necessary to pave the path through the historical perspective of mathematical evolution of geometry and topology. Because the topic of this work can only be explored more deeply thanks to the above-mentioned findings.

The mathematical side of our topic is not all we want to look at. Of course the specific applications of the given mathematical apparatus and results in theoretical physics are important. The scope of our results falls into the currently developing area in advanced string theory. Specifically, we are dealing with applications in topological and geometrical essence of string theory. Therefore, it makes sense to give an introduction to the topological string theory and string theory in general. However, let us leave the summary of the fundamental principles of string theory and topological string theory up to the second part of this work, where we will introduce the applications of our results of the specific Cheeger problem.

2 Formulation of the Cheeger problem and its beginnings

The purpose of this part is to provide some motivations to the reader. We shall overview the important theorems that are closely related to the Cheeger problem. This summary also includes a historical overview of the findings that led to the precise formulation of the related problems. The topic we are dealing with in this work belongs to the class of variational problems with topological background. Formulation of these problems is generally based on the variation, that is defined over some class of closed sets in the specific region [10]. This region is generally defined on the Riemannian manifold that has certain properties. The historical motivation of the Cheeger problem is based on the isoperimetric problem [11]. Especially on its solutions and on the inequalities, that are called isoperimetric. In general, isoperimetric problems can be very simple, but also very complex. Many isoperimetric problems in two dimensional cases are to some extent solvable, but if we go to the higher dimensional manifolds with specific properties, things can quickly become very complicated [12]. Isoperimetric problems have a long history that goes back to antiquity, where the foundation stones were laid to build today's modern theory that describes the essence, which is hidden behind isoperimetric problems in the speech of modern mathematical objects.

Class of variational problems that is related to the Cheeger problem is wide. Now, we will restrict it to a few variational problems, the essence of which is most important for properly defining the Cheeger problem. However, this restriction will not bet the main approach we choose to define the formal structure of the Cheeger problem. We will mention the reasons and motivations in the next section.

2.1 Introduction and some motivations

Normally, the introduction to the Cheeger variational problem is conceived in terms of inequalities, which provide estimates of the smallest eigenvalue of the Laplace operator [10]. Using the above estimates, the so-called Cheeger constant is defined. The given ratio, which represents the constant, is a limit case, within the p-Laplace operator applications [13]. The associated eigenfunction of the smallest eigenvalue converges to a function of a specific set that clearly minimizes the respective ratio. Then it is possible to move to the formulation of particular properties of a given constant, its geometrical and topological consequences, generalizations and applications.

However, in my point of view, this approach is technically demanding and requires knowledge of structures that are closely related to the issue, but without knowing how the variational problem actually looks like. The mentioned variational problem directly stems from geometry itself, whose perception may be more direct than the perception of the algebraic structures hidden behind the geometric arrangement. Therefore, let us now provide a formal scheme of the Cheeger problem.

Definition 1 ([13]). Let $\Omega \subset \mathbb{R}^d$ be an open bounded set, with $d \geq 2$. Given a Borel set $S \subset \mathbb{R}^d$, which varies over all non-empty, bounded and smooth subdomains of Ω. We denote $|S|$ its d-dimensional Lebesgue measure of S (the volume of S). Next, we use the notation $|\partial S|$ that in variational way represents the $(d-1)$-dimensional Hausdorff measure of the smooth boundary ∂S. We assume that ∂S is smooth or Lipschitz to the whole class of Borel sets. Then we can define the *Cheeger constant* of Ω as

$$h(\Omega) := inf \left\{ \frac{|\partial S|}{|S|} : S \subset \Omega, |S| > 0 \right\}. \tag{1}$$

Any set $A \subset \Omega$, which minimizes the ratio $\frac{|\partial S|}{|S|}$ is called a *Cheeger set* of the region Ω. Next, we call the relation $Q(S) = \frac{|\partial S|}{|S|}$ the *Cheeger quotient* of S. Then if the set A is a minimizer of (1), then $Q(A) = h(\Omega)$. We also call the *Cheeger problem* any situation, where we are concerned in a direct computation, estimation of (1), or characterization of the Cheeger set of (1).

2.2 The smallest eigenvalue of the Laplace-Beltrami operator

The variational problem that we are investigating in our work first appeared in isoperimetric-type inequality that was first proved by Jeff Cheeger in [14]. This isoperimetric inequality was defined on the n-dimensional, compact Riemannian manifold without boundary. Cheerer's proof focused on the estimation of the smallest eigenvalue of p-Laplace operator. This fact implies that the Cheeger problem must be related to the p-Laplacian. The most important partial differential equation of second order in the history of mathematics is the Laplace equation. This is the best prototype for linear elliptic equations. This shape of the Laplace equation is well known. But that's not all. It has less well-known counterpart which is called *p*-Laplace equation. This equation can be represented by *p*-Laplacian. J. Cheeger noticed in his proof that as a consequence, one can obtain the Poincaré inequality, which has a certain property [14]. It has an optimal constant that is uniformly bounded from below by a geometric constant. This is the mentioned interconnection between isoperimetric problems and *p*-Laplacian.

Let M bet he n-dimensional, compact Riemannian manifold without boundary. Let $\lambda_2(M)$ be the least non-zero eigenvalue of the Laplace operator on the manifold M. Then J. Cheeger proved the following inequality

$$\lambda_2(M) \geq \inf_{S \subset\subset M} \frac{P(S)^2}{4min\{V(S), V(M \setminus S)\}}, \tag{2}$$

where $V(S) = |S|$ and $P(S) = |\partial S|$ denote the volume and perimeter of S. This is the wording of the theorem that J. Cheeger proved in his paper [14]. The result is so significant in topology and variational geometry that the mechanism behind this important fact is worth mentioning.

The spectrum of the Laplace-Beltrami operator on Riemannian manifold is very significant in the field of differential geometry and topology [15]. The purpose of Cheerer's paper was to give a lower bound for the smallest eigenvalue of the Laplace operator on M. As we have mentioned before, the bound is defined as a constant in the isoperimetric inequality (2). In other words, it is a geometric invariant. For the sake of fairness, let us note that the result (2), holds also for compact Riemannian manifolds with boundary, not only for Riemannian manifolds without boundary. In describing the mechanism behind proof of the inequality for the smallest eigenvalue of the p-Laplacian, a specific isoperimetric constant rises to the surface. Let us lay the foundations for the definition of this constant.

Definition 2 ([14]).

a) Let M be a compact Riemannian manifold, $\partial M \neq \emptyset$. Let's have a constant defined as $h(M) = \inf_{S \subset M} \frac{P(S)}{\min V(M_i)}$, where $P(S)$ denotes $(n-1)$-dimensional Hausdorff measure of the boundary of S. Simply put, $P(S)$ is $(n-1)$-dimensional area. $V(M_i)$ is the n-dimensional Lebesgue measure of M_i. So $V(M_i)$ denotes volume, and the infimum is taken over all compact $(n-1)$-dimensional submanifolds S. This provides the dividing of manifold M into submanifolds with boundary M_1, M_2. Then the whole manifold M is given as unification of these two submanifolds, so $M = M_1 \cup M_2$. Also $\partial M_i = S$.

b) If $\partial M \neq \emptyset$, we can present the following equality

$$h(M) = \inf_{S} \frac{P(S)}{V(M_1)},$$

where $S \cap \partial M = \emptyset$, and there is a submanifold with boundary M_1 such that $S = \partial M_1$. It is assumed that ∂M, M_1, M_2, S does not have to be connected.

Theorem 1 ([14]). *With the validity of Definition 1, the following estimate applies to the smallest eigenvalue of the Laplace operator on the previous mentioned manifold* $\lambda \geq \frac{1}{4}h(M)^2$.

Proof of Theorem 1 ([14])

The proof is generally based on the gradient method that is used in conjunction with the integration of functions over a particular manifold. These functions correspond to the eigenvalues of the Laplace operator on the underlying manifold. Critical points of this function play an important role. The proof is quite technical on the whole, but it can be intuitively understood.

Assume that the manifold M is not orientable. Consider a covering map. The covering map is a continuous function from one topological space to another such that each point in the first topological space has an open neighbourhood evenly covered by the covering function. The first topological space in the covering map is called a covering space. The topological space which it is mapped on is called based space of the covering projection. The important thing is that the nature of this definition implies that every covering map is a local homeomorphism. Topic of covering spaces plays very important role in general Riemannian geometry. One of the things that is used in the proof is to look at the orientable cover of the manifold. Particularly, we are looking on the n-fold orientable cover of the manifold M. So, denote the first topological space in the covering mapping as Σ. And the second topological space denote as ϕ. For every $\varphi \in$ ϕ, the structure that is called a fiber over σ is a subset of Σ. The fibers have unique property and that is their homeomorphism on every component of the topological space ϕ. Requiring property of the components in ϕ is their connectivity. If ϕ is connected topological space, then there exist a discrete space Ξ such that for every element $\varphi \in$ ϕ the fiber over this element φ is homeomorphic to the space Ξ. The consequences of homeomorphism is logical because the fibers are homeomorphic on every connected component the space ϕ. The connectivity is a source for homeomorphism of the fiber over element φ to the space Ξ. This implies, that if every fiber has n elements, then the covering structure is technically called n-fold covering.

So in our case we are interested in what is the n-fold covering, if we generally assume that manifold M is non-orientable. M is a compact n-dimensional Riemannian manifold, and the infimum in (1) that is taken over all submanifolds S divides M into two submanifolds with boundary, therefore every fiber that is homeomorphic to the topological space Ξ has two elements. Consequently, the covering structure is 2-fold cover. Moreover, it is 2-fold orientable cover. Now, we need to assign a function to the proper eigenvalue of Laplace operator on M. Let f be the eigenfunction corresponding the the least eigenvalue λ. We make the assumption that the eigenfunction f has isolated critical points. It's worth noting the fact that if the essential assumption in definition 1 holds true $\partial M \neq \emptyset$, then we can assume that $f * \frac{df}{\partial M} = 0$. However, the assumption may not be so narrow. We can make a generalization. Because for any region G, such that $G * \frac{dG}{\partial M} = 0$, we can write the integral equality for λ as follows

$$\lambda = \frac{\int_G \Delta^2 f \cdot f}{\int_G f^2} = \frac{\int_G \|grad(f)\|^2}{\int_G f^2} = \frac{\left(\int_G \|grad(f)\|^2\right)}{\left(\int_G f^2\right)^2}\left(\int_G f^2\right)$$
$$\geq \frac{\left(\int_G |f| \cdot \|grad(f)\|\right)^2}{\left(\int_G f^2\right)^2} = \frac{1}{4}\frac{\left(\int_G \|grad(f^2)\|\right)^2}{\left(\int_G f^2\right)^2}. \qquad (3)$$

Now come the critical points of function f. Assume that the critical value of f is non trivial. If the critical value of f is zero, then the arguments turns into a triviall case. The infimum that is contained in definition relation of constant h is taken over all compact $(n-1)$-dimensional submanifolds S. This submanifolds divides the underlying manifold M into two previously mentioned submanifolds with boundary. Therefore we have to introduce a manifold that divides the Riemannian manifold M into two submanifolds, whose unification is again manifold M. So let's define a manifold $Q = \{x|f(x) = 0\}$. Manifold Q divides M into two n-dimensional submnaifolds with boundary $M_1 = \{x|f(x) \geq 0\}$ and $M_2 = \{x|f(x) \leq 0\}$. We can use the information that $\lambda \neq 0$, this fact implies that the eigenfunction f is nonconstant, hence it must take on positive and negative values. Let h, h_1 and h_2 be the isoperimetric constants corresponding to manifolds M, M_1 and M_2. For technical correctness, let us say that the isoperimetric constant h defined by the relation (1) is generally called *Cheerer's constant* in honor of its creator. As we shall see now, monotonicity of the Cheerer's constant is very important property. The monotonicity interconnection between these three constants depends directly on the n-dimensional Lebesgue measure of the Riemannian manifold M. In other words, the monotonicity logically depends on the volume of each manifold M_i. If $V(M_1) \leq V(M_2)$, then $h_1 \geq h$. Here we can see that there will be no problem in the formulation of the proof of the estimation for the submanifold M_1. This argument can be further extended to the estimation for any manifold with boundary. The regions of the submanifold M_1 lying between the critical levels of function f^2. If we focus on the nature of the critical levels, then we find that natural product structure for the critical levels of function f^2 is given by the level surfaces and the orthogonal trajectories. Level surfaces is a set of all real-valued roots of an equation in three variables. In other words, it is a surface, where the function acquires constant values for all its variables. Therefore, it is intuitive that the natural product is given as $L \times I$. This product has a product coordinates (x, t). We choose the local coordinates on L and put $t = f^2$. This parametrization allows us to introduce the elementary volume form in the mentioned coordinates as $dV = v_1(x,t)dx + v_2(x,t)dt$. It is possible to write the volume formula in this form because dt are orthogonal to the local coordinates dx_i on the level surface L. Since $t = f^2$, then if we consider the same integral as in the relation (3), but the difference is that in this case we do not integrate across region G, but we integrate over the whole submanifold M_1.

Let $V(t)$ be the volume of the set on M_1, where $f(x)^2 \geq t$. From the definition of this set we know, that the corresponding vector field which represents the volume of the mentioned set is continuous. Then it must be differentiable. This implies the following integral equality

$$\int_{M_1} \|grad(f^2)\| \cdot dv = \int_L \left(\int_0^\infty \|grad(f^2)\| \cdot v_1 \cdot v_2 \cdot dt \right) dx \tag{4}$$

Since $t = f^2$ and $v_1(x,t) = \left\| \frac{\partial}{\partial t} \right\|$, we can change the proper integration of v_2.

$$\int_L \left(\int_0^\infty v_2 \, dt \right) dx = \int_0^\infty \left(\int_L v_2 \, dx \right) dt \geq h_1 \int_0^\infty V(t) \, dt = -h_1 \int_0^\infty t \frac{dV(t)}{dt} \cdot dt \tag{5}$$

This relation (5) for integration of v_2 can provide us the exact formula for the volume

$$V(t) = V(M_1) - \int_0^t \left(\int_L v_1(x,t) \cdot v_2(x,t) \, dx \right) dt. \tag{6}$$

Formula for $V(t)$ is intuitively given by the difference between volume of the whole submanifold M_1 and the double integration in variable t and the integration over the level surface L of v_1 and v_2. Assuming the equality $t = f^2$ we get

$$-h_1 \int_0^\infty t \frac{dV(t)}{dt} \cdot dt = h_1 \int_0^\infty t \left(\int_L v_1(x,t) \cdot v_2(x,t) \, dx \right) dt = h_1 \int_{M_1} t \, dV. \tag{7}$$

Relation (7) implies the following estimate for the isoperimetric constant h_1

$$\frac{\left(\int_M \|grad(f^2)\| \right)^2}{\left(\int_{M_1} f^2 \right)^2} \geq {h_1}^2 \geq h^2. \tag{8}$$

By deepening the gradient formulas, the estimation (3s) for the least eigenvalue of Laplace operator λ and the above mentioned estimation (8) for the constant h, we obtain the previously mentioned inequality for the least non-zero eigenvalue of the Laplace-Beltrami operator on M. We have thus summarized the proof of the fundamental theorem which constitutes the cornerstone of the Cheeger problem and the important isoperimetric constant h.

2.3 Typical eigenvalue problem and crucial estimate of the Cheeger constant

J. Cheerer's findings give rise to all problems that fall within the field of variational analysis, especially geometrical problems, that are related to Laplace-Beltrami operator [16]. Therefore, let us mentioned a well-known result in the field of nonlinear partial differential equations. This problem is closely related to the estimation of the least non-zero eigenvalue of the Laplace operator.

Let us consider the following. Suppose that $p \in (1, \infty)$ and a region $\Omega \subset \mathbb{R}^d$ is simply connected and bounded. The region Ω has sufficiently smooth boundary. The following eigenvalue problem [10]

$$\begin{gathered} \Delta_p u + \lambda |u|^{p-2} u = 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega. \end{gathered} \tag{9}$$

This problem has a weak solution in Sobolev space $W_0^{1,p}(\Omega)$, which is in general a vector space of functions. Where the norm of these functions is a mixture of L^p norms of the functions and its derivatives up to a given order [17]. Especially these derivatives, makes the space complete. In other words, Sobolev space is constructed as a space of functions as the elements of vectors space, which possesses arbitrary many derivatives, which makes the space Banach, for some specific domain. The eigenvalue that corresponds to the solution is simple. Operator Δ_p is called a p-Laplace operator. Which is a generalization of the known Laplace operator. Δ_p that acts on arbitrary function f a can be written by the divergence operator as $\Delta_p f := div(|\nabla f|^{p-2} \nabla f)$. Where $|\nabla f|^{p-2}$ is defined by the generalized relation for the basic Laplacian $|\nabla f|^{p-2} = \left[\left(\frac{\partial f}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2\right]^{\frac{p-2}{2}}$. If we put $p = 2$, we obtain the usual relation for Laplace operator. In general, the solutions of differential equations that involves p-Laplacian do not have derivatives of the second order [18]. This implies that the solutions to these equations must be understood as weak solutions. In other words, in general, the weak solution is a function, where we cannot rely on the existence of all derivatives, but the mentioned function behaves like appropriate solution of the differential equation. A general differential equation may have solutions which are not differentiable. Then comes the weak formulation that allows us to find some convenient solutions.

We have showed how the estimation for the least non-zero eigenvalue of the Laplace operator looks like. Let us now generalize this concept and give an exact definition of the eigenvalue of generalized p-Laplace operator. Generalization is done, because of the connection of the p-Laplacian problem with the estimation of the Cheeger constant by the least eigenvalue, as we have mentioned in the introduction to the variational problem of Laplace operator and Cheeger problem. A closer analysis of this connection will be mentioned in a while. The essence of the definition lies in the formulation of the minimization problem [10]

$$\lambda_p(\Omega) := \min_{0 \neq f \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla f|^p dx}{\int_\Omega |f|^p dx}, \tag{10}$$

where u is a minimizer. The eigenfunction is scaled to $\|u\|_\infty = 1$. $\lambda_p(\Omega)$ is composed from the minimum operator and very interesting part, which is called Rayleigh quotient. This quotient is a functional. Therefore, (10) is a minimization of functional, which is used for computation of the exact values of all eigenvalues. The concept of Rayleigh quotient may be better known for the Rayleigh quotient approximation theorem. The main idea is to approximate the specific eigenvalue by the approximation of the corresponding eigenvector. This is mediated by the iteration of the given coefficient [19].

We have mentioned one of the many ways in which we formulate the exact definition of $\lambda_p(\Omega)$. Another method is to introduce the construct of the energy functional [10]. The reason for choosing this formulation is the nature of the eigenfunction itself in the energy functional structure.

Let the original assumptions apply, then the energy representation can be defined by the following relation

$$J_p(f) = \int_\Omega |f|^p \, dx \text{ on the set } H := \{f \in W_0^{1,p}(\Omega) \,|\, \|f\|_{L^p(\Omega)} = 1\}. \tag{11}$$

In the sense of the energy functional, the eigenfunction can be characterized as a minimizer of (11). For further specification, the eigenfunction is a critical point of $J_p(f)$ [10]. Equivalent expression of (10) can be written by infimum operator and the $L^p(\Omega)$ norm of f as

$$\lambda_p(\Omega) := \inf_{f \in W_0^{1,p}(\Omega)} \frac{\|\nabla f\|_p^p}{\|f\|_p^p}, \tag{12}$$

with u as a minimizer from the eigenvalue problem (9) and $f \neq 0$. There is no simple algorithm for obtaining $\lambda_p(\Omega)$ explicitly [13]. If we choose a specific function f as a function that satisfies the Rayleigh quotients (10) and the relation (12) with minimizer u, then we get a upper bound of $\lambda_p(\Omega)$. On the other hand, obtaining the lower bound is not nearly as easy.

From (12) it is possible to show that the following inequality holds for all $1 \leq p < \infty$,

$$\lambda_p(\Omega) \geq \frac{h(\Omega)^p}{p^p}, \tag{13}$$

where $h(\Omega)$ is defined by (1). The proof of (13) is almost straightforward, rather it falls into the category of intuitive proofs, since it evokes the geometric nature of the constant $h(\Omega)$ [13].

Let us choose the function f from the Sobolev space $W_0^{1,p}(\Omega)$ with a positive Sobolev norm. Set $q = \frac{p}{p-1}$. Now, the situation offers a clear use of the elegant Holder's inequality as we are working in the measure space and the chosen function $f \in W_0^{1,p}(\Omega)$ is measurable. It is a fundamental integral inequality that uses the essential properties of L^p spaces. Then we obtain the following relation

$$\frac{\int |\nabla u|^p}{\int |u|^p} \geq \frac{\left(\int |u|^{p-1} |\nabla u|^p\right)^p}{\left(\int |u|^p\right)^p} = \frac{\left(\int |\nabla |u|^p|\right)^p}{p^p \left(\int |u|^p\right)^p}. \tag{14}$$

Put. $f = |u|^p$ and consider a coarea formula, which states that the integration over region $\Omega \subset \mathbb{R}^d, d \geq 1$ of the Lipschitz function u can be expressed as a improper integration over $\mathbb{R}$ of $(d-1)$-dimensional Hausdorff measure of the inverse u^{-1}, we get

$$\int |\nabla f| \geq h(\Omega) \cdot \int_0^\infty |\{f > t\}| dt = h(\Omega) \cdot \int f. \tag{15}$$

From (14) we can obtain the important inequality [13]

$$\frac{\left(\int |\nabla |u|^p|\right)}{\left(\int |u|^p\right)} = \frac{\int |\nabla f|}{\int f} \geq h(\Omega), \tag{16}$$

which implies that the inequality (15) and (16) provides the desired lower bound for $\lambda_p(\Omega)$.

The proof can also be formulated in another way, which consists not only in the use of coarea formula, but also in the use of Cavalier's principle, which demonstrates greater elegance and a direct reference to the geometric nature of the situation [10]. We could see the specification of the proven inequality (13), where we give a summary of the original result in

J. Cheerer's paper [14]. There, the result treated also manifolds without boundary and it was proved for $p = 2$. The generalization was done in [20]. The following definition of the Cheeger constant differs from the definition (1), because it is derived from the proven inequality (13) by means of the convergence of the eigenvalue of the p-Laplacian.

Remark 1 ([13]). Let $p \to 1$ in (13), then the left hand side of the inequality (13) converges to $\lambda_1(\Omega)$ and the right hand side tends to the $h(\Omega)$. Then the following equality applies

$$h(\Omega) = \lambda_1(\Omega), \tag{17}$$

suggesting that inequality (17) becomes sharp as $p \to 1$.

Verification of Remark 1, is not problematic [13]. In general, it is necessary to prove that the inequality $\lambda_1(\Omega) \leq h(\Omega)$ holds as the opposite inequality directly follows from (13). Then let f in (15) be a function from a Sobolev space $W_0^{1,1}(\Omega)$, so $p = 1$ and f is selected to approximate the suitable characteristic function of a set $S \subset \Omega$. The approximation should be designed so that the relation $\frac{|\partial S|}{|S|} \approx h(\Omega)$ is met. Since this is merely an approximation of (1), without prejudice to generality, it can be assumed that the set S is relatively compact in Ω and that the perimeter ∂S is smooth. These assumptions provide a specific definition to the function f, which can be used for approximation of (1). So, f can be considered as a regularization of the function χ_f in a such way that $0 \leq f \leq 1$, $|S| \approx \int f$ and perimeter can be represented by integration over gradient as $|\partial S| \approx \int |\nabla f|$. Then we get

$$\int |\nabla f| \approx \frac{|\partial S|}{|S|} \int f \approx h(\Omega) \int f.$$

By construction this approximation, inequality the Definition 2. can be considered as proven.

Let us give an example for a specific idea, which is based on inequality (13). If $\Omega := B_a$, where B_a is a d-dimensional ball of radius a, then $h(B_a) = \frac{d}{a}$ and the *Cheeger set* can be directly identified with B_a. This gives a lower bound for the eigenvalue of p-Laplacian as $\lambda_p(B_a) \geq \left(\frac{d}{a_p}\right)^p$. Note that when we take a p-th rooth of the estimation of λ_p for balls, the right hand side of this relation goes to zero as $p \to \infty$ and the left hand side converges to $\frac{1}{a}$ [21].

3 General results on the Cheeger problem and Cheeger constant

After recalling some definitions, basic facts, important inequalities and the origin of the Cheeger problem, we shall present some general results on the Cheeger problem in $\mathbb{R}^d$. However, this is not the entire content of the next section, as there exist a number of unsolved hypotheses that provide an extension of the Cheeger problem to the Riemannian manifolds with constant sectional curvature. Within this extension we will be interested in tubular neighbourhoods on the respective manifolds. We will also provide some brief insight into the possible solutions of the mentioned hypotheses.

The Cheeger problem is sometimes preceded by technical definitions of perimeter as a Borel set or relative perimeter, which are based on the characteristic function of a given Borel set in. The purpose of redefinition is to identify the concept of perimeter with total variation of the distributional gradient of the characteristic function of the given Borel set. All theorems contained in the above mentioned analysis of the perimeter redefinition are purely technical. It is therefore not necessary to list them here. All the findings regarding these facts can be found in [13]. Let us focus directly on the general properties of the Cheeger constant and the Cheeger sets inside the region Ω, which are valid for any dimension $d \geq 2$ (one can see, [10, 22]).

Theorem 2 ([13]). *Let $\Omega, \tilde{\Omega} \subset \mathbb{R}^d$ be open and bounded sets. Then the following properties hold.*

1) *If $\Omega \subset \tilde{\Omega}$ then $h(\Omega) \geq h(\tilde{\Omega})$.*
2) *For arbitrary $\beta > 0$ and any isometry $\nu : \mathbb{R}^d \to \mathbb{R}^d$, one obtain $h(\beta\, \nu(\Omega)) = \frac{1}{\beta} h(\Omega)$.*
3) *The possibly non-unique Cheeger set is allowed to exist $A \subset \Omega$, such that $Q(A) = h(\Omega)$.*
4) *If A is Cheeger set in the region Ω then $|A| \geq \omega_d \left(\frac{d}{h(\Omega)}\right)^d$.*
5) *If A and B forms the Cheeger set in Ω, then $A \cup B$ and $A \cap B$ forms also the Cheeger set in the region Ω (the non-emptiness* of *A and B must be satisfied).*

Let us show briefly the proof of Theorem 2 [13]. The property 1) (directly follows from the Definition 1. of the Cheeger constant, the isometry and the property 2)) is no longer so straightforward, because it includes the properties of two Borel sets of a finite perimeter with a given isometry. Suppose the sets A and B are Borelian and have finite perimeter. Next, let's have a constant $\beta > 0$ and a isometry $\nu : \mathbb{R}^d \to \mathbb{R}^d$. Based on these assumptions we get

$$|\partial(\beta\, \nu(A))| = \beta^{d-1} |\partial A|.$$

If we combine this equality with scaling $|\beta\Omega| = \beta^d|\Omega|$, we get 2). Detailed proof of the remaining properties can be found in [22, 23]. It is worth mentioning that the properties 4) and 5) allow finding the minimal Cheeger sets in the region Ω and a unique maximal Cheeger set. I fit is possible to find the maximum Cheeger set in Ω, then its explicit construction can be done as the union of all minimal Cheeger sets of Ω.

An important and predicted property of the Cheeger constant is its continuity. The continuity can be divided according to the characteristics of the region Ω. Which we will show in the following theorem.

Theorem 3 ([13]). *Let $\Omega, \Omega_i \subset \mathbb{R}^d$ are open nonempty and bounded sets for all indices $i \in \mathbb{N}$. Consider χ_{Ω_i} to be a characteristic function of Ω_i. If $\chi_{\Omega_i} \to \chi_\Omega$ in L^1 space, then the following limit holds*

$$\liminf_{i\to\infty} h(\Omega_i) \geq h(\Omega) \tag{18}$$

If the assumptions are amended and $\Omega, \Omega_i \subset \mathbb{R}^d$ are sets of finite perimeter and $|\partial\Omega_i| \to |\partial\Omega|$ as $i \to \infty$, *then*

$$\lim_{i\to\infty} h(\Omega_i) \geq h(\Omega) \tag{19}$$

The importance of this theorem is considerable, so let us provide complete proof. Its essence lies in the choice of the Cheeger set in the sequence Ω_i. Then it is sufficient to verify the assumptions of the Theorem 3, [24].

Let A_i be a Cheeger set in Ω_i. We know, that its existence (possible not uniqueness) is guaranteed by the Proposition 3 in Theorem 2. We assume that $\liminf_{i\to\infty}|\partial A_i| < \infty$. We also automatically expect the convergence $\chi_{A_i} \to \chi_A$ for $i \to \infty$ in L^1 space up to the subsequence for some Borel set A with $|A| > 0$. The sequence A_i must be a subset of Ω_i and $\chi_{\Omega_i} \to \chi_\Omega$ in L^1 for $i \to \infty$. From which we can deduce that $A \subset \Omega$. It is also true that $|A_i| \to |A|$, then we can estimate the Cheeger constant its upper bound as

$$h(\Omega) \leq \frac{|\partial A|}{|A|} \leq \liminf_{i\to\infty} \frac{|\partial A_i|}{|A_i|}.$$

This inequality proves (18). To prove the second limit relation (19), it is necessary to consider that $|\partial\Omega_i| \to |\partial\Omega|$ as $i \to \infty$. Let a Borel set A forms the Cheeger set in Ω. Based on this assumption, the sequence A_i can be directly constructed as $A_i = \Omega_i \cap A$. Now, if we a limit $i \to$

∞, then it is obvious that $A_i \to A$ and $A \cup \Omega_i \to \Omega$ in L^1 space. From which it is possible to estimate the limes inferior of a sequence of perimeters $|\partial A_i|$ from above, then the procedure is straightforward. However, we need to know how to estimate the sum of the perimeters of intersection and union of two Cheeger sets in $\mathbb{R}^d$.

Given two Borel sets $A,B \subset \mathbb{R}^d$. Assume that A and B have finite primeter. Then we can provide an estimation by $|\partial(A \cup B)| + |\partial(A \cap B)| \le |\partial A| + |\partial B|$. Therefore, we are able to provide the estimation of the upper bound of the sequence $|\partial A_i|$ by

$$\limsup_{i\to\infty} |\partial A_i| \le |\partial A| + \limsup_{i\to\infty} |\partial \Omega_i| - \liminf_{i\to\infty} |\partial(A \cup \Omega_i)| \le |\partial A| + |\partial \Omega| - |\partial \Omega| = |\partial A|,$$

so the relation $\limsup_{i\to\infty} |\partial A_i| \le |\partial A|$ combined with the inequality (18) gives the desired relation (19) [24].

3.1 Regularity of the Cheeger set

From Theorem 2, specifically from property 3), we know that the existence of the (possibly non-unique) Cheeger set is guaranteed. We can now be interested in the shape of that set. Let $A \subset \Omega$ represents the Cheeger set, so $A = S$ in (1). Then $|A|$ represents a d-dimensional Lebesgue measure, i.e. volume of A. So, $|A|$ is a volume constraint. The Cheeger problem reflects the effort to find the subdomains of Ω which minimizes the surface are $|\partial S|$ in (1). Variational problems that fall into this category have already been formulated and studied in [22] and [25]. These papers have shown a specific property of the boundaries of the Cheeger sets and that is smoothness as the boundary $|\partial \Omega|$. The boundaries of the optimal domain A also smoothly touches $\partial \Omega$ and are analytic. The exception is a set of $(n-8)$-dimensional measure [25]. There is a direct relationship between the nature of the domain Ω and the corresponding optimal Cheeger set in Ω. Which is shown in the exmaple where we assume that the domain is C^1, then the Cheeger sets are globally of the class C^1.These properties reflect the behavior of the Cheeger constant towards the so-called „corners“ in a given domain. We know that if the ideal set A can be found, then it minimizes (1). On the basis of the existence of A and the aforementioned characteristics of the Cheeger set, it can be said that the following set $\partial A \cap \Omega$ is surface of constant mean curvature, which is directly equal to $h(\Omega)$. When the Cheeger set is globally of the class C^1, then in the spatial restriction to two dimensions means that the boundaries have only finitely many singular points. Clearly, in this case, the surface $\partial A \cap \Omega$ must consist of circular arcs [26]. Here we have provided a rational argument why the geometric mechanism behind the concept of the Cheeger constant (1) literally „avoids corners “in the given domains. This unique feature of minimizer in (1), which we introduce in terms of regularity, plays an important role in the computation of (1) especially in the case of planar domains [27]. Consequently, the construction and guessing of the shape of the Cheeger set become an almost straightforward matter. Which we will show explicitly on the solvable models in the section 8. The relevant property is also widely used within the characterization of the Cheeger sets for convex subsets of the plane [23]. Thus, guessing the Cheeger set proceeds as follows.

Once $h(\Omega)$ is determined, we can consider a ball of radius $(d-1)/h(\Omega)$, where $d \in \mathbb{N}$ and cover Ω with it. Then in $d=2$, as we have already mentioned in the previous paragraph, the ideal minimizer of (1) can be defined by $S := \bigcup_{x \in \Omega, dist(x,\partial\Omega)>(d-1)/h(\Omega)} B\left(x, \frac{d-1}{h(\Omega)}\right)$ [10]. The reason we mention this definition is for validity of this minimizer up to $d < 3$. Once $d \geq 3$ the mechanismus of how we are guessing the shape of the Cheeger set is no longer valid in general. There exist sets where if we sweep it from inside with balls of radius h, then the final set has mean curvature h near the corners (rounded corners since $\partial A \cap \Omega$ must consist of circular arcs in $d=2$) and $(d-1)/h$ near rounded edges [10].

3.2 Monotone dependence between $h(\Omega)$ and the region Ω.

Monotonicity can be directly deduced from the initial definition (1). The nature of this definition is variational, which implies the following inequality $h(\Omega_1) \geq h(\Omega_2)$ if $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ and $\Omega_1 \subset \Omega_2$ [10]. Considering the case of sharp inequality $h(\Omega_1) > h(\Omega_2)$ of the corresponding Cheeger constants, the strict inclusion of the regions $\Omega_1 \subset\neq \Omega_2$ is not immediately a source of mentioned inequality. As a concrete example, it is possible to analyze the Cheeger problem on square. For simplicity, let's start with the result from [21, 26]. Then if the region Ω corresponds to square of the side a, so $\Omega := S_a = (-a,a)^2$,straightforward calculations provides the explicit formula $h(S_a) = \frac{4-\pi}{(4-2\sqrt{\pi})a}$. Put $a=1$ and modify the square S_1 near one of its corners. Then the Cheeger set, whose shape is in this case ($d=2$) given by the mechanism of circular arcs, and the Cheeger constant are not affected by this modification. This example clearly confirms that the strict inclusion of the domains does not necessary implies sharp inequality of the corresponding Cheeger constants [26].

3.3 Convexity interconnection between Ω and the Cheeger set

First, let's specify convexity directly on the Cheeger set of convex domain Ω. So we will talk generally about the convexity of the minimizer (1) in the convex region Ω [28].

Subsequently we move on to restricting the generality to characterization of the Cheeger sets for convex subsets of the plane.

Let $A \subset \Omega$ forms the Cheeger set. Then it divides into two parts, where we are interested in their mean curvature. This will help us specify the spaces, where the convexity of the region Ω clearly determines the convexity of A. It is possible to divide the Cheeger set A into surfaces $\partial A \cap \Omega$ and $\partial A \cap \partial\Omega$. Both subsets have constant mean curvature. The first subset $\partial A \cap \Omega$ has constant mean curvature $h(\Omega)$ and $\partial A \cap \partial\Omega$ has identical mean curvature as $\partial\Omega$ [10]. Which implies a limit in terms of the upper limit for dimensions of a given space. So, for convex domain Ω the Cheeger set is also convex if $d = 2$. If $d \geq 3$ and Ω is convex, then ∂A has nonnegative mean curvature, which means that the perimeter of the Cheeger set is mean-convex. More detail on the general knowledge about the convexity relation between the region Ω and the Cheeger set A is given in [29].

3.4 Cheeger sets and convex subsets of the plane

Here we only briefly mention the most important facts and theorems concerning the Cheeger problem on convex sets in the plane. More detailed information can be found in [22]. Let's give a simple example to begin with. If the region Ω is rectangle or triangle, then its Cheeger set C_Ω can be directly obtained from Ω by roundign and smoothing all the corners [23]. We have already discussed this mechanism in detail in the section 3.1. It is also possible to come across the limits of this method, since its functionality is not guaranteed for all general polygons. However, there exist a class of polygons where the method of „rounding the corners", can be relied upon. These are the so-called Cheeger-regular polygons [23]. It is interesting that for convex sets Ω, where $C_\Omega = \Omega$, there is exist an explicit characteristization of those convex sets, which can be demonstrated by the following theorem.

Theorem 4 ([23]). *Let Ω be any convex set, $\tilde{\kappa}$ is the maximum value of its curvature. Then $C_\Omega = \Omega$ if and only if*

$$\tilde{\kappa}|\Omega| \leq |\partial\Omega|. \tag{20}$$

The deduction is straightforward, because if $\partial\Omega$ is not of class C^1, then $\tilde{\kappa}$ is infinite and we get the desired fact that $C_\Omega \neq \Omega$. The origin and proof of Theorem 4. can be registered in [30] and [31].

Let’s go back to the polygons that are Cheeger-regular. The method of „rounding the corners“ suggests that the Cheeger-regular is such polygon whose Cheeger set touches every side of Ω [23]. The following theorem quantifies this property and at the same time presents an explicit formula for computing the Cheeger constant of polygons under certain assumptions.

Theorem 5 ([23]). *A polygon Ω is Cheeger-regular if and only if*

$$|\Omega| - r_0|\partial\Omega| + {r_0}^2(T(\Omega) - \pi) \leq 0, \tag{21}$$

where $T(\Omega)$ is defined as $T(\Omega) := \sum_{i=1}^{n} tan(\alpha_i)$, which is the sum of the tangents of half inner angles α_i of general convex polygon, see [23]. *And r_0 is given by*

$$r_0 := \min_{1\leq i\leq n} \frac{l_i}{tan(\alpha_i) - tan(\alpha_{i-1})}.$$

The term l_i represents the distance of two adjacent vertices of the general convex polygon, see again [23]. *In that case, the area and perimeter of C_Ω are given by*

$$|\partial C_\Omega| = |\partial\Omega| - 2(T(\Omega) - \pi)r, \qquad |C_\Omega| = |\Omega| - (T(\Omega) - \pi)r^2 = r|\partial C_\Omega|,$$

where $r = 1/h(\Omega)$ represents the smallest root of equation $(T(\Omega) - \pi)r^2 - r|\partial\Omega| + |\Omega| = 0$. Then the Cheeger constant of Ω can be computed as

$$h(\Omega) = \frac{2(T(\Omega) - \pi)}{|\partial\Omega| - \sqrt{|\partial\Omega|^2 - 4(T(\Omega) - \pi)|\Omega|}} = \frac{|\partial\Omega| + \sqrt{|\partial\Omega|^2 - 4(T(\Omega) - \pi)|\Omega|}}{2|\Omega|}. \tag{22}$$

The formula (21) can be found in [23]. Thus we have a direct mechanism to find a solution to the Cheeger problem for Cheeger-regular polygons, which respects the conditions given by the Theorem 5. We will use this result later for demonstration of the solution to some simple models. The essence of the resulting formula (22) can be found already in [32].

Probably the most interesting and elegant feature that is typical for any convex polygons in Ω is the existence of a precise computational method for finding the Cheeger constant [23]. This direct algebraic algorithm has a finite number of steps that can be realized. This number does not exceed the number of sides of the polygon. The elegance of this algorithmic method also lies in its validity for Cheeger-irregular polygons. Which are polygons where their optimal Cheeger set does not touch all their sides. Opposite to Cheeger-regular polygons. The situation will be greatly simplified if the condition (21) holds, then we are done. On the other hand, if (21) no longer holds, the detailed procedure consists in defining a newer polygon and in the subsequent approximation to approach the conditions in Theorem 5. As we would expect from a constructive algorithm. Zoom in on this method into [23, Section 5]. The pitfalls of the algorithm are its unwillingness to generalize. It is known, that it can't be generalized for domains of any shape in arbitrary dimensional space [10]. However, it can be used to find an approximation for the Cheeger set of an arbitrary planar convex set Ω [23]. It is therefore a restriction to $d = 2$. The methodology is simple. Replace Ω by an appproximative polygon D that satisfies $D \subset \Omega \subset (1+\epsilon)D$ for $\epsilon > 0$ sufficiently small. Then we are done. Looking back to section 3.2, about the monotonicity of $h(\Omega)$, we find that the Cheeger set monotonically depends on Ω. But we know, that this dependence is not strict, so if $\Omega \subset \tilde{\Omega}$ implies $C_\Omega \subset C_{\tilde{\Omega}}$, then the equality $C_\Omega = C_{\tilde{\Omega}}$ does not always determines the equality $\Omega = \tilde{\Omega}$. Which was shown in [23]. But it can be directly deduced from monotone dependence in section 3.2.

3.5 Coherence between quasilinear parabolic equations and Cheeger sets

In general context, parabolic equations are specific type of partial differential equations. Thus, quasilinear equations are their subtypes. Quantitative study of these equations provides an excellent exact description of many time-dependent phenomena in real world [33]. The mentioned quantitative analysis of general parabolic equations is based on the definition using the elliptic operator, which is simply the Laplace operator equipped with minus sign [33]. If we focus on the probable most important part of the parabolic equation and that is the time derivative of the sought function, we find that it can be identified with the action of a second-order elliptic operator on the original function. The solution and classification of the equation therefore depends on the mentioned operators. This is a modest summary of quantitative approach. On the contrary, a way to find a connection between Cheeger problem and this type of differential equations is through qualitative analysis. This is the second way to understand the nature of the solution and its evolution with time t.

In publications [34] and [35] it is shown, how the Cheeger sets plays very important role in the quantitative study of specific quasilinear parabolic equations. Let's suppose that we have a domain $\Omega \times (0, \infty)$, where $u(x,t)$ solves the equation

$$u_t - div\left(\frac{\nabla u}{\sqrt{1 + |\nabla u^2|}}\right) = 1,$$

where the boundary and initial information vanish. Assume also that $h(\Omega) < 1$. Under these assumptions, it is possible to estimate the growth of the respective solutions over time t. Solutions grows in time with speed proportional to $1 - h(\Omega)$ [10]. Of course, this growth takes place on a minimizer of $h(\Omega)$, i.e. on the Cheeger set. Here again a very close connection between the class of parabolic equations and the Cheeger set is evident.

The approach to this issue is partly opposite from the rest of problems, as we assume the existence of $h(\Omega)$ and its minimizer. Since it is possible to estimate the qualitative course of the solutions according to these quantities. Other more detailed findings relate to non-existence of classical stationary solutions. We refer to [33] and [35].

4 Variational problems that are closely related to Cheeger problem

In order not to constantly talk about the pure mathematical facts provided by the formal scheme of the Cheeger problem, let us turn back to Definition 2, because the following problem has many different and interesting applications. One of these applications relates to specific class of functions, so-called *BV* functions. In other words, these are functions of bounded variation [36]. The Definition 2. makes it possible to identify the Cheeger constant with the first eigenvalue of p-Laplacian, where $p \to 1$.

4.1 Torsional problem

Based on the convergence $\lambda_1(\Omega) := \lim_{p \to 1+} \lambda_p(\Omega) = h(\Omega)$ we can formulate the following problem [26], where we define the first eigenvalue $\lambda_1(\Omega)$ (Cheeger constant) by the divergence operator formula

$$-div\left(\frac{\nabla u}{|\nabla u|}\right) = \lambda_1(\Omega) \quad in\, \Omega, \quad u = 0 \quad on\, \partial\Omega. \tag{23}$$

This is a typical variational problem. In the section 1, we mentioned that the Cheeger problem itself is very closely related to different classes of variational problems, and here we see that it is possible to formulate a variational problem that contains the Cheeger constant itself. Of course, we are mainly interested in the properties of (23) in relation to $h(\Omega)$.

Assume that there exist a classical solution u of (23). Then the validity of the assumption that $f(u)$ solves (23) is also ensured for any Lipschitz-continuous f. Now, we need to write the p-Laplace operator in intrinsic coordinates. Which is only possible when $|\nabla u| \neq 0$ in the neighbourhood $U(j)$, where $j \in \Omega$. Let us take this property as fulfilled. Then the p-Laplacian can be written as

$$\Delta_p u = (p-1)|\nabla u|^{p-4}\langle D^2 u \nabla u, \nabla u\rangle - |\nabla u|^{p-2}(d-1)H(j)|\nabla u|,$$

where we can observe that this relation depends on the mean curvature of the level surface of the function u in point $j \in \Omega$ [26]. In this notation of p-Laplacian we record the presence of previously neglected geometry within the level surface. When we apply the limit $p \to 1$, the variational problem (23) transforms to simple equation $(d-1)H(j) = h(\Omega)$. This implies that every level set $\Omega_e := \{\, j \in \Omega : u(j) > e\}$ has a boundary with specific property. This property is related to mean curvature. Because the mean curvature of the boundary of any level set Ω_e corresponds to the Cheeger constant $h(\Omega)$ itself and is independent of e. Then, it can be argued that (23) has no classical solutions because its level sets Ω_e would be nested, so $\Omega_e \subset\subset \Omega_f$ for $e > f$ [37]. By the way, the level sets are generalization of level surfaces. It can also be shown that if the right hand side of (23) is constant and positive but differs from $\lambda_1(\Omega) = h(\Omega)$ then it is not possible to expect the solution in $BV(\Omega)$ [26]. Let us give a small comment on this fact.

The case $p \to 1$ in (13) leads to very interesting free boundary problems. It is possible to represent problem (23) by Euler equation [38], which is associated with the following energy functional

$$J_p(u) = \int_\Omega \left\{\frac{1}{p}|\nabla u|^p - u\right\} dx$$

Generally, we have already mentioned the representation of the energetic functional by relation (11) in the section 1.2. Let $p = 1$ and minimize the corresponding energy functional $J_p(u)$ on the Sobolev space $W_0^{1,1}(\Omega)$. If we consider space $W_0^{1,1}(\Omega)$ as a locally convex topological vector space, we find that this space do not coincides with continuous dual space. Simply put, the space is not reflexive. In general, the problem of the existence of a solution often occurs in non-reflective spaces. Here it is specifically $W_0^{1,1}(\Omega)$. This is where the tool of functions of bounded variations (*BV*-functions) comes into play. Because a classic way to overcome the difficulties, which are closely related to the non-existence of solution in $W_0^{1,1}(\Omega)$ can be overcomed by working in the space $BV(\Omega)$ [17].

4.2 *ROF* model and calibrable domains *Ω*

Sometimes the methodology behind the *ROF* model is called stable shapes for total variation minimization, see [39]. This is one of the many practical applications of total variation, which involves the deep interconnection between variational problems and the geometric background of the Cheeger constant (1). The *ROF* model is generally used for regularization of noisy images. The model can also be known as total variation denoising. The general functionality of the model is based on that signals with excessive detail may have high total variation, which implies that the integral of the absolute gradient of the signal is high [40]. Practically, the model removes the unwanted details while leaving alone the important details such as the information on edges [41]. Based on these facts about functionality of *ROF* model, one can formulate and solve the following variational problem [13].

Let $q \in L^2(\mathbb{R}^2)$ be an image. The mentioned regularization is ment to be performed on q. Then the minimization problem can be formulated as follows

$$\min_{u \in L^2(\mathbb{R}^2) \cap BV(\mathbb{R}^d)} \int_{\mathbb{R}^d} |Du| + \frac{1}{\rho} \int_{\mathbb{R}^d} |u - q|^2, \tag{24}$$

where $|Du|$ corresponds to the total variation measure. This measure is related to the definition of the total variation. Because the total variation can be stated as a norm, which is defined on the space of measures of bounded variation. The mentioned measure is also related to the distributional gradient of the function u. ρ corresponds to a positive real parameter. There can be no doubt about the existence and uniqueness of the solution of this functional because (24) is convex. Associated with the functional is Euler-Lagrange equation, which has the form

$$\rho \, div\left(\frac{Du}{|Du|}\right) = u - q.$$

However, it is necessary to include a tool of convex analysis. Because the gradient of the solution vanishes on domains of positive Lebesgue measure. Then the ideal way to write the Euler-Lagrange equation as a convex functional. Therefore, the total variation can be defined by

$$|Du|(\mathbb{R}^d) = \int_{\mathbb{R}^d} |Du| := sup\left\{\int u \, div(\Sigma) : \Sigma \in C_c^1(\mathbb{R}^d; \mathbb{R}^d), |q| \leq 1\right\}.$$

Now, we want to define $J[u]$ as a functional, where it is possible to define its subdifferential at $u \in L^2(\mathbb{R}^d)$. Then it must be true that $J[u]$ is convex and set $J[u] = |Du|(\mathbb{R}^d)$. Consider the subdifferential of $J[u]$ as a partial derivative, that is defined as a following set

$$\partial J[u] = \{v \in L^2(\mathbb{R}^d) : J[u + w] \geq J[u] + \langle v, w \rangle \; for \; \forall w \in L^2(\mathbb{R}^d)\}.$$

Based on the shape of the subdifferential it can be argued that the desired Euler-Lagrange equation is given by $(q - u)/\rho \in \partial J[u]$. It is interesting that the vector field with specific properties can be included in the concept of this Euler-Lagrange equation. The vector field has already been introduced in the relation for total variation of u. Now, let's give it the necessary properties. The subdifferential $\partial J[u]$ consists of the divergence of vector field $\Sigma_u \in L^\infty(\mathbb{R}^d)$ such that Σ_u has unit size, so $|\Sigma_u| \leq 1$, then $div(\Sigma_u) \in L^2(\mathbb{R}^d)$ and the total variation can be characterized by the vector field as $Du = \Sigma_u |Du|$. Where the divergence can be explicitly expressed by the relation

$$div(\Sigma_u) = \frac{u-q}{\rho}, \tag{25}$$

where $q = \chi_\Omega$ and χ_Ω represents the characteristic function of some bounded domain Ω. This domain is also Lipschitz. It is worth mentioning that the possibility of expressing the concept of the subdifferential $\partial J[u]$ as the divergences of vector fields, conspicuously corresponds with the vector field method [42], which we will later use to formulate the lower bound of the Cheeger constant of a particular region Ω. But it is not just a simple math of concepts. From relation (24), one can see the link between the Euler-Lagrange equation in the framework of image regularization and the upper bound of the Cheeger constant, because $h(\Omega) \leq \frac{|\partial\Omega|}{|\Omega|}$. We will put a comment on this comparison later.

The main goal of the *ROF* model solution is to characterize the domain Ω for a particular solution u. In fact, we require that the regularization produced by the *ROF* model (24) determines a change in, for example the contrast of an image, but at the same time not to change the shape of the original image, which is defined as $q = \chi_\Omega$. The characterization of Ω therefore consists in the assumption that the solution u of (24) with the initial image $q = \chi_\Omega$ is a mere scaling of q. Then u can be obtained in th form $u = \varsigma q = \varsigma \chi_\Omega$, with $\varsigma \geq 0$.

Another technical, but very important feature is the calibrability of Ω. In [43] a Lipchitz domain Ω is calibratable if its perimeter is finite and if there exist a vector field, such that $\Sigma \in L^\infty(\mathbb{R}^d; \mathbb{R}^d)$. The vector field must also satisfy the unit size limitation $|\Sigma| \leq 1$ and $\Sigma = \nu_\Omega H^{d-1}$ almost everywhere on $\partial\Omega$, as stated in [13]. Then $-div(\Sigma) = \frac{|\partial\Omega|}{|\Omega|}\chi_\Omega$. There we can directly observe that the divergence of the vector field Σ is defined by a combination of the upper bound of (1) and the initial image in terms of the characteristic function. Which suggests that we understand the divergence in the distributional sense. Since $q = \chi_\Omega$ and $\Sigma = \nu_\Omega H^{d-1}$, where H^{d-1} denotes the corresponding $(d-1)$-dimensional Hausdorff measure. The condition of calibrability of Ω is extremely important as it guarantees the existence of an explicit function that is a minimizer of (24) with the initial image defined as $q = \chi_\Omega$. This fact can be summarized in theorem, with constructive proof, see [44].

Let us bring the elegance of the close connection between ROF regularization and the Cheeger problem. The elegance lies in the calibrability of Ω and in the concept of mean-convexity. Let $\Omega \subset \mathbb{R}^d$, $|\partial\Omega| < \infty$, then Ω is considered to be mean-convex, if for arbitrary Borel set $Y \subset \mathbb{R}^d$, where $Y \subset \Omega$ and $|\partial\Omega| \leq |\partial Y|$ holds for finite perimeter. The property of mean-convexity implies that Ω is a minizer of the perimeter with the respect to outer variations.

5 The Cheeger constant of curved tubes

5.1 Introduction

We study the Cheeger constant of domains obtained like tubular neighbourhoods of complete curves on an arbitrary dimensional Riemannian manifold with sectional curvature identically equals to zero. In other words, we are interested in the Cheeger constant of the tubular neighbourhood in the arbitrary dimensional Euclidean space, which was our original hypothesis (see [. It is worth mentioning that if the curve is simple, closed, complete and finite than the tubular neighbourhood itself is the Cheeger set [45]. Which is also proven in the next passage.

Let $\Omega \subset \mathbb{R}^d$ be an open connected set with $d \geq 1$. The Cheeger constant of Ω is defined by the relation (1). For our practical needs, let us recall one technical fact that is related to Definition 1. If there exist a minimizer of (1) (e.g. if Ω is bounded domain) then, as we know, it is called a Cheeger set of Ω and it is denoted by symbol C_Ω.

As discussed in detail in sections 1 and 2, which provides the introduction to the Cheeger problem and general results on the Cheeger problem, there are very few known domains Ω, where the Cheeger constant $h(\Omega)$ can be expressed explicitly. Given the nature of our following theorem and its relevant proof, let us mention here a particular domain that is universal in the sense of its generalization. In all dimensions there are a priori only balls $B_a = \{x \in \mathbb{R}^d : |x| < a\}$ with $a > 0$, for which we have the explicit expression of the Cheeger constant

$$h(B_a) = \frac{d}{a} \tag{27}$$

and the Cheeger set is directly equal to the set B_a, so $C_\Omega = B_a$ [46].

It was shown in [47] that there exist another large class of planar domains for which the Cheeger constant can be computed explicitly. These planar domains are called *curved strips*. If we have smooth closed planar curve Ψ and a positive number a, then we define a strip of radius a as tubular neighbourhood $\Omega_a \coloneqq \{x \in \mathbb{R}^2 : dist(x, \Psi) < a\}$. If we assume that a is so small that Ω_a does not overlap itself, then the main result of [47] says

$$h(\Omega_a) = \frac{1}{a}, \tag{28}$$

where the Cheeger set coincides with tubular neighbourhood itself, so $C_\Omega = \Omega_a$. In this case, it would be good to pause and describe in more detail the precise estimation of Cheeger constant and explicit characterization of Cheeger set for general strips.

5.2 The Cheeger constant of curved strips

The key idea here is that if the underlying planar curve is finite and complete, then the domain obtained as planar tubular neighbourhood of the mentioned curve is equal to the Cheeger set itself [47]. This result is very important because it provides an explicit mechanism for obtaining the Cheeger constant of *curved strips* in the plane. When we mentioned the assumption that the reference curve is complete and finite, we meant that the domain was defined as curved annulus. We will also comment later on the *unbounded strips* and we also consider a case where the requirement for the underlying planar curve is changed. The new requirement is that the curves must be finite but non-complete. Now, let's provide the geometric background to this problem.

5.2.1 The underlying geometry

In this part we set the basics of the notations for the geometrical situation that we will address. Let Ψ be a C^2 curve that is connected in $\mathbb{R}^2$. The fact that the curve Ψ is of class C^2 implies that it is a homeomorphic image of 1-sphere under a C^2 function. Further, let us denote by $|\Psi| = \int_\Psi dq$ its length, dq being the arclenght element of curve Ψ. Let's define a map $M\colon \Psi \to \mathbb{R}^2$ be a C^1 vector field giving the normal vector in the points of curve Ψ, and let the map $\kappa : \Psi \to \mathbb{R}$ be the its corresponding curvature. It is worth noticing, that if we change the sign of the curvature κ, the orientation of the previous define vector field M will be different. For the precise definition of κ, we can take a unit-speed parametrization φ of Ψ, than we have

$$\kappa(q) = \ddot{\varphi}\big(\varphi^{-1}(q)\big) \cdot M(q). \qquad (29)$$

Consider a map from $\Psi \times \mathbb{R}$ to $\mathbb{R}^2$, that can be defined by the following relation

$$O(q,t) := q + tM(q), \qquad (30)$$

and for any positive a we introduce the set

$$\Omega_{\Psi,a} := O\big(\Psi \times (-a,a)\big)$$

Set $\Omega_{\Psi,a}$ represents non-self intersecting tubular neighbourhoods of the underlying curve Ψ. The map O is injective in $\Psi \times (-a, a)$. From the relation for curvature κ we derive the following equation, which is the bilinear form [47, Section 1.1]

$$dO^2 = (1 - \kappa(q)t)^2 dq^2 + dt^2 \tag{31}$$

Knowledge of injectivity of the map O implies that a must be small compared to the curvature κ. In other words, we obtain this formula $a|\kappa(q)| \leq 1$ for any arbitrary $q \in \Psi$, that the boundary of the previously defined set $\Omega_{\Psi,a}$ is of the class $C^{1,1}$[47]. But that's not all what we can deduce from the injectivity of the map O. If we know that the map between $\Psi \times \mathbb{R}$ and $\mathbb{R}^2$ is injective in $\Psi \times (-a, a)$, than it is obvious that the same mapping must be a C^1 diffeomorphism between the sets $\Psi \times (-a, a)$ and $\Omega_{\Psi,a}$. The assumption that the map O is injective the mentioned set directly implies that $\Omega_{\Psi,a}$ is in geometrical sense an open non-self-intersecting strip. The strip is contained between the parallel curves $q \to q \pm aM(q)$, with arbitrary $q \in \Psi$ [47, Section 1.1]. This geometric nature of the set $\Omega_{\Psi,a}$ corresponds to the identification with the Riemannian manifold $\Psi \times (-a, a)$.

As we mentioned in the introductory section, where we discuss the historical context of topology and manifolds, each Riemannian manifold is assigned with the corresponding metric. Thus, by the metric on this Riemannian manifold, we mean the map O, which is injective in $\Psi \times (-a, a)$. Riemannian manifold $\Psi \times (-a, a)$ with the metric (31) is called a curved strip. In general, any set $\Omega_{\Psi,a}$ with the injective metric O is called curved strip. If the curve Ψ is contained in a line then the curved strip $\Omega_{\Psi,a}$ reduces to rectangle. The most interesting as non-convex cases that occurs just when the curve Ψ has more complicated geometry. Then it is not possible to cover this problem by the known result for the Cheeger problem. We can characterize four situations occurring for a curved strip. The division of the curved strip into four types depends directly on the geometrical properties of the reference curve. If Ψ is not finite, the curved strip will be either infinite or semi-infinite. In other words, if the curved strip is not finite then the strip is complete. But in the semi-infinite case, the strip is not finite and non-complete. The symbolism remains identical, we denote the mentioned types of *curved strips* as $\Omega_{\Psi,a}$. However, the type of curved strip does not only imply, completeness of the strip but also compactness. When the curve Ψ is finite, then it can be homeomorphic to a circle or the homeomorphism can be targeted to an open segment. Homeomorphism to a circle implies compactness of $\Omega_{\Psi,a}$. Otherwise, when we talk about the homeomorphism to an open segment then the curved strip is non-compact. In the *curved strips* typology, the finite *curved strips* can be either compact or non-compact. *Curved annulus* corresponds to the case when the curve is circle [47]. The homeomorphism to an open segment can be identified with the *finite curved strip*.

6 The proofs

Theorem 6 ([47, Section 3.1]). *Let the curve* Ψ be *infinite*, *compact* or *semi-infinite*. *Then*

$$h(\Omega_{\Psi,a}) = \frac{1}{a}. \tag{32}$$

If Ψ is compact, then the Cheeger constant is given by the relation (32) and the unique Cheeger set can be identified with the *curved annulus* itself, $C_{\Omega_{\Psi,a}} = \Omega_{\Psi,a}$. *Infinite* and *semi-infinite* curves are also an interesting case. As we have mentioned before, the set $\Omega_{\Psi,a}$ is *infinite* or *semi-infinite curve strip*, then the infimum of the Cheeger constant (1) is not attained. Under these conditions, it is possible to construct and optimize a local sequence of $\Omega_{\Psi_{L,a}}$ that converges to $\Omega_{\Psi,a}$ for $L \to \infty$.

6.1 Proof of Theorem 6: the upper bound

The method for getting a good estimation of upper bound of (32) is mostly straightforward (see, [47, Lemma 4]. The main problem is the estimation of the lower bound of (32), which stems both from its definition and from its geometric nature.

The main idea in the estimation of the upper bound for the Cheeger constant for particular cases of *curved strips* is the correct choice of test domain in (1). In the case of Theorem 6, Ψ can be compact, then $\Omega_{\Psi,a}$ is exactly a curved annulus as we pointed out above in relation to homeomorphisms of Ψ. Or it may happen that $\Omega_{\Psi,a}$ is *semi-infinite* or *infinite curved strip*. Our particular choice of test domain depends on the type of the *curved strip*. Let's focus on the case where $\Omega_{\Psi,a}$ is a curved annulus. The test domain is identical to the whole *curved strip* $\Omega_{\Psi,a} = S$. Considering the metric (31), we obtain precise formulas for perimeter and volume of the test domain.

$$|S| = \int_{\Psi} \int_{-a}^{a} (1-\kappa(q)t)dt\, dq = \int_{0}^{L} \int_{-a}^{a} (1-\kappa(q)t)dt\, dq \tag{33}$$

$$|\partial S| = \int_{\Psi} (1+\kappa(q)a) + \int_{\Psi} (1-\kappa(q)a)dq \tag{34}$$

The upper bound of $h(\Omega_{\Psi,a})$ can be obtained as a ratio of $|\partial S|$ and $|S|$ since the $h(\Omega) \leq \frac{|\partial S|}{|S|}$. Then,

$$\frac{|\partial S|}{|S|} = \frac{1}{a}. \tag{35}$$

An important fact is the independence of the ratio on the curvature. The curvature term both cancels due to the symmetry of the test domain S. The reason is not only symmetry, but the underlying geometry of the problem that induces that symmetry of the test domain.

Now let's move to infinite and semi-infinite types of curve Ψ. The set $\Omega_{\Psi,a}$ transforms to *infinite* or *semi-infinite curve strip* and the infimum of the Cheeger constant (1) is not attained. Under these conditions it is possible to construct and optimize a local sequence $\Omega_{\Psi_{L,a}}$ and the corresponding Cheeger constant of $\Omega_{\Psi_{L,a}}$ that converges to the upper bound of $h(\Omega_{\Psi,a})$ as $L \to \infty$. The whole *curved strip* cannot be attained because it has both infinite area and perimeter. The access we can select is particular choice of the *finite curved strip* $S = \Omega_{\Psi_{L,a}}$ as a test domain, where $L > 0$. This local segment is contained inside the original *infinite* or *semi-infinite curved strip* $\Omega_{\Psi,a}$. Then we can evaluate the ration of area and perimeter as follows

$$\frac{|\partial S|}{|S|} = \frac{4a + \int_{\Psi} (1 + \kappa(q)a) + \int_{\Psi} (1 - \kappa(q)a)dq}{\int_0^L \int_{-a}^{a} (1 - \kappa(q)t)dt\, dq} = \frac{4a + 2|\Psi_L|}{2a|\Psi_L|} \xrightarrow[L\to\infty]{} \frac{1}{a}. \tag{36}$$

The term $4a$ corresponds to two parts of the diameter of the local finite segment $\Omega_{\Psi_{L,a}}$. First part of the diameter of length $2a$ is located at the beginning of the *curved strip* and the second part of the diameter of length $2a$ is situated at the end of segment $\Omega_{\Psi_{L,a}}$.

Given the definition (1), formulas (35) and (36) provides us the proper upper bound for the *curved annulus*, *infinite* and *semi-infinite curved strips*. The determination of the lower bound is much more complicated. Both the definition and the geometry itself are the reasons for the complexity of the estimation. The approach for the lower bound can be constructed in various different ways. One of the possibilities is straightforward and depends on the particular theorem. However, in certain situations it is quite technically demanding. In the guise of the following argument, let us hold the first possibility of estimating the lower bound of (32) and mention the theorem from which our method stems. For our purposes, we will name this approach as a vector field method [42].

6.2 Proof of Theorem 6: the lower bound

6.2.1 Vector field method

Theorem 7 ([42]). *Let a map $V:\Omega \to \mathbb{R}^2$ be a smooth vector field on Ω and $c \in \mathbb{R}$. Assume that the pointwise inequalities $|V| \leq 1$ and $div(V) \geq c$ hold in the region Ω. Then $h(\Omega) \geq c$.*

We can use Theorem 7, for direct approach for establishing the lower bound of $h\left(\Omega_{\Psi_{L,a}}\right)$. The vector field is constructed so that its divergence satisfies the relevant requirements in the Theorem 7. The exact approach to the construction depends directly on the first term of the metric (31) and also depends on the particular value of the curvature $\kappa(q)$. Let us introduce the function $V_t: \Psi \times (-a, a) \to \mathbb{R}$ as follows [47, Remark 9]

$$V_t(q,t) := \begin{cases} \dfrac{(1-\kappa(q)a)(1+\kappa(q)a)-(1-\kappa(q)t)^2}{2a\kappa(q)(1-\kappa(q)a)} & (37) \\ \dfrac{t}{a} & (38) \end{cases}.$$

Let's notice that the first equality (37) is constructed for cases of non-zero curvatures $\kappa(q)$ and the second equality (38) is constructed for $\kappa(q) = 0$. The relation for vanishing curvature corresponds to taking the limit $\kappa(q) \to 0$ in the first equality. The components of the vector field are considered with respect to the coordinates (q,t). It is necessary to verify the relevant assumptions of Theorem 7 in order to further calculate the divergence. It can be done easily, because our vector field is given as $V(q,t) := \left(0, V_t(q,t)\right)$. Then it satisfies $\|V\|_{L^\infty(\Psi\times(-a,a))} = 1$. Fulfilling the first assumption of Theorem 7, entitles us to perform a calculation of divergence

$$\left(div(V)\right)(q,t) = \frac{1}{1-\kappa(q)t}\partial_t[(1-\kappa(q)t)V_t(q,t)] = \frac{1}{a}. \tag{39}$$

This relation is satisfied for every $(q,t) \in \Psi \times (-a,a)$ [47, Remark 9]. As can be seen, divergence of the vector field (37), (38) gives the desired lower bound of the Cheeger constant (37). For a deeper understanding, let us better describe the structure of the vector field and divergence.

The structure of (39) is equivalent to the general formulation of the divergence operator on the Riemannian manifolds [5]. Following the introductory section of this work, where we talked about Riemannian manifolds and the corresponding volume forms on these manifolds, we point out the connection between the generalization of the divergence operator and the differentiable manifolds. The divergence of a vector field can be extended simple to arbitrary differentiable manifold of dimension d [5]. The basic assumptions about the manifold include not only differentiability, but also the existence of a volume form. Riemannian manifolds meet these requirements precisely [48]. This directly implies that on a Riemannian or pseudo-Riemannian manifolds, the divergence operator is defined with respect to the metric volume form and can be computed in terms of the Levi-Civita connection. However, the equivalent and simpler formulation of the divergence operator can be made without using connection. Then we get

$$div(V) = \frac{1}{\sqrt{det(j)}} \partial_t \left(\sqrt{det(j)} V_t \right), \tag{40}$$

where j is the metric and ∂_t denotes the partial derivative with respect to the coordinates of the constructed vector field V_t [5, 48]. In the context of our proof, the general metric j corresponds to the metric dO^2, that is given by the relation (31). So, the $\sqrt{det(j)}$ denotes square root of the first term in (31). This expression is very important as it encodes the behavior of geodesics against the reference curve Ψ of the *curved strip* $\Omega_{\Psi,a}$. It can be concluded that the vector field itself depends on the geometry of $\Omega_{\Psi,a}$ and, above all, on the topology of the underlying Riemannian manifold. In the case of c*urved strips*, the Riemannian manifold corresponds to $\mathbb{R}^d$. The sectional curvature is then $K^{\mathbb{R}^d} = 0$. In (37) we see the direct result of the zero sectional curvature of the manifold $\mathbb{R}^d$. None of the terms of (37) is dependent on the curvature of the *curved strip* or the manifold. The corresponding vector fields (37), (38) and its divergence (39) are also constructed in this sense. The essential nature of a vector field method and general curvilinear coordinates that greatly support the description of our original problem will be further analyzed as soon as we generalize the Theorem 6 up to d dimensions.

The vector field method is not the only approach to the estimation of the lower bound of (32). So, let us begin with the "stripization" method.

6.2.2 Stripization method

Stripization is an operation, that appropriately modifies the test domain S. The modification consists in smoothing out the distortions of the test domain, which interfere with the behavior of the Cheeger constant (1) [47, Lemma 4]. Let us take an open set $S \subseteq \Omega_{\Psi,a}$. Define the restriction of the curve Ψ as

$$\Psi_S := \{ q \in \Psi : O(\{q\} \times (-a, a)) \cap S \neq \emptyset \}.$$

Let's also define the maps $f_\pm : \Psi_S \to [-a, a]$ as

$$f_+(q) := sup\{t \in (-a, a) : (q, t) \in S\}, \qquad f_-(q) := inf\{t \in (-a, a) : (q, t) \in S\}$$

The reason for introducing these functions lies in their properties. If we graph f_+ and f_-, we find that the test domain S is included between them. Thanks to the above mentioned relations for functions $f_\pm$, we can now define a set whose properties play an important role in the estimation of the lower bound.

Definition 4 ([47, Section 2]). Let S be an open subset of the *finite curved strip* $\Omega_{\Psi,a}$. Let Ψ_S and $f_\pm$ satisfies the definitions above. Then

$$S^* := \{O(q, t) \in \Omega_{\Psi,a} : q \in \Psi_S, f_-(q) < t < f_+(q)\}. \tag{41}$$

Theorem 8 ([47, Lemma 6]) (properties of the set S^*). *Let S be an open, connected and bounded subset of Ω. The region Ω has finite perimeter. Then*

$$|S^*| \geq |S|, \qquad |\partial S^*| \leq |\partial S|.$$

And $f_\pm \in BV(\Psi_S)$, the following relation is satisfied

$$\begin{aligned} |\partial S^*| = &\int_{\Psi_S} \sqrt{\left(1 - \kappa(q) f_+(q)\right)^2 + f_+'(q)^2} dq \\ &+ \int_{\Psi_S} \sqrt{\left(1 - \kappa(q) f_-(q)\right)^2 + f_-'(q)^2} dq + |D_S f_+|(\Psi_S) + |D_S f_-|(\Psi_S) \\ &+ \left(f_+(q_0) - f_-(q_0)\right) + \left(f_+(q_1) - f_-(q_1)\right). \end{aligned} \tag{42}$$

Terms $f_\pm'(q)dq$ and $f_-'(q)$ are the absolute continuous part of Df_+, Df_- and $D_S f_+$, $D_S f_-$ its singular part.

Brief proof [47]. Without the loss of generality, we will not give a complete proof of Theorem 6, since its essence is more technical. Therefore, most steps are not as essential as general outline, which is most important in terms of methodology and approach that we choose. The main objective is to point out the difference in approach by vector field method and by stripization.

From the definition of S^* (41), we know that $S^* \supseteq S$. This implies the first inequality in $|S^*| \geq |S|$ in Theorem 8. Within the proof of the relation for perimeter, we use the Compactness Theorem for BV functions [36]. This theorem states that a sequence of functions has a convergent subsequence if these functions are locally of bounded total variation and uniformly bounded at a point. We will use the mentioned convergence. Let us take a sequence S_i of smooth sets converging in the L^1 space to S in a such way that the sets of perimeters converges to the original perimeter $\partial S_i \to \partial S$. If we take a look on the definition 4, of the set S^*, we concluded that ${S_i}^* \to S^*$. Let's do the same thing within the sequences of the perimeters. By the lower semicontinuity of the perimeter, we can obtain a upper bound as follows $\partial S^* \leq \liminf \partial {S_i}^*$. The final step is to prove an appropriate estimate for the general case, where we consider smooth sets. Then, in full generality $|\partial S^*| \leq |\partial S|$ for smooth S. Using previous considerations and exact formula for perimeter (42), it is possible to formulate a direct estimation of $|S^*|$ and $|\partial S^*|$. For simplicity of the upcoming estimate, let us introduce the two following functions

$$t_+ := sup\{f_+(q) : \; q \in \Psi_S\} \qquad\qquad t_- := inf\{f_-(q) : \; q \in \Psi_S\}.$$

These auxiliary functions allow us to construct the desired estimation

$$\begin{aligned} |S^*| &= \int_\Psi \int_{f_-(q)}^{f_+(q)} (1-\kappa(q)t)dt\, dq = \int_\Psi \big(f_+(q) - f_-(q)\big)\left(1 - \kappa(q)\frac{f_+(q)+f_-(q)}{2}\right)dq \\ &\leq (t_+ - t_-)\int_\Psi \left(1 - \kappa(q)\frac{f_+(q)+f_-(q)}{2}\right)dq. \end{aligned} \tag{43}$$

The perimeter can be estimated from above using the relation (42) as follows

$$\begin{aligned} |\partial S^*| = &\int_{\Psi_S} \sqrt{\big(1-\kappa(q)f_+(q)\big)^2 + f_+^{'}(q)^2}dq \\ &+ \int_{\Psi_S} \sqrt{\big(1-\kappa(q)f_-(q)\big)^2 + f_-^{'}(q)^2}dq + |D_S f_+|(\Psi_S) + |D_S f_-|(\Psi_S) \\ &+ \big(f_+(q_0) - f_-(q_0)\big) + \big(f_+(q_1) - f_-(q_1)\big) \\ &\geq 2\int_{\Psi_S}\int_\Psi \left(1 - \kappa(q)\frac{f_+(q)+f_-(q)}{2}\right)dq. \end{aligned} \tag{44}$$

Now we use the proven inequalities from Theorem 8, to construct the lower bound of (32) by the following ratios

$$\frac{|\partial S|}{|S|} \geq \frac{|\partial S^*|}{|S^*|} \geq \frac{2}{t_+ - t_-} \geq \frac{1}{a}. \tag{45}$$

The reason for the last inequality lies in the boundaries of the functions t_+ and t_-. Because t_+ is defined as supremum of $f_+(q)$ for all $q \in \Psi_S$ and t_- as a infimum of $f_-(q)$ for all $q \in \Psi_S$, then it must be true that $t_- < t_+$. We note that the test domain S is contained between graphs of f_+ and f_-. Therefore, if we consider t_+ and t_- as their supremum and infimum, then the inequality $t_- < t_+$ can be completely bounded by the diameter of the *curved strip*. The trivial bounds are $-a \leq t_- < t_+ \leq a$. In conclusion, we were able to prove the relations for upper and lower bound of the Cheeger constant (32) for a *curved strip* of any kind [47, Lemma 7]. The last part of the proof concerns the geometric nature of the Cheeger constant (32), namely the identification of the Cheeger set. This set is the minimizer of the ratio (1). If, based on previous arguments, we accept that $h(\Omega_{\Psi,a}) = \frac{1}{a}$. Then there is some Cheeger set $C = C_{\Omega_{\Psi,a}}$ as a minimizer of (1). Any inequalities in the proof of the upper and lower bound will got to equalities if the test domain S itself can be matched with the Cheeger set. Then $S = C_{\Omega_{\Psi,a}}$. From this equality immediately follows that the functions f_+ and f_- which contains the test domain S are constant. Based on the features of the mentioned functions, it can be deduced that their infimum and supremum corresponds to their lower and upper bound, so $t_+ = a, t_- = -a$. This argumentation gives us the right to match the Cheeger set and the *curved strip* $C = \Omega_{\Psi,a}$.

A complete proof of the Theorem 6, was submitted, i.e. for the cases when the reference curve Ψ is compact, then $\Omega_{\Psi,a}$ corresponds to the curved annulus and $C = \Omega_{\Psi,a}$. In the case when Ψ is infinite or semi-infinite, then $\Omega_{\Psi,a}$ is *infinite* or *semi-infinite curved strip*. The infimum of (1) is not achievable and the proof of the upper bound of (32) provides a minimizing sequence $\Omega_{\Psi_{L,a}}$ for $L \to \infty$. From the characterization of the lower bound proof, we know that the unique Cheeger set is $C = \Omega_{\Psi,a}$. Since the set $\Omega_{\Psi,a}$ has an infinite area and perimeter, then there cannot exist a minimizer set for the case when Ψ is infinite or semi-infinite. However, there exist the previous mentioned sequence $\Omega_{\Psi_{L,a}}$ which is a minimizing sequence for $L \to \infty$.

We have provided the proof that there exist a large class of planar domains for which the Cheeger problem and the Cheeger constant can be computed explicitly. Only a modest number of domains can provide explicit proof and calculation of the Cheeger constant (1). One of the most interesting open problems concerns the Cheeger constant of a three-dimensional cube. Unfortunately, already a three-dimensional case does not admit an explicitly known Cheeger constant. Also, there is no explicit analytical description of its Cheeger set [49, Open problems 1 and 5]. These facts give us an idea of how unfavorable the problem would be in the case of arbitrarily dimensional cubes. Generally speaking, very little is known about the behavior of the Cheeger problem (1) in higher dimensions, regardless of the geometrical domain and underlying topology. In spite of all the difficulties mentioned above, which are contained in the formulation of the Cheeger problem (1) in higher dimensions, there exist one class of geometrical domains where the Cheeger problem (1) can be correctly formulated, but not only. It is also possible to provide complete proof and explicit formula for the Cheeger constant. The mentioned class of geometrical domains is a product of generalization of Theorem 6, and is called a *curved tubes* [45]. In generalizing the Theorem 6, we naturally move from *curved strips* to *curved tubes* However, the topology of the underlying manifold also changes. From the two-dimensional plane we move to the d-dimensional manifold with zero constant sectional curvature. This topological description of the underlying manifold directly corresponds to the definition of $\mathbb{R}^d$. As we heve already mentioned several times, the values of sectional curvature, is very important. Since it reflects the essence of the $\mathbb{R}^d$ topology. This radically translates into the concept of generalized curvilinear coordinates that will be used in parametrization of the *curved tubes*. In general, the *curved tubes* will be defined topologically using the concept of the tubular neighbourhood of the reference geodesic on $\mathbb{R}^d$.

The metric of the geometry is closely related to the description of the tubular neighbourhood. As we have pointed out in relation (31) for *curved strips* in two-dimensional plane, the first term of (31) represents some sort of field that describes the behavior of infinitesimally close geodesics in the relationship to reference geodesic of tubular neighbourhood. This construct is called a Jacobi field. This tool plays a key role in the parametrization of the tubular neighbourhood in d-dimensions [50]. The topology of the Riemannian manifold itself determines the nature of the Jacobi field, because the formulation depends directly on the concept of tangent space [51]. In the upcoming argumentation we generalize the concept of Jacobi field, allowing us to construct explicit proof of the Cheeger constant (1) and will provide us a deeper understanding of the underlying geometry. We note that the construction of the vector field for estimating the lower bound of (32) is not trivial. However, as soon as we successfully construct the vector field correctly, we find that its divergence is given by (40). The relation (40) gives the general definition of the divergence operator that acts on a given field on a Riemannian manifold using the Jacobi field conception. We see that the geometry of the underlying manifold directly determines the metric that is needed to describe our problem. However, the metric is closely related to the Jacobi field around the reference geodesic of the tubular neighbourhood in $\mathbb{R}^d$. In the end, this chain of facts seamlessly connects to the divergence and construction of the vector field itself. Thus, generalization of Theorem 6 is not just a direct causeless reasoning, but an internal connection of ideas that are closely linked to the formulation of our problem. The importance of the above considerations is not misleading, as understanding the following proof and its applications, which will be discussed in more detail in the second part of this work, is based on the links between the general concepts we have just described. Let us move on to a detailed formulation of the Cheeger problem (1), within the following geometrical shape.

Given a closed smooth curve Ψ in $\mathbb{R}^d$, $d \geq 2$ we introduce a *curved tube* as a map $\mathbb{R}^2 \to \mathbb{R}^d$ by [45]

$$\Omega_a \coloneqq \{x \in \mathbb{R}^2 : dist(x, \Psi) < a\}. \tag{46}$$

We say that Ω_a does not overlap itself if the map $\Psi \times (0, a) \ni (q, t) \to q + tN(q)$ induces a smooth diffeomorphism for any smooth vector field N along the geodesic Ψ. For the reason that Ψ is compact, this condition holds for sufficiently small a. Then the theorem is formulated as follows.

Theorem 9 ([45])**.** (Generalization of Theorem 6) *Let Ψ be a closed smooth curve in $\mathbb{R}^d$ with $d \geq 2$ and a be a positive number. Let Ω_a be defined by* (46). *If a is so small that Ω_a does not overlap itself, then*

$$h(\,\Omega_a) = \frac{d-1}{a} \tag{47}$$

and $C_{\Omega_a} = \Omega_a$.

This theorem summarizes our original result. It is a generalization of the result for *curved strips* [47]. We will see later that our result of Theorem 9 emerges from even more general structure of d-dimensional Riemannian manifolds with constant sectional curvature. However, we have not formulated an exact proof yet.

Taking a deeper look at Theorem 9, we conclude that the Cheeger constant of a d-dimensional *curved tube* of radius a on the manifold with zero sectional curvature ($\mathbb{R}^d$) corresponds to the Cheeger constant of the $(d-1)$-dimensional ball of the same radius in $\mathbb{R}^d$ [45]. Which tells a lot about the geometric nature of the Cheeger constant and its behavior and monotonicity in higher dimensions.

Another fact that is directly visible at first glance is the independence of $h(\Omega_a)$ on the shape of the underlying submanifold Ψ. An identical situation was encountered in the case of Theorem 6, where Ψ was one-dimensional submanifold of $\mathbb{R}^2$. This can be interpreted as the first eigenvalue of the non-linear 1-Laplacian [45]. The independence of the isoperimetric constant $h(\,\Omega_a)$ on the shape of Ψ stems from the topology of $\mathbb{R}^d$. Hence from the particular formula of the Jacobi field within the estimation of the upper bound and the vector field within the estimation of the lower bound of (47). Further details and explanations are given in the following proof.

7 The proof of Theorem 9

The proof strategy was precisely described in the previous paragraph. For correctness, let us just point out that the following proof differs from the *curved strips* argumentation in some details. It is no longer possible to generalize and then use a "stripization" method. Therefore, in estimating the lower bound of (47), we rely only on the generalization of Theorem 7 and on the usage of the vector field method [47].

Proposition 1. (Generalization of Theorem 7) *Let $\Omega \subset \mathbb{R}^d$ be an open connected set. If there exist a smooth vector field $V: \Omega \to \mathbb{R}^d$ satisfying pointwise inequalities*

$$|V| \leq 1, \qquad div(V) \geq c$$

in Ω with some constant $c \in \mathbb{R}$, then $h(\Omega) \geq c$.

Upper bound of (47) can be obtained directly by using suitable test domains. The same approach was used in the two-dimensional case of Theorem 6. Now we generalize it to test domains within submanifold $\Omega \subset \mathbb{R}^d$.

7.1 Proof of Theorem 9: the upper bound

Take the whole tube Ω_a itself as a test domain in (1). For a good grasp of our problem, it is necessary to describe some basic facts about the geometry of *curved tubes*.

Assuming that Ω_a does not overlap itself implies that Ψ is an embedded submanifold of $\mathbb{R}^d$. Which means that if $M = \mathbb{R}^d$ is a smooth manifold, then there is an inclusion $i : M \to \Psi$ that represents the appropriate embedding of $\Psi \subset M$. For Ψ there is a local parametrization that can be defined by a smooth map $\gamma : I \to \mathbb{R}^d$, where I is an open interval. In particular, $\gamma(s) \in \Psi$ for all $s \in I$. In general, $\gamma(s)$ can be considered as a unit speed parametrization. Then $|\dot{\gamma(s)}| = 1$ for all $s \in I$. We denote $e_1 := \dot{\gamma}$ and $\kappa := |\ddot{\gamma}|$, the tangent vector field and curvature of the submanifold Ψ, respectively.

Note the important fact that we allow curves whose curvature may vanish on a subset of I. There is no coordinate system that we urgently need to describe the behavior of such curves. Specifically, we are talking about a usual Frenet frame. The reason for the lack of Frenet-Serret apparatus is that the precise relations, which describes the behavior of geodesic, can only be constructed for non-degenerate cases [52]. Which roughly means that these curves (or geodesics) have nonzero curvature. In our case this assumption cannot always be fulfilled, as we have already mentioned.

However, there exist a frame defined by parallel transport [52]. The mentioned parallel transport induces the following smooth map $e_2, \dots, e_d : I \to \mathbb{R}^d$ and a map for each curvature $\kappa_1, \dots, \kappa_{d-1} : I \to \mathbb{R}$ such that $|e_\mu(s)| = 1$ for all $s \in I$, $\mu \in \{2, \dots, d\}$. From which we obtain the formulas for Frenet-Serret frame in d- dimensions [53]. The following equality applies to curvatures ${\kappa_1}^2 + \dots + {\kappa_{d-1}}^2 = 1$. Set $\{e_1, \dots, e_d\}$ form an orthonormal vector field along Ψ and that the vectors $e_2, \dots, e_d$ forms a basis of the normal bundle. The tube Ω_a can be locally parametrized by using the general curvilinear coordinates called Fermi coordinates [51]. This system and Fermi fields are tools that work very well for general cases, where a given tube is a submanifold of any Riemannian manifold M. So it is a mathematical construct with a very broad field of application. Understanding these concepts requires a description of the geometry of a underlying Riemannian manifold M in a neighbourhood of the mentioned tube, which is considered as a submanifold of M.

Probably, the normal coordinates system is more familiar to readers, as a tool that can be used for describing the behavior of a geodesic. Normal coordinates are based on the principle of exponential mapping. Coordinates at a point x in a differentiable (Riemannian) manifold M equipped with a affine connection, which is symmetric, the normal coordinates forms a local coordinates system in a neighbourhood of a given point x [51]. The specific way and how to raise these coordinates lies in the before mentioned exponential map to the tangent space at the point x. Here, we are talking about the tangent space of a manifold, which reflects a generalization of vectors from affine spaces. More precisely, geodesic normal coordinates are local coordinates on a manifold M with a affine connection, that is given by the exponential mapping $exp_x : T_xM \supset V \to M$ [51]. Also an isomorphism is required $\omega : \mathbb{R}^d \to T_xM$. This map is defined by any basis of the space T_xM. If the existence of the Riemannian metric on the manifold M is required, then the basis can be defined as orthonormal. This provides better properties of normal coordinates within the specific computational methods.

Let's go back to the concept of general curvilinear coordinates, because on the basis of the foregoing, it can be argued that Fermi coordinates are a generalization of normal coordinates. Let us therefore give a formal definition of Fermi coordinates.

Definition 5 ([51, Section 2.1]). *The Fermi coordinates* $(y_1, \dots, y_d)$ *of a submanifold* $N \subset M$ *centered at point* p *are defined by*

$$y_a\left(exp_\sigma\left(\sum_{j=q+1}^{d} t_j E_j(p')\right)\right) = y_a(x') \quad (a = 1, \dots, q), \tag{48}$$

$$x_i\left(exp_\sigma\left(\sum_{j=q+1}^{d} t_j E_j(p')\right)\right) = t_i \quad (i = q+1, \dots, d), \tag{49}$$

for $p' \in V$, *provided the numbers* $t_{q+1}, \dots, t_d$ *are small enough so that*

$$\sum_{j=q+1}^{d} t_j E_j(p') \in O_N.$$

Here, σ denotes the tangent bundle. O_N is a subset of the tangent bundle σ defined as the largest neighbourhood of the zero section of σ. For which $exp_\sigma : O_N \to exp_\sigma(O_N)$ is a diffeomorphism. The index q refers to the arbitrary system of coordinates $(y_1, \dots, y_q)$ defined in a neighbourhood $V \subset N$ of the point $p \in N$. Orthonormal sections are also linked to this arbitrary coordinates system as $E_{q+1}, \dots, E_d$ of the restriction of the tangent bundle σ to V.

We know that the exponential map exp_σ is a diffeomorphism on O_N. It can then be argued that equations (48) and (49) define a proper coordinate system near the point p. The meaning of these equations is that they provide a mechanism for construction of the local coordinates that are adapted ideally to a geodesic. More precisely, for small t, the coordinates $(t, 0, \dots, 0)$, represents the geodesic near the point $p \in V$.

The nature of the Fermi coordinates is that they allow not only the construction of an exactly defined local system that is ideally adapt to the behavior of geodesic, but mainly provides a deeper understanding of the geometry of the general Riemannian manifold M. To put it simply, Fermi coordinate system provides a measurement of the geometry of M in a neighbourhood of a submanifold N [51]. Which tells a lot about the topology of M. Because of equations (48) and (49) provide measurement of the given geometry, then it does not matter at all for a particular choice of coordinates on the submanifold N. The described situation depends only on the selected system being the normal coordinates.

Now, we briefly mention two simple results, which emerges automatically from the previously mentioned facts. If we have a system of Fermi coordinates $(y_1, \ldots, y_d)$, which are centered at point p and defined by the relations (48) and (49), then the restrictions of the coordinate vector fields to the submanifold N are orthonormal. By the restrictions of the coordinate vector field, we mean the field $\frac{\partial}{\partial x_{q+1}}, \ldots, \frac{\partial}{\partial x_d}$ [51, Lemma 2.4]. This is obvious, because the Fermi coordinates are generalized normal coordinates, where we can afford the orthogonality property because of the existence of metric. The second result is not so obvious. It concerns the formulation of direct relations for Fermi coordinates of geodesic.

Consider a unit-speed geodesic, which is normal to the submanifold N. Its star is situated in point p. Let us denote the geodesic as α.Then $\alpha(0) = p$ and $v = \dot{\alpha}(0)$. Under such conditions, there exist the Fermi coordinates, that for small t satisfies the following relations

$$\frac{\partial}{\partial x_{q+1}}_{(\alpha(t))} = \alpha'(t), \tag{50}$$

and

$$\frac{\partial}{\partial x_{a}}_{(p)} \in N_p, \quad \frac{\partial}{\partial x_{i}}_{(p)} \in N_p^{\perp} \tag{51}$$

which are fulfilled for $1 \leq a \leq q$ and $q + 1 \leq i \leq d$ [51, Lemma 2.5].

The specific choice of Fermi coordinates was based on the previous relations. Local parametrization looks like this

$$\begin{gathered} \phi : I \times B_a \to \mathbb{R}^d, \\ (s,t) \to \gamma(s) + t_\mu e_\mu(s), \end{gathered} \tag{52}$$

where $B_a := \{t \in \mathbb{R}^{d-1} : |t| < a\}$ is the $(d-1)$-dimensional ball, $t := (t_2, \ldots, t_d)$ [45, Section 2.1]. Within the Greek indices $2, \ldots, d$, we use a Einstein summation convention. Relation for d-dimensional formulas for Frenet-Serret frame [53], gives us an insight into the properties of the metric. Because with the frame, we can find that the metric $G := \nabla\phi \cdot (\nabla\phi)^T$ has the diagonal form as $G = diag(J^2, 1, \dots, 1)$. Where the first term J represents the Jacobian

$$J(s,t) = 1 - \kappa_\mu(s)t_\mu. \tag{53}$$

The tube Ω_a does not overlap itself, then J must be positive, so $a\|\kappa\|_\infty < 1$. In the term of Jacobian, the concept of Jacobi field is directly encoded.

As we have mentioned before, the Jacobi field is very important tool in topology and geometry, which provides the description of the difference between the behavior of the reference geodesic and a very close geodesic. It forms a vector fields (Jacobi fields) along the geodesic. These vector fields, provides the tangent space to the geodesic. In our description, Jacobi field is the first term in diagonal metric, and at the same time it is Jacobian of the local parametrization of the tube. Jacobi field can be generally defined using the following Theorem.

Theorem 10 ([5])**.** *Let M be a Riemannian manifold equipped with the metric g, with non-positive sectional curvature. Then any geodesic, which forms a submanifold on M is locally minimizing.*

Assume that γ is a geodesic and l is a geodesic variation of γ. Let J be its variational field. Then we obtain $\nabla_{l_t} l_t = \nabla_{\dot{\gamma}_s} \dot{\gamma}_s = 0$. The commutator of l_t and l_s is identically zero. Thus $\nabla_{l_t}\nabla_{l_t} l_s = \nabla_{l_t}\nabla_{l_s} l_t = -\nabla_{l_s}\nabla_{l_t} l_t + \nabla_{l_t}\nabla_{l_s} l_t + \nabla_{[l_s,l_t]} l_t = R(l_s,l_t)l_t$. Take $s = 0$, then we have the following relation

$$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}} J + R(\dot{\gamma}, J)\dot{\gamma} = 0. \tag{54}$$

R denotes the Riemannian curvature tensor, $\dot{\gamma}$ is a tangent vector field and $\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}$ corresponds to the covariant derivative with respect to the Levi-Civita connection [50]. This equation forms the essence of the Jacobi field concept since a vector field J along a geodesic γ is called a *Jacobi field* if it satisfies the equation (54). In other words, the variational field of a geodesic is a Jacobi vector field. Many important theorems are closely related to this tool, such as the existence of a unique Jacobi field along the geodesic or the theorem, which states that the set of Jacobi fields along the geodesic forms a linear space of double dimension compared to the dimension of the underlying manifold [54]. All these findings can be deduced directly from the definition itself (54) and from the properties of the Riemannian curvature tensor. In our case, we specialize in only one particular property of Riemannian tensor that gives us an explicit prescription of Jacobi field.

Because our manifold is $\mathbb{R}^d$, then the sectional curvature is identically zero. Then we know from the identity of fundamental forms for curvature [54] that the Riemannian tensor is given by

$$R(J,Y)Z = K^M(\langle J,Z\rangle - \langle Y,Z\rangle J). \tag{55}$$

From this equation we obtain the relation for normal Jacobi field, which is defined as a perpendicular field to $\dot{\gamma}$ along γ. Normal Jacobi field can be directly computed from

$$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}J + K^M J = 0. \tag{56}$$

To solve this equation, we have to take the orthonormal basis $\{e_i(t)\}$ of the tangent space $T_\gamma M$, such that each $e_i(t)$ is parallel along the geodesic γ and $e_1 = \dot{\gamma}$. Naturally we suspect that the Jacobi field will be given as a linear combination of particular fields and the components of the base of the tangent space. Then,

$$J = \sum_{i=2}^{d} J^i e_i.$$

This will give us the equation for coefficients J^i as follows [54]

$$\ddot{J} + K^M J = 0, \qquad 2 \le i \le d. \tag{57}$$

Initial conditions are required to obtain a specific solution to this equation. Their choice depends purely on us and on the particular description of our problem. The conditions are given as follows $J(\cdot\,,0) = 1, \ddot{J}(\cdot\,,0) = -\kappa$. Under these conditions, the solution of Jacobi equation is

$$J(s,t) = \begin{cases} cos\left(\sqrt{K^M}t_\mu\right) - \dfrac{\kappa_\mu(s)}{\sqrt{K^M}} sin\left(\sqrt{K^M}t_\mu\right), & K^M > 0 \\ 1 - \kappa_\mu(s)t_\mu, & K^M = 0 \\ cosh\left(\sqrt{|K^M|}t_\mu\right) - \dfrac{\kappa_\mu(s)}{\sqrt{|K^M|}} sinh\left(\sqrt{|K^M|}t_\mu\right), & K^M < 0 \end{cases}, \tag{58}$$

where K^M denotes the constant sectional curvature of manifold M. If $M = \mathbb{R}^d$, that the sectional curvature is zero, and we have $J(s,t) = 1 - \kappa_\mu(s)t_\mu$. As we can see, this relations, is identical to the Jacobian of the map (52). We have verified that the Jacobi vector field, which is represented in the metric G, stems from the topological nature of the manifold $\mathbb{R}^d$ and at the same time, it is a Jacobian of our local parametrization, which we accomplished using the system of Fermi coordinates. This is the reason why it is possible to use the difference between the reference geodetic Ψ and arbitrary close geodetic for estimation of the upper bound of $h(\Omega_a)$.

Now we can compute the volume of the section $\Omega_a^I := \phi(I \times B_a)$ of the tube Ω_a. For every $r \in (0, a]$, we can integrate the obtained Jacobi field (58) over the local parametrization, which give us

$$|\Omega_r^I| = \int_I \int_{B_r} J(s,t)dt\, ds = |I|\,|B_r|, \tag{59}$$

where the second equality follows by the fact that 0 is the center of mass of the ball B_r [45]. This implies that $\int_{B_r} t\, dt = 0$. Here $|I|$ denotes the length of the interval (section of Ω_a) and $|B_r|$ represents the volume of the $(d-1)$-dimensional ball.

Let $|\partial\Omega_r^I|$ represents the $(d-1)$-dimensional Hausdorff measure of the surface that is defined by $\phi(I \times \partial B_a)$. It is possible to express $|\partial\Omega_r^I|$ as the derivative of the volume $|\Omega_r^I|$ with the respect to the variable $r \in (0, a]$ [3, Lemma 3.13]. We can also scale the volume of the ball B_r as the dimension goes up $|B_r| = |B_1| r^{d-1}$. By this scaling we get

$$|\partial\Omega_r^I| = \frac{d}{dr}|\Omega_r^I| = (d-1)r^{d-2}|I||B_1| = \frac{d-1}{r}|I||B_r|. \tag{60}$$

The reference geodesic Ψ is parametrize by its arc-length (locally as γ) by which it can be argued that formulas (59) and (60) can be extended to global form [45]. Then we have the following identities

$$|\Omega_a^I| = |\Psi||B_a| \quad \text{and} \quad |\partial\Omega_a^I| = \tfrac{d-1}{a}|I||B_a|.$$

At the beginning of the section 2.1, we mentioned that we would choose the whole tube as the test domain. Let then $S := \Omega_a$ in (1), we therefore get the desired upper bound [45, Section 2.1]

$$h(\Omega_a) \leq \frac{d-1}{a}. \tag{61}$$

7.2 Proof of Theorem 9: the lower bound

In order to prove the lower bound of (47), we must move on to the different parametrization of the tube Ω_a. The following parametrization is based on a different geometric view of the tube, since we use the vector field method. By this field we try to cover the relevant parametrization so that the assumptions in Proposition 1, are met. Let us define a map

$$\tilde{\phi} : I \times U \to \mathbb{R}^d,$$
$$(s, \theta) \to \gamma(s) + a\sigma_k(\theta) e_k(s),$$

where $\sigma : U \to S_+^{d-1} \subset \mathbb{R}^d$ is a parametrization of the half-sphere $S_+^{d-1} := \{x \in \mathbb{R}^d : |x| = 1 \wedge x_1 > 0\}$ and for $\theta := (\theta_2, .., \theta_d) \in U$ is possible to choose the hyperspherical coordinates [45]. Again, we apply the Einstein summation convention to the indices $1, .., d$. Using the formulas for Frenet-Serret frame in d- dimensions [53], we obtain the Jacobi matrix

$$J := \nabla\tilde{\phi} = \begin{pmatrix} (1 - a\sigma_\mu \kappa_\mu) e_1 + a\sigma_1 \kappa_\mu e_\mu \\ a\partial_2 \sigma_k e_k \\ \vdots \\ a\partial_d \sigma_k e_k \end{pmatrix},$$

where the terms e_k are arranged as row vectors. The following choice of the vector field is constructed according to parametrization $\tilde{\phi}$.

Let $x \in \mathbb{R}^d$ and $(s, \theta) \in I \times U$, then the relationship between x and the parametrization is given by $x = \tilde{\phi}(s, \theta)$. These assumptions allow us to define a vector field that reads, locally,

$$V(x) := \frac{x - \gamma(s)}{a} = \sigma_k(\theta) e_k(s), \tag{62}$$

where the first assumption of Proposition 1, is fulfilled, because $|V| = 1$ [45, Section 2.2]. The remaining step is to compute the divergence.

As we have already mentioned, the general relation for divergence operator on the Riemannian manifold is given by (40). So, if we take the first equality of (62), we get

$$div(V) = \frac{1}{a}(d - e_1 \cdot \nabla s), \tag{63}$$

where s represents the first component of the inverse $\tilde{\phi}^{-1}(x)$ [45]. Now we need to compute the inverse to the gradient of the parametrization $\tilde{\phi}$.

Using that $|\sigma|^2 = 1$, so that $\sigma \cdot \partial_\mu \sigma = 0$ for all indices $\mu \in \{2, \dots, d\}$, then

$$J^{-1} = \left(\frac{\sigma_k e_k}{\sigma_1}, *_2, \dots, *_d\right),$$

where the components e_k and $*_\mu$ are arranged as column vectors. From this formula we can deduce that the gradient of the first component of inversion $\tilde{\phi}^{-1}(x)$ is given by the relation

$$\nabla s = \frac{\sigma_k e_k}{\sigma_1}$$

and thus

$$div(V) = \frac{d-1}{a}. \tag{64}$$

It is clear that all the prerequisites and essence of Proposition 1, are met. So we therefore get the desired lower bound [45, Section 2.2]

$$h(\Omega_a) \geq \frac{d-1}{a}. \tag{65}$$

Note that the upper (61) and lower bound (65) coincide. Which implies that we have just established (47) in Theorem 9. However, this is not the only fact that the proof provides. It is obvious from the way how (47) was proven that the minimizer in (47) corresponds to the tube itself. Thus, in the formalism of the Cheeger problem, the Cheeger set is identical to the d-dimensional *curved tube,* so $C_{\Omega_a} = \Omega_a$ [45]. This completely concludes the proof of our original hypothesis.

It is interesting to note the case, when the reference geodesic of the tube is not finite, it will be complete, but not bounded. So, let us now consider the same definition of the tube Ω_a , where the geodesic Ψ is an unbounded complete curve. Then proceed in the same way as in the previous case. Consider the same test vector field as in Section 2.2. Then this field yields the lower bound (65). So let's move to the upper bound. Here the situation at the first glance appears the same. Choose the test domain $S := \phi(I \times B_a)$ in the relation (1). We run an identical local algorithm on this domain and obtain the following upper bound

$$h(\Omega_a) \leq \frac{\frac{d-1}{a}|I||B_a| + 2|B_a|}{|I||B_a|}.$$

If we send the length $|I|$of the tube to infinity, then we get the desired equality (47) as in Theorem 9. It is therefore clear that the situation within the explicit formula of the Cheeger constant (47) is the same for both closed, smooth Ψ and unbounded and complete Ψ. However, the situations, does not coincide with the achievability of an infimum in (1). Then in the case of *unbounded tubes*, there is no Cheeger set C_{Ω_a} [45. Remark 2.3].

7.3 Alternative proof of Theorem 9

The formulation of the alternative proof is based on the previously mentioned problem (23) in section 4.1. The problem is often called a torsional problem [26]. We know that the Cheeger constant can be generally defined as the first eigenvalue of p-Laplacian, according to relation (17). Using this definition, we formulate the problem (23), where we set $\Omega := \Omega_a$. So the domain directly corresponds to the *curved tube*. For practicality, let's reiterate some facts related to solution of the torsional problem.

Let there exist a classical solution u of (23), then also the function $f(u)$ solves the corresponding equation. But we must assume that the function is continuous in the sense of Lipschitz definition of continuity. We can rewrite the p-Laplacian in intrinsic coordinates, if we make an assumption that $|\nabla u| \neq 0$ in $U(j)$, where $j \in \Omega$.
So, the intrinsic coordinate interpretation looks like this

$$\Delta_p u = (p-1)|\nabla u|^{p-4}\langle D^2 u \nabla u, \nabla u\rangle - |\nabla u|^{p-2}(d-1)H(j)|\nabla u|.$$

Take $\lim_{p\to 1} \Delta_p u = -|\nabla u|^{-1}(d-1)H(j)|\nabla u|$. Then the equation (23) transforms into simply solvable equation $(d-1)H(j) = h(\Omega)$. This equation determines the Cheeger constant of Ω by the mean curvature of the level surface of u in a given point $j \in \Omega$. We made the choice that Ω is identical to *curved tube*. For the reasons of the alternative proof, it is sufficient to determine the type of level surface with the respect to Ω_a and to the equation (23). Then it just remains to compute the mean curvature $H(j)$ of the corresponding level surface. The equation for $h(\Omega)$ says that every level set $\Omega_e := \{\, j \in \Omega : u(j) > e\}$ has a boundary with constant mean curvature $h(\Omega)$ independent of e [10].

The quantity $H(j)$ represents the mean curvature of the level surface of the function u in j. By definition, the level surface is given as $L_c(u) := \{j : u(j) = c\}$. By the term surface we mean a clear categorization of the number of variables of the function u. In the case of surface, u can be considered as a function of three variables. Within problem (23) our three variable function $u = 0$ on $\partial\Omega := \partial\Omega_a$. Under these conditions, the mean curvature $H(j)$ of a surface can be specified by an implicit equation $u = 0$ on the given surface $\partial\Omega_a$. Then H can be obtained by unit normal $\nabla u/|\nabla u|$. Which provides the divergence formula for the mean curvature $H(j) = -\frac{1}{2} div\left(\frac{\nabla u}{|\nabla u|}\right)$. Comparing this equation and the first equation in (23) we find that

$$-2H(j) = div\left(\frac{\nabla u}{|\nabla u|}\right) = \lambda_1(\Omega) = h(\Omega).$$

However, the divergence formula for $H(j)$ applies only to surfaces given by the implicit equation $u = 0$, which in our case is satisfied for the surface $\partial\Omega_a$. The level surface is defined in j, so we are interested in the solution of (23) that has the form $u(j) = c$, where $c = const$. Such solution of the equation (23) can be found only on Ω_a. Since $u = 0$ on $\partial\Omega_a$. We know that the formula $H(j) = -\frac{1}{2} div\left(\frac{\nabla u}{|\nabla u|}\right) = -\frac{1}{2} h(\Omega)$ applies only for the implicit surface $u = 0$. Then it is straightforward that the level surface corresponds to specific solution of (23) on the *curved tube*. From the geometrical situation related to Ω_a itself, it can be clearly conclude that the mean curvature of such level surfaces is directly equal to the inverse of the width of the *curved tube*. Thus

$$H(j) = \frac{1}{a} \text{ of } L_c(u);\ u(j) = c \text{ on } \Omega_a. \tag{66}$$

Employing the equality (66), we have

$$(d-1)H(j) = \frac{d-1}{a} = h(\Omega_a). \tag{67}$$

This relation directly corresponds to our result from Theorem 9.

7.4 Spherical shells

Another interesting result refers about the Cheeger constant of spherical shells. Let the spherical shell by defined as $A_{r,R} := \{x \in \mathbb{R}^d : r < |x| < R\}$ for $r, R > 0$ [45]. Then we can formulate the Cheeger problem in the following theorem.

Theorem 11 ([45]). *Given two positive values $r < R$ and the spherical shell defined as mentioned. Then the spherical shell itself corresponds to minimal Cheeger set, and*

$$h(A_{r,R}) = d\frac{R^{d-1} + r^{d-1}}{R^d - r^d}. \tag{68}$$

A detailed proof of Theorem 11 is given in [45, Section 7.3]. Perhaps more interesting, than the proof itself is the similarity of Theorem 9 and 11. If we assume within Theorem 11 that the closed smooth curve Ψ in $\mathbb{R}^d$ is identical to circle and set $d = 2$, then we get the result in Theorem 9. In both cases, we restrict the generality. As in the case of Ψ or in the case of dimensionality of the underlying manifold. By this we mean that by the particular choice of smooth closed curve Ψ and dimensionality d, various interesting problems can arise. Note also that the result of the Cheeger constant of spherical shells extends the result for annuli to higher dimensions [31].

Generally speaking, our main Theorem 9 provides some insight into the formulation of the problem whose purpose is to determine the Cheeger constant (1) of tubular neighbourhoods of general submanifolds of $\mathbb{R}^d$. Which leads us to present a new open problem as an extension of Theorem 9 to a d-dimensional Riemannian manifolds with constant sectional curvature.

Conjecture 1 (Constant curvature manifolds). *Let M be a d-dimensional Riemannian manifold with constant curvature $K_{sec}^M = const$. Then the Jacobi field equation* (57) *admits explicit solutions, which is given by* (58). *Next, let Ω_a be the d-dimensional a-tubular neighbourhood of a closed smooth geodesic on M. Then the Cheeger constant of Ω_a is given by*

$$h(\Omega_a) = \begin{cases} \sqrt{K_{sec}^M}(d-1)a^{d-2} cotg\left(\sqrt{K_{sec}^M} a^{d-1}\right) \ , \ K_{sec}^M > 0 \\ \dfrac{d-1}{a} \ , \ K_{sec}^M = 0 \\ \sqrt{|K_{sec}^M|}(d-1)a^{d-2} cotgh\left(\sqrt{|K_{sec}^M|} a^{d-1}\right) \ , \ K_{sec}^M < 0 \end{cases}.$$

Obviously, the case $K_{sec}^M = 0$ reduces to proof of Theorem 9. Notice also, that the positive, respectively negative curvature cases contain the Cheeger constant of *curved tubes* on manifolds, with $K_{sec}^M = 0$, since

$$(d-1)a^{d-2} = \left(\frac{d-1}{a}\right) a^{d-1}.$$

Let us denote $\widetilde{h(\Omega_a)} = (d-1)/a$, then

$$h(\Omega_a) = \begin{cases} \sqrt{K_{sec}^M}\ \widetilde{h(\Omega_a)} a^{d-1} cotg\left(\sqrt{K_{sec}^M} a^{d-1}\right) \ , \ K_{sec}^M > 0 \\ \widetilde{h(\Omega_a)} \ , \ K_{sec}^M = 0 \\ \sqrt{|K_{sec}^M|}\ \widetilde{h(\Omega_a)} a^{d-1} cotgh\left(\sqrt{|K_{sec}^M|} a^{d-1}\right) \ , \ K_{sec}^M < 0 \end{cases}.$$

Redundant terms in the relations for $K_{sec}^{M} \neq 0$ corresponds to the contributions of the geometry and topology of the given manifold. This implies that the probable nature of the proof of Conjecture 1, could be potentially based on the natural topological and geometrical properties of M. The methodology of the proof may therefore be fundamentally different.

However, there should not be such a dramatic difference. Since the potential proof of the upper bound of Conjecture 1 should be identical with the proof of Theorem 9, except for the use of a different Jacobi field, according to the relation (58). In particular, the proof should therefore differ in estimating the lower bound using the vector field method. The method itself will be preserved, but the explicit formula for vector field will of course be different. There could also be a possibility where the proof loses the divergence of the sought vector field, but can therefore be based on the alternative approach from section 7.3.

Based on the previous considerations, it is possible to extend Conjecture 1, to the most general level, namely to the Riemannian manifolds M with variable curvature K^{M}. In any case, it is also possible to consider the generalization of Conjecture 1 not only within the curvature, but also within the type of manifold. An interesting example could the Kähler manifolds. The reason lies in their interesting structure (see, [56]). This could drastically affect the geometrical behavior of $h(\Omega)$. From the point of view of the methodology of potential proof, the structure of comparison theory within the Riccati differential equation, or shape operator on manifolds appears to be applicable [51].

8 Simple solvable models

Let us take a brief look at simple solvable models within the Cheeger constant. In other words, we will be talking about the basis of several examples of *curved strips* about circles and circular arcs. The following facts can be directly deduced from Theorem 9, if we modify the appropriate set Ω_a to the extent that we receive the desired geometric shape.

8.1 Annulus

Probably the simplest solvable model is given by strips built about full circles, i.e. annulus. Annulus is one of the few non-convex geometric shapes, where the Cheeger set is the strip itself and the Cheeger constant is equal to the half of the distance between the boundary curves of annulus [31]. The question remains what is the constant and the minimization set in the general case. Well, we proved that in the case of *curved strips*, the Cheeger constant only depends on the width of the strip, irrespectively of the curvature of the curve. More precisely, the Cheeger constant is the inverse of the half-width. It follows from this argumentation that the situation remains unchanged for general curve annuli, the Cheeger constant equals the half of the distance between the boundary curve. Special cases of curved annuli are discs with various properties. One of them is a disk without the central point. These types of discs, has the same minimization Cheeger set and Cheeger constant as the discs with central point. Plus, if we take a limit for $\delta \to 0_+$ of the annulus which is built about the circle of radius $a + \delta$, the former set can be understood as this limit case. In other words, annulus and the disc can be considered as its limit case in the sense of the Cheeger set and Cheeger constant [47].

8.2 Rectangles

Rectangles are again specific cases of *curved strips* that are built about the planar segment, which is defined as $\Psi := (-b, b) \times \{0\}$. The rectangle is formally defined as a multiplication of two intervals $R_{a,b} := (-b, b) \times (-a, a)$, with $a, b > 0$. Using an explicit formula from Theorem 5, for the Cheeger constant of general bounded convex subsets of $\mathbb{R}^2$, we get the following equality [23]

$$h(R_{a,b}) = \frac{a + b + \sqrt{(a-b)^2 + \pi ab}}{2ab}.$$

From the general properties of the Cheeger constant we know that a given mechanism within convex sets does not like corners. This implies, that the Theorem about Cheeger constant of convex sets [23, Theorem 3] determines the Cheeger set of $R_{a,b}$ as the rectangle, where its corners are rounded off by circular arcs of radius $h(R)^{-1}$. Note also the scaling of the Cheeger constant. Consider that the base of the rectangle is 1 and the vertical side is b/a, then scaling can be observed as $h(R_{a,b}) = h(R_{1,b/a})$ [47, Section 4].

If we take a better look at $h(R_{a,b})$, we find that it is possible to divide it into two parts. It is a connection of two concepts that we have proved. So, the Cheeger constant can be written as follows

$$h(R_{a,b}) = \frac{1}{a} + \frac{k(a,b)}{|\Psi|}, \qquad k(a,b) = \frac{a - b + \sqrt{(a-b)^2 + \pi ab}}{a},$$

where $|\Psi| = 2b$ [47, Section 4]. Notice, that the first part is made up of the Cheeger constant of *curved strips* in $\mathbb{R}^2$ and the second part is an algebraic complement whose essence lies in the geometric difference between the *curved strip* and rectangle $R_{a,b}$. Thus, it can be argued that the Cheeger constant of $R_{a,b}$ in a plane is a combination of two geometries that look different, but in a result, they are unifying. It can be seen that the above mentioned scaling also applies in the split relation $k(a,b) = k(1, b/a)$. This scaling directly implies that a map $b/a \to k(a,b)$ represents a decreasing function with the a following limit $k(a,b) \to 2$ for $b/a \to 0$. And $k(a,b) \to \pi/2$ as the ratio $b/a \to \infty$.

Identically, Theorem 5 can be used to compute the Cheeger constant of square or triangle [23]. We can also obtain the shape of their Cheeger sets by „ rounding the corners “.

8.3 Finite curved strips

Here we only briefly mention the Cheeger constant of *finite curved strips* as a complement of the section 7, where we provide a proof of the Cheeger constant of *curved strips* in $\mathbb{R}^2$ and the generalization, which is the Cheeger constant of *curved tubes* in $\mathbb{R}^d$.

Theorem 11 ([47]). *Let Ψ be a non-complete a finite curve, hence the structure $\Omega_{\Psi,a}$ is a finite curved strip. Then there exists a positive constant c such that*

$$\frac{1}{a} + \frac{c}{|\Psi|} \leq h(\Omega_{\Psi,a}) \leq \frac{1}{a} + \frac{2}{|\Psi|}. \qquad (69)$$

One can put $c = 1/400$. The infimum in (1) is achievable for some connected set $C_{\Omega_{\Psi,a}} \subset \Omega_{\Psi,a}$.

For simplicity, we present Theorem 11 without proof. Very interesting is the comparison between the Cheeger constant of the rectangle $R_{a,b}$ and the *finite curved strip* $\Omega_{\Psi,a}$, where the reference curve Ψ is non-complete and finite. Specifically, these are the properties of the above mentioned limits in relation for $h(R_{a,b})$, which is differentiated into the Cheeger constant of the *curved strip* (28) in $\mathbb{R}^2$ and the algebraic complement $k(a,b)$. Because these limits directly imply that the upper bound of Theorem 11 becomes sharp in the limit of very narrow rectangles [47].

8.4 Link between graphs with mean curvature and the Cheeger constant

There is also an area of solvable models that deviates sharply from the mentioned cases. The field of application lies in the broad area of discrete mathematics, specifically in graphs with prescribed mean curvature and their existence [13]. The analysis of this problem is essentially based on modified quasilinear partial differential equation, which is to some extent closely related to the equation from the section 3.5. The modification makes the original equation transform to a nonlinear elliptic partial differential equation. And the right hand side is equal to prescribed mean curvature of the graph. For more details, see [55].

To recapitulate, in the first part of our work, we have dealt here purely with the Cheeger problem, its general properties and especially the Cheeger constant (1) of specific geometric domains. The most important point of the examined problems was Theorem 9, which basically formed a key hypothesis. Equally important was also the proof of (47). However, it is worth it to pay the same attention to alternative proof of (47), because in the verification of validity of Theorem 9, the alternation connects various geometric concepts and the definition of the Cheeger constant using the smallest eigenvalue with the formulation and solution of the variational problem (23). Which, in my opinion, provided a deeper understanding of the nature of the main hypothesis about *curved tubes*. The reason sought lies in the elegant interconnection of closely related problems and formulations, which we gradually touched on in the sections 2.2, 2.3 and 4.1.

In the second part of our work, which now follows, we will focus mainly on the application of the mentioned theorem, its proof and other consequences. These consequences show a close coherence with the information give in the first part. The application falls into the field of string theory and freely passes to the most fundamental variational problem of the modern view of theoretical physics. Its specification will be gradually introduced and discussed.

Chapter 2

9 Application in string theory

9.1 Introduction and historical development of string theory

String theory is considered as one of the most interesting, challenging and controversial areas of modern theoretical physics and mathematics. The main reason for building and creating the structure of string theory was the desire for a unified theory of strong nuclear interaction. This happened in 1960's [57]. But everything is not as it seems. In the sense that some structures that were reintroduced in string theory arose much earlier. In particular, some constructs already appeared in classical unification field theory that was first introduced by Albert Einstein. Within the popular conception of string theory, it is generally known that this potentially unifying theory works with multidimensional models of reality. The concept of multidimensional theories goes back to the period of general relativity as a unifying theory of gravity, because this theory was the first to add the fifth dimension [58]. The reason was the unification of gravity and electromagnetism, which was mathematically possible in five dimensional model. The basic problem of extra dimensions was that no one was quite sure what the topological and geometric properties these mathematical structures had. Thus no one had any idea of the physical consequences of the mentioned properties. Everything was partially corrected in 1920's, when Oskar Klein was the first to provide a physical interpretation of artificially created extra dimensions [59]. From topological and geometrical point of view, extra dimensions were incorporated into an infinitesimally small circle in multidimensional spacetime. Another achievement in the pre-string era was, for example, the introduction of scalar components and a non-symmetrical metric tensor into theory of gravity. All these findings, which were discovered especially in the first and early second half of the twentieth century, were extensively used partially redefined in the string theory apparatus [57]. The most important impetus, was that not only did the mathematical understanding of these tools change, but also the understanding of the physical context and consequences that are the building blocks of current string theory.

Let's move back to the times of the string theory creation. Mainly, it was realized that the spectrum of a fundamental string contains an unknown massless spin-two particle [60]. However, string theory did not prove adequate to describe the essence of strong interaction. The theory that shows as correct was Quantum chromodynamics, that also contained the description of hadrons [57]. Here, again, we will refer to the historical background of the

1940’s, because at that time the S-matrix theory was developer by Werner Heisenberg. Its essence lay in the non-local understanding of spacetime. It was believed that the local notions of spacetime break down at the scale of nuclear force. This theoretical approach gave birth to ideal platform for a theory of quantum gravity [61]. Again, these facts opened new doors for string theory, because it was proposed to identify the previously mentioned massless spin-two particle in the spectrum of strings with the graviton. The field particle that is probably responsible for gravitational interaction. Predicting the existence and identification of graviton gave string theory the necessary boost and raised it to the level of a potential unifying theory. String theory also developed into one of the most interesting theories of high-energy physics. Thus, it can be argued that the string theory has gradually evolved into a competent candidate for a quantum theory of gravity unified with other forces [57].

The detail understanding of inner concepts of string theory has evolved over decades. But in some periods of history, the scientific progress in string theory was much more rapid than in others. Progress can be differentiated into two streams, the first and the second superstring revolution [57]. It is worth noting that the first superstring revolution was precede by concepts of tachyons, open and closed strings, bosonic string theory in 26 dimensions, string field theory and many other brilliant ideas. Of particular note is the generalization and modern formulation of the path integral that was discovered by Alexander Polyakov [62].

The first superstring revolution was led by one of the best theoretical physicist and mathematician of our time, Edward Witten. The essence of his discovery is based on the proven permission that the most potential theories of quantum gravity are not able to accommodate fermions like neutrino. This led Witten and other theoretical physicist to study possible violations of the conservation laws in quantum theories of gravity with some particular inconsistent parts [57]. Subsequent studies gave birth to the first superstring revolution as Witten was a scientist of considerable renown, causing many scientists to work in the field of string theory. In the period of the first superstring revolution, it was confirmed that the specific group within the closed strings theory reflects the laws of the standard model [60]. Another problem that has been solved is the identification of a particular types of topological manifolds, which in a certain sense are responsible for compactification of extra dimensions. These topological manifolds also able to preserve a realistic amount of supersymmetry. We are talking about the Calabi-Yau manifolds [63]. The period of the first superstring revolution was not only positive in the sense of discoveries. Specifically, it was a perturbation string theory, in which some unexpected divergences appeared [57]. When compared to field string theory, it turned out that the relevant aspects of the theory diverged much faster. So the theory lacked new non-perturbative objects. If the string theory provides a solution to gravitational singularities, these can only be achieved by using a higher-dimensional objects that are called *D*-branes. It is possible to identify these branes with solutions for black holes. Here we refer to the mentioned problems with perturbative divergences. *D*-branes are considered as objects that are suggested by these divergences [64]. The true conception on which the string theory actually stands, comes to light. The main content is based on mathematical properties of *D*-branes and other type of brane objects, that directly implies the physical interpretation of the string theory.

One of the most important contributions to string theory and unifying theories in general is the holographic principle [65]. With the framework of string theory, Leonard Susskind postulated the holographic principle for the first time. Susskind's postulate of the holographic principle provides equivalence between the long highly excited string states with thermal black hole states [66]. But this is not the only consequence that is essential for understanding that the excited states of strings correspond to thermal states of black holes. By the way, the same principle applies to *D*-branes whose physical interpretation corresponds to specific types of black holes [57]. So, the main consequence of the holographic principle is that it describes correctly the degrees of freedom of the black hole, using the world-sheet or world-volume theory. Our topic is closely related to this particular part of string theory, since our application of the Cheeger constant is based on the concepts of world-sheet, world volume and holographic principle. In 1990's a second superstring revolution came. Edward Witten provided a unifying description of the five string theories that existed at that time [67]. He showed that within the eleven-dimensional model, these five theories can be considered as an identical picture of a single complex whole. This higher-dimensional manifestation is called M-theory [67]. Within this new theory, the holographic principle, which worked using certain types of branes, was also extended. Subsequent developments have made desired interconnections between different kinds of theories, such as cosmology, particle physics phenomenology and various types of conformal field theories. Despite all expectations, string theory provides very important results also in the field of pure mathematics [68].

Like many modern and developing ideas in theoretical physics, string theory originally had no consistent and rigorous mathematical formulation or framework, where all of its essential concepts and theorems can be defined precisely. Most parts of the rigorous structure of string theory was built on the basis of particular conjectures, which were stated individually for some parts of the theory and then proven. By gradual establishment of connections between the individual blocks of theory, the rigorous constructions of string theory were formulated with the help of proof, thus creating new results in various parts of mathematics as topology, symmetries, conformal field theories, various kinds of algebras, number theory, especially modular functions and group theory [57].

As we have already mentioned before, the application of the proven Theorem 9, is carried out in the context of the world-sheet, world volume, holographic principle and the underlying topological nature of these concepts. Specifically, we will later get into the problems of closed strings action, topology and geometry of the given spacetimes, the context of p-branes and other very interesting things that appear in connection with proven Theorem 9. The consequences of our application will also gradually move us to the core of Hamilton's principle of least action. Or as this principle is sometimes called, the Hamilton variational principle. Since the Cheeger problem itself is based on a variational analysis, the connection with the geometric nature of Hamilton's principle within string theory will be most obvious. Therefore, let us analyze the knowledge and facts we need for a detailed analysis of the Cheeger constant and its application in the mentioned areas of string theory.

9.2 Main features and shortcomings of string theory

We will briefly comment on the shortcomings of the theory, because it is on this information that we build a possible summary of the basic building blocks of the theory. Simply put, string theory cannot yet be fully and consistently formulated. There are parts and theoretical concepts, the description of which is a mystery from the point of view of string theory. These are mainly cosmological models of the early universe, various spacetimes with the presence of singularities, emergence of the standard model of elementary particles at low energies and others. However, there is a different field of application of string theory, where the apparatus is inner consistent and fully functional. Therefore, in our own interest let us stick to the use of Occam's razor and deal with the areas just mentioned, as this is where our application of the Cheeger problem of *curved tubes* falls.

9.2.1 General relativity is included

Probably the biggest benefit of string theory is that its apparatus directly includes the general relativity. Relativity itself emerges from the core of string theory. The inclusion of the general relativity is based on its modification at small spacetime dimensions and high energies [69]. Within macroscopic distances, string theory continuously transforms into the classical model of Einstein's theory. However, what mechanism makes it possible to include a model of gravity and at the same time to allow a fleeting transition to a different mathematical apparatus during a diametrical change of energy and spatial scales? General relativity is directly rising from a consistent quantum model [57]. In other words, we need an inner consistent quantum theory that is directly included in the framework of string theory. This is made possible by string theory. The main difference is that classical quantum field theories do not assume the existence of gravity [57]. Then, the first great victory of string theory is therefore the inclusion of consistent gravity.

9.2.2 Strings and their size

In classical theory of gravity, or in any classical theories, the fundamental particles are infinitesimal mathematical points. This notion of particles is ideal for certain physical systems. Which has been true for centuries. However, when unifying large theoretical structures as quantum field theory, theory of strong interaction, or theory of electroweak interaction, this idea turned out to be completely wrong. At least in the mathematical description of a unified theory [70]. The idea of using one dimensional loops of zero thickness as a direct description of elementary particles can be considered as a construct that comes partly from topology or abstract geometry, as it seems to bet the most ideal object to describe the properties of particles and their interactions. Of course, the description is not so narrow.

Strings can be characterized by simple mathematical parameters, such as scale, length or tension [60]. The length of the string can be directly determined by dimensional analysis. Denote it as l_S. If we want to determine the size of the string and thus characterize it as a one dimensional loop, it is necessary to include the fundamental constants that give the string theory a physical dimension. We mentioned that string theory involves gravity. It also includes a relativistic model and, of course, quantum theory [60, 71]. Which provides the consistent framework for quantum gravity. This evokes that the fundamental constants must consist of the speed of light c, the Planck's constant $\hbar$. And also universal Newton's gravitational constant G. Using these constants, an initial estimate of the length scale of the string can be computed as $l_P = \left(\frac{\hbar c}{c^3}\right)^{1/2}$. Then $l_P \sim 10^{-33}$ cm. Which is called a Planck length. We can also determine the scale of the mass for strings by $m_P = \left(\frac{\hbar c}{G}\right)^{1/2}$, so $m_P \sim 10^{19}$ Gev/c^2, which is called a Planck mass. The Planck scale seems ideal as a rough estimate of the fundamental string length and mass scale [60]. We now know that size and mass of the fundamental strings can be roughly estimate by Planck scale, which implies the fact that the conventional quantum field theory can provide great description of the real world. At energies high than Planck scale, strings can be optimally approximate by infinitesimal point particles. Thus, string theory transforms into quantum field theory for energies that do not fall within the Planck scale. One-dimensional loops smoothly collapse into point approximation for elementary particles [60]. This is not the only consequence, which we can deduct from the estimations of the spatial dimension of the string. The spatial dimension provides rough estimation of the characteristic size of extra dimensions, which are essential within the mathematical apparatus. These extra dimensions are compactified [57]. Then we get a proper estimation of the size of the compactified dimensions. Which can provide many useful insights into topological and geometrical nature of string theory.

9.2.3 Extra dimensions

It is certain that the theory is consistent in more dimensions that we perceive in the everyday world. There are exact values for dimensions where the theories can be considered functional [57]. As we have already mentioned, the extra dimensions must be compactified. This compactification is intuitively performed on manifolds; whose properties directly corresponds to the required spatial dimension of strings. At the same time, the requirement that these manifolds are not normally detectable must be satisfied. On the one hand, this requirement is illogical, since we assume the existence of a mathematical structure that bears excess compactified dimensions with such properties that its geometry is undetectable. On the other hand, this assumption is intuitive, as we do not normally register these extra dimensions and the mathematical apparatus is consistent throughout their existence [57, 60, 71].

We mentioned that there are exact values for the number of dimensions in which the theory is consistent. For example, the theory is consistent in a ten-dimensional spacetime. In some particular cases, the eleven-dimensional spacetime is also possible [72]. Thus, six or seven extra dimensions are compactified on an internal manifold with specific properties, which are important for the topological nature of the strings. This in turn determines, for example, the vibrating or winding modes of the string [57]. That is, the type of particle itself in the standard model. A slightly different situation may arise, within the extension of string theory to cosmological scales. Relevant cosmological models may have specific, time-dependent geometries, where the compactification works a little bit differently [73]. Based on these considerations, it is necessary to clearly determine the type of manifold for compactification. It turned out that the so-called Calabi-Yau manifolds offer a suitable classification [63]. These manifolds were first considered for compactifying six extra dimensions within the ten-dimensional spacetime model. It has also been proven that they are phenomenologically promising in the sense of topological background for the description of interactions of particles and various fields [63]. Of the interest is due to the classification of manifolds the fact, that

Calabi-Yau manifolds do not have isometries. The absence of isometries on these manifolds causes from a mathematical point of view a breakdown of the symmetries [63]. The presence of the symmetries has far-reaching physical consequences. So, the Calabi-Yau manifolds serves more to break symmetries. Nevertheless, their usage as topological object for compactifiying extra dimensions is the most advantageous and elegant. It can be partly argued that symmetry, or even supersymmetry, are rather local features of the theory [74].

9.2.4 Consistency in terms of supersymmetry

One of the basic preconditions for the consistency of string theory is the presence of supersymmetry [57]. To begin with, let’s also mention that this is primarily conjectured relationship between two types of elementary particles. By the term conjectured we mean that it was originally a mathematical structure that was applied in solving the unifying problems of the standard model [75]. This mathematical structure is in its core closely related to group theory. Within the physical properties of string theory, supersymmetry is applied as a type of spacetime symmetry which is an optimal candidate for elegant solution to many current problems. Rather, it serves to predict the existence of new undiscovered particles [74]. Specifically, the mentioned conjectured relationship applies to bosons and fermions and relates these two basic classes of elementary particles together.

The mathematical consistency of string theory provided by supersymmetry follows from local supersymmetry of string theories, which includes fermions [74]. Because there are few possibilities which allows the existence of a so-called non supersymmetric string theories [76]. These theories are strictly bosonic. They do not include fermions. Simply put, they are completely unrealistic. If we consider a string theory that properly includes both bosons and fermions, it is necessary that at least the assumption of local supersymmetry is met. The apparatus of string theory can provide a plethora of various types of these theories. This property is called landscape of theories [77]. However, each of them shares one significant feature and that is the presence of supersymmetry, which is probably a universal feature of all potential realistic string theories.

We mentioned that the concept of supersymmetry is in fact mathematical. This offers a number of interesting connections that can be observed within the coherence of purely mathematical structures and theories, which describes various parts of physics. An example is the description of complex fields and quantities that satisfy the condition of holomorphism [78]. If this property is met, then it is possible to perform, in most cases, a direct computation of these complex quantities using the tool of supersymmetry [74]. In my opinion, one of the most remarkable consequences of supersymmetry is its usage in the proof of Atiyah index theorem. The essence of the theorem lies in the field of differential geometry, so it is a part of pure mathematics. Put simply, the theorem deals with behavior of the elliptical differential operator a compact smooth manifold [79]. The behavior concerns the so-called analytical index, which in some sense represents the dimension of the space of all solutions. Theorem says that the analytical index is directly equal to the topological index [80]. Which is a construct that depends on topological data of the compact manifold. However, we have stated the wording of the theorem only from a technical point of view, as it is not so important for our intention. The proof itself is the most important, as the supersymmetry makes it possible to significantly simplify. But the context of the proof is not so clear-cut. In particular, the supersymmetric formulation of quantum mechanics makes it possible to construct a greatly simplified form of the proof of the mentioned theorem [81].

It can be clearly seen that the consequences of the concept of supersymmetry within theories in physics have implicit tendencies that manifest themselves in the context of purely mathematical structures. Perhaps this is why the string theories that include supersymmetry contributes significantly to both fields as pure mathematics and theoretical physics.

In supersymmetry, each existing and potentially existing particle have an associated particle. This particle is called as superpartner [57]. The parameters of particles and superpartners may or may not differ. It depends purely on whether the given supersymmetry is unbroken or not [74]. Within the standard model, it is therefore possible to ask whether this supersymmetry is experimentally verifiable. There are few facts that indicates, that the characteristic energy scale of supersymmetry breaking or the masses and other parameters of superpartners of elementary particles are above experimentally feasible energy limits of current technologies. Various theories predict that the characteristic energy lower bound for possible supersymmetry breaking is maybe connected to energetic scale of electroweak interaction. So we move in orders up to 100 GeV or maybe TeV's [74]. These energies were accessible after starting the Large Hadron Collider. So far, it has not been proved that any superpartners exists at all [82].

9.3 Basics of string theory

The most basic postulates of string theory go back to the apparatus of theoretical mechanics, or variational calculus. The considerable contributions of quantum theories, thermodynamics, black hole theory, relativity and, from a mathematical point of view, also topology, number theory, conformal field theory, duality and many others cannot be neglected.

Let's start with basic description of the behavior of the strings themselves, as almost all the remaining parts of the string theory stems from this knowledge.

The following concept is based on the variational, or Hamiltonian view of the relativistic particle. However, string theory is interesting in that, because in my opinion, there is no point analyzing the whole variational view of the principle of least action for relativistic particle. This whole concept is only narrow restriction of the more general apparatus, which is based on the topological concept of action. It is therefore worthwhile to proceed deductively.

If we follow the evolution of the string over time, we will find that after projecting it into spacetime, it draws a two-dimensional surface. This surface in generally called the string world-sheet [57]. It is a multidimensional extension of the world line for a point particle. This approach is widely used especially in the area of quantum field theory, and perturbation theories, where contributions from perturbations acts on amplitudes, which are associated with a tool of Feynman diagrams [57]. These diagrams greatly describe possible configurations of world lines, which directly depict the spacetime trajectories of particles [83]. An identical approach is widely used in the perturbation expansions in string theory, where the string world sheets of various topologies are included.

The concept of world sheet is so universal that it also includes a description of the interactions between strings. It is here that the conceptually deeper theoretical view of physics is reflected, since the existence of different types of interactions in string theory can be understood as a direct implications of world sheet topology [57]. Which mainly differs in from the concept of interactions as local singularities in world sheet topology.

9.3.1 p-branes, bosonic strings and action

Here we will introduce the apparatus of bosonic string theory, topological objects such as *p*-branes and then move on to the principle of action on these objects. From the point of view of the fundamental consistency of string theories, the bosonic theory is completely incompatible with reality and suitable for phenomenology [60]. Nevertheless, it is very advantageous to start with the apparatus of this theory, as much will be used in the description of much more realistic types of string theories, where we will implement our application.

The fundamental string is in its nature a dimensional restriction of a *p*-dimensional extended object moving through spacetime [57]. This object is technically called a *p*-brane. For further specification, *p*-brane forms a submanifold that is implemented on a pseudo-spacetime manifold with different topology. Within this concept a point particle or relativistic particle is a particular case of *p*-brane, when $p = 0$ [57]. Then the particle sweeps out the previously mentioned world line, which is one dimensional submanifold of arbitrary dimensional pseudo-spacetime manifold. Set $p = 1$, then this situation corresponds to one-branes, or simply strings. Generally, p-branes as a topological object were chosen to suggest a simple generalization of membranes, which freely flows though spacetime [57, 60, 71].

Strings, naturally share many interesting properties with higher-dimensional objects at classical level. But there exist a specific properties of quantum theories that are built as two-dimensional world volume quantum theories. In other words, as quantum theories with apparatus defined by p-branes with $p = 2$. There is no need to analyze the unique property mentioned above. It is enough to point out the importance that multidimensional branes do not exhibit that property. Therefore, a completely functional quantum theory cannot be built on the basis of arbitrary dimensional branes.

The motion of relativistic particle can be fully described on a pseudo-manifold, which is often considered to be a D-dimensional Minkowski spacetime $\boldsymbol{R}^{1,D-1}$[57]. However, it is not a difficulty to conceive the motion of a point particle as a variational problem, which formulation can be defined in a curved D-dimensional spacetime. Because the motion of mentioned particle creates a one-dimensional submanifold (geodesic) in D-dimensional spacetime, then the action must be proportional to $(D-1)$-dimensional Lebesgue measure of particles trajectory. Simply put, action is directly proportional to the length, which is an invariant. This type of action can be explicitly generalized to action of p-brane [84]. Submanifold as p-brane maps a $(p+1)$-dimensional world volume on D-dimensional spacetime. In the case of a point particle, we observe one-dimensional geodesic as a trajectory whose length is proportional to action. The analogous approach is extended to p-branes, where $p < D$ and the action essentially takes the following form [57, 84]

$$S_p = -T_p \int d\omega_p. \tag{70}$$

The term $d\omega_p$ denotes the $(p+1)$-dimensional volume element of p-brane and T_p is generally called the p-brane tension. Since the p-brane forms a submanifold on the pseudo-spacetime manifold, which in some sense forms a pseudo-Riemannian manifold. As in the case of Minkowski spacetime$\boldsymbol{R}^{1,D-1}$. In the section 1 (Chapter 1) we talked about the key property of Riemannian manifolds and that is the existence of metric. Our case is no different. For example, world sheet is a curved surface embedded in spacetime. So, when the world sheet is said to be embedded in another (higher-dimensional) object, the embedding is given by some natural injective structure that preserves mapping from world sheet to the mentioned spacetime.

In other words, there exist a morphism from one submanifold to the higher-dimensional underlying manifold.

Based on these topological properties it can be argued that the induced metric $G_{\alpha\beta}$ on the $(p+1)$-dimensional world volume is in fact the „pull back" of the flat matric on Minkowski space. Then

$$G_{\alpha\beta} = g_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu, \tag{71}$$

where $\alpha, \beta = \{0, \dots, p\}$ [57, 60, 84].

Now we can determine the $(p+1)$-dimensional volume element of *p*-brane as

$$d\omega_p = \sqrt{-det(G_{\alpha\beta})} d^{p+1}\sigma. \tag{72}$$

In order to write the action of the string according to formula (70), parametrize the *p*-brane world volume by spacetime coordinates that consist of one time-like coordinate $\sigma^0 = \tau$ and p space-like coordinates σ^i. This coordinate system forms a natural parametrization of the *p*-brane, which is nested in *D*-dimensional spacetime.

A very important property of equation (70) is its invariance under reparametrization $\sigma^\alpha \to \sigma^\alpha(\tilde{\sigma})$ of the $(p+1)$-dimensional world volume coordinates (see [84]). For many purposes, there are many typed of formulas for string action. Now let's specify some of them for their importance.

The following considerations specialize to the case when $p = 1$. So, on one-brane, which is implemented in *D*-dimensional spacetime, with constant sectional curvature equal to zero. However, the spacetime can be classified as pseudo-manifold whose metric has a time component, i.e. Minkowski spacetime. By definition, a one-brane is a string that is nested in this spacetime and maps out a two-dimensional world sheet. Within the parametrization by spacetime coordinates, we perform a restriction by $\sigma^0 = \tau$ and $\sigma^1 = \sigma$. Where σ^0 is considered as time-like and σ^1 as space-like coordinates. Bosonic string theory admits the existence of only closed strings [57, 60]. Which implies that the σ coordinate can be called as circumferential coordinate (on a two-dimensional surface). Then take σ to be periodic, with range

$$\sigma \in [0, 2\pi). \tag{73}$$

For easier manipulation with coordinates, it is possible to cumulate the two coordinates together as $\sigma^\alpha = (\tau, \sigma)$. The surface that is drawn by the string (one-brane) actually defines a map from the two-dimensional world sheet to Minkowski spacetime. From a topological point of view, the map forms a injective structure, which preserves morphism from world sheet to $\boldsymbol{R}^{1,D-1}$. Thus, we see a clear correspondence between the definition of embedding and natural parametrization σ^{α}.

Let's denote the relevant function as $X^{\mu}(\sigma,\tau)$, where $\mu = \{0, \dots, D-1\}$. We consider only closed strings, so we required periodicity

$$X^{\mu}(\sigma,\tau) = X^{\mu}(\sigma + 2\pi, \tau). \tag{74}$$

Which implies that the following type of action can be formulated as a special case of *p*-brane action. Technically, the action is often called the Nambu-Goto action [57, 60, 63] and is defined by

$$S_{NG} = -T \int d\sigma d\tau \sqrt{-det\left(G_{\alpha\beta}\right)} \tag{75}$$

The metric $G_{\alpha\beta}$ can be written more explicitly, as it will intuitively be a 2×2 matrix,

$$G_{\alpha\beta} = \begin{pmatrix} \dot{X}^2 & \dot{X} \cdot X' \\ \dot{X} \cdot X' & X'^2 \end{pmatrix}, \tag{76}$$

where $\dot{X}$ and X' are defined as a partial derivative according to coordinates, so $\dot{X}^{\mu} = \partial X^{\mu}/\partial\tau$ and $X'^{\,\mu} = \partial X^{\mu}/\partial\sigma$. Employing the relation (75), we get the desired formula [84]

$$S_{NG} = -T \int d\sigma d\tau \sqrt{\left(\dot{X} \cdot \dot{X}'^2 \right) - \dot{X}^2 X'^2}. \tag{77}$$

The scalar product inside the integral (77) is defined as a classical product in a flat spacetime. Most importantly, the relation (77) directly reflects the geometry of world sheet. Because the integral, which appears in Nambu-Goto action describes the area of the surface embedded in $\boldsymbol{R}^{1,D-1}$. Which, after all, we demanded after the restriction of the action for *p*-branes.

For the sake of accuracy, let us give a direct proof that the action (77) is really proportional to the area of the world sheet, except for the constant T. The proof itself will differ in its structure, as we will not include the metric $G_{\alpha\beta}$ straightforwardly, but we will use simple curvilinear geometries. This will give us a better insight into understanding the connection between Theorem 9 and the essence of the principle of least action in string theory.

Simply put, let's implement a closed string into Euclidean space. We will thus get rid of the complicated time-dependent structure of the pseudo-Riemannian manifold, which reflects the properties of Minkowski spacetime. Under these conditions, it is slightly easier to consider an embedding of the surface. Let σ and τ denotes the coordinates in euclidean space, then the embedding function is $\vec{X}(\sigma,\tau)$. The area of the infinitesimal region can be computed by the vectors, which are tangent to the boundary

$$\overrightarrow{dv_1} = \frac{\partial \vec{X}}{\partial \sigma} \quad , \quad \overrightarrow{dv_2} = \frac{\partial \vec{X}}{\partial \tau}.$$

Let the angle between these two vectors be represented by γ, then we obtain the area by

$$ds^2 = \left|\overrightarrow{dv_1}\right|\left|\ \overrightarrow{dv_2}\right| sin(\gamma) = \sqrt{{dl_1}^2 {dl_2}^2 - \left(\overrightarrow{dv_1} \cdot \overrightarrow{dv_2}\right)^2}, \tag{78}$$

which is identical with the integrand of the Nambu-action (77). We have proved that action (77) represents an area of the two dimensional world sheet [84].

The desired result is, that the classical string motion minimizes the world sheet area, just as in the case of point particle, where its motion makes the length of the particle geodesic extremal by moving along. In the formula (70) it still remains to describe the meaning of T. If we take the concept of the closed string action in general, then the world volume element $d\omega_p$ has units of lenght $(lenght)^{p+1}$. Then T represents a tension of *p*-branes and its dimension is

$$\left[T_p\right] = (l_S)^{-p-1} = \frac{m_S}{(l_S)^p}. \tag{79}$$

Which can be interpreted also as energy per unit *p*-volume of the *p*-brane [57]. In the case of world sheet, $p = 1$, so $[T] = (l_S)^{-2} = \frac{m_S}{l_S}$.

Tension provides insight into the potential energy of string that can be obtained through action. We rewrite coordinates in Minkowski spacetime as $X^\mu(\vec{x},t)$. The reason lies in the chosen gauge with $X^0 \equiv t = Q\tau$, where $Q$is a constant and will drop out at the end of the following argument. The gauge naturally means that me make a cross section by hypersurface in terms of the *p*-brane action. In terms of world sheet action, we make a cross section by planar surface. This approach will give us some sort of static photo shoot at a fixed time $\frac{d\vec{x}}{d\tau} = 0$. So we have created the conditions for the kinetic energy to disappear. Then

$$S_{NG} = -T \int d\sigma d\tau Q \sqrt{\left(\frac{d\vec{x}}{d\sigma}\right)^2} = -T \int l_S dt \,. \tag{80}$$

It is striking that if the kinetic energy vanishes, then the Nambu-Goto action is proportional to the time integral of potential energy [84]. This implies that if we choose a specific gauge and shrink the are mapped by the closed string in $\boldsymbol{R}^{1,D-1}$. We formulate a certain type of minimization problem, where the solution is action (80) as a time integral of potential energy. Which gives

$$V_S = T l_S \tag{81}$$

Equation (81) suggests that if we minimize the world sheet of closed string by fixing time, which is implied by gauge, then the energy of string decreases in direct proportion [84]. It is clear that the presence of the tension T of the string gets its expected reason, which is the elastic behavior of the string and its energy increases linearly with length. The minimization problem in relation to potential energy therefore says that the string wants to shrink to zero size, to point.

Once we have a proper formula for action we can obtain the equations of motion. To derive the equations of motion for strings, which are described by Nambu-Goto action, we first must to compute the momenta from Lagrangian structure. However, this approach, which is an obvious analogy of theoretical mechanics, is very impractical. The reason lies in the difficult solvability of the obtained equation of motion as they form a system of non-linear differential equations [57, 60, 63, 84]. Starting from (75), it is possible to make the variation of a determinant by

$$\delta\left(\sqrt{-G_{\alpha\beta}}\right) = \frac{1}{2}\sqrt{-G_{\alpha\beta}}G^{\alpha\beta}\delta\left(G_{\alpha\beta}\right).$$

Employing the pull-back metric, gives rise to the desired equations of motion

$$\partial_\alpha \left(\sqrt{-det\left(G_{\alpha\beta}\right)}G^{\alpha\beta}\partial_\beta X^\mu\right) = 0. \tag{82}$$

We have made some changes, but equations (82) written in the variables X^μ are still very difficult to solve [84]. It is therefore necessary to re-establish the analogy to the Nambu-Goto action at classical level, because it gives rise to the equivalent equations of motion. The following approach is generally called a string sigma model action. But it is better known as Polyakov action [62].

Define auxiliary world sheet metric $m_{\alpha\beta}(\sigma,\tau)$. Let $m = det(m_{\alpha\beta})$ and $m^{\alpha\beta} = (m^{-1})_{\alpha\beta}$. Then the Nambu-Goto action transforms to

$$S_{\sigma} = -\frac{1}{2}T\int d^2\sigma\,\sqrt{-m}m^{\alpha\beta}\partial_{\alpha}X\cdot\partial_{\beta}X. \tag{83}$$

At the classic point of view, the Polyakov action is equivalent to the Nambu-Goto action [57, 84]. However, if we pass to a much smaller scale, we find that action (83), is much more suitable for quantization, see [62, 84]. The equations of motion can be obtained in the same way as in the previous case. Since there is no kinetic term for the metric $m_{\alpha\beta}$, then the world sheet energy momentum tensor vanishes

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-m}}\frac{\delta S_{\sigma}}{\delta m^{\alpha\beta}} = 0.$$

The next steps lie in the varying of the action with respect to metric. Details can be found in [60, 63, 84]. The last necessary equation of motions follows directly from Euler-Lagrange condition

$$\Delta X^{\mu} = -\frac{1}{\sqrt{-m}}\partial_{\alpha}\left(\sqrt{-m}m^{\alpha\beta}\partial_{\beta}X^{\mu}\right) = 0.$$

In a generalized Polyakov action to the p-brane action for $p \neq 1$, it is necessary to include a cosmological constant term. Then the equation of motion for the world-volume metric as the embedded submanifold, is obtained exactly as in the previous cases [57].

Like the Nambu-Goto action, Polyakov action has various symmetries that allows the existence of a gauge, such as the static gauge discussed earlier in the context of minimization problem, that is closely related to the potential energy (81). Specifically, for the action of closed strings in Minkowski spacetime, there exist a Poincaré symmetry, which forms a global symmetry. This is followed by reparametrization symmetry. In other words, the change of parametrization of the world sheet does not change the action. So, there exist a needed invariant. The last symmetry is the so-called Weyl transformation, which provides the rescaling of action, under which is the action invariant [57, 60, 63, 84].

9.3.2 Gauge of the closed string described by Polyakov action

The auxiliary field, or in other words the dynamical metric on the world sheet, can be described by 2×2 matrix, since we are talking about two dimensional wordl sheet, which is embedded in $\boldsymbol{R}^{1,D-1}$. The field has also three independent components

$$m_{\alpha\beta} = \begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix},$$

where $m_{10} = m_{01}$ [63]. We will now use the necessary symmetries. Reparematrization invariance allow us to make a choice of two components of the metric. Then, only one independent component remains. By Weyl transformation, and the invariance of the action under Weyl rescaling, the remaining component vanishes, because of gauge. As the result, the dynamical metric can be chosen by gauge fix as

$$m_{\alpha\beta} = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $\eta_{\alpha\beta}$ denotes a flat world sheet metric [57]. Only this metric is possible since there is no topological obstruction. Under these conditions, it can be stated that from a topological point of view, the world sheet has for example the shape of torus or cylinder. For choosing the gauge fixation, one can directly argue that the following string action describes propagation in flat Minkowski spacetime [84]

$$S = \frac{T}{2} \int d^2\sigma \left(\dot{X}^2 - X'^2\right). \tag{84}$$

9.3.3 Equations of motion and boundary conditions with flat world sheet metric

Let there be a flat world sheet metric on the submanifold of $\boldsymbol{R}^{1,D-1}$. From a topological point of view, the most ideal shape of the world sheet of closed strings will be an infinite cylinder. In the opposite case of the open strings, we choose an *infinite curved strip*. In both cases, the movements of the strings are described by an action (84), which is determined by the gauge fixation, due to existence of the above mentioned symmetries of Polyakov action. It turns out that closed and open strings behave in this case approximately as classical objects and are described by the wave equation [60, 84]

$$\partial_\alpha \partial^\alpha X^\mu = 0 \quad or \quad \left(\frac{\partial^2}{\partial \sigma^2} - \frac{\partial^2}{\partial \tau^2}\right) X^\mu = 0. \tag{85}$$

Then the energy-momentum tensor satisfies $T_{\alpha\beta} = 0$, due to the gauge fixation of the metric on the world sheet. In the gauge $m_{\alpha\beta} = \eta_{\alpha\beta}$, we obtain the following components of the energy-momentum tensor [84]

$$T_{01} = T_{10} = \dot{X} \cdot X' \quad and \quad T_{00} = T_{11} = \frac{1}{2}\left(\dot{X}^2 + X'^2\right) \tag{86}$$

Given that we want to define the problem of equations of motion from a variational point of view, then it is necessary and sufficient to add the boundary conditions. Here, the desired connection of Theorem 9, and the boundary conditions slowly comes to play. The reason is that in Theorem 9, we are talking about the Cheeger constant of tubular neighbourhood, which forms, from a topological sense, as a neighbourhood of the closed smooth geodesic in $\mathbb{R}^d$. Based on this fact, it is necessary to work only with closed string, because for example if we consider a world sheet, that is mapped out by the closed string, it creates a cylinder or torus. Both cases form specific restrictions of (46). We can thus refer back to relations (73) and (74), as these just represent the required boundary conditions for closed strings.

Let us comment on the explicit solution of equations of motion and the closed string mode expansion. As one can see in [57, 60, 84]. The solution of closed string wave equation is provided. A new parameter was introduced in the solution, and is called the Regge slope parameter α' [69]. Then the tension can be redefined by the fundamental length of string, because

$$\frac{1}{2} {l_S}^2 = \alpha', \tag{87}$$

then

$$T = \frac{1}{2\pi\alpha'} = \frac{1}{\pi {l_S}^2}. \tag{88}$$

We mentioned relation (88) for practical reasons, as it will play a very important role in our application.

9.3.4 Generalization of flat spacetime action

Due to the scale of our application, let's generalize the action which acts on a flat Minkowski spacetime to the curved pseudo-Riemannian manifolds, which represents a curved spacetimes. To generalize, it is sufficient to consider the classical relation for Polyakov action (83) and extend it by the appropriate metric of the spacetime manifold.

Then

$$\begin{aligned} S_\sigma &= -\frac{1}{2} T \int d^2\sigma \sqrt{-m} m^{\alpha\beta} \partial_\alpha X^\mu \cdot \partial_\beta X^\nu W_{\mu\nu} \\ &= -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-m} m^{\alpha\beta} \partial_\alpha X^\mu \cdot \partial_\beta X^\nu W_{\mu\nu}, \end{aligned} \tag{89}$$

where $m_{\alpha\beta}$ represents the world sheet metric [63]. The action itself describes a map from the world sheet into a curved spacetime with general metric $W_{\mu\nu}$. The explicit construction of the metric $W_{\mu\nu}$ is based on the gravitons. Which are theoretical fundamental intermediate particles of gravitational interaction. The reason why the mechanism of construction of $W_{\mu\nu}$ is based on gravitons is given by the specific form of the so-called vertex operator [60, 63, 84]. The geometric reason may be better understood, as in the perturbative string theory, gravitons are closed strings in a very particular low-energy vibration states.

It is gravitons that are interesting to us, because as closed strings, their world sheet forms a curved cylinder or torus in spacetime. However, this particular case can be generalized, as we can consider the existence of so-called multidimensional gravitons, which can be described by the structure of *p*-brane. The *p*-brane itself sweeps out the $(p + 1)$-dimensional hypersurface, which freely propagates through generally curved spacetime. Let us stick to these initial considerations as they will form the building blocks of the following section.

The facts stated so far about the particular and key parts of string theory and bosonic string theory are largely sufficient for us to understand the following application. Subsequently, we will illuminate certain hidden connections within the topological nature of the principle of the least action, gravitons, *p*-branes, M-theory and the exact solution of equations of motion within the mentioned action (70). All these findings will be generalized to curved spacetimes. Many interesting parts of string theory are connected to these topics, but for the sake of clarity we will specifically mention only some of them.

9.4 Interconnection between the Cheeger constant of curved tubes and the essence of strings actions

The interconnection of concepts that we consider to be our application will be based purely on a mathematical description of the situation. Then we will move on to the derivation of physical consequences within the Framework of string theory and Hamilton's variation, which is conceived from a topological point of view.

At the beginning of section 9.3.1, we showed that the action of a closed string can be generalized by the relation (70) to arbitrary dimensional p-brane [60, 63, 84]. The basic precondition for a correct topological definition of p-brane is to guarantee its free propagation on the pseudo-Riemannian manifold, which has a generally curved geometry. Inequality $p < D$ is also fulfilled. D denotes the dimension of the underlying manifold.

Let $\sigma^0 = \tau$ bet he time-like coordinates and σ^i are p space-like coordinates and let M denotes the D-dimensional spacetime manifold. Then there exist a map $X^\mu(\sigma^i, \tau)$, which maps the $(p+1)$-dimensional world volume to M. Also $\mu = \{0, \ldots, D-1\}$ [57]. Since we consider only closed strings, we require that the condition of periodicity of the given representation be met, which is provided by relation (74). The mentioned map provides a natural spacetime parametrization of the p-brane as a submanifold in M. In order to the structure of the p-brane to be topologically consistent, its embedding must exist in M.

Let's denote the p-brane as P. Then there exist a map $\varsigma : P \to M$, which is injective and the embedding structure preserves the map ς. In other words the structure preserving map ς corresponds to morphism [5, 85]. The existence of this embedding structure can be taken for granted, as form a physical point of view, the map $X^\mu(\sigma^i, \tau)$ provides ideal support for the formulation of various types of closed strings actions, for example (75) and (83). From a geometric point of view of the situation, it is clear that $(p+1)$-dimensional world volume, which is mapped on M corresponds to the generalized $(p+1)$-dimensional tubular neighbourhood. This tubular neighbourhood is defined around imaginary time-like geodesic $\Sigma(\tau)$, which is smooth and unbounded. In the sense of our terminology, which is related to Theorem 9, it is an infinite *curved tube* or *unbounded curved tube*.

Within the proof 7.1 and 7.2, the reference geodesic Ψ, which is complete and unbounded in the case of *unbounded curved tubes*, directly corresponds to the unbounded time-like geodesic $\Sigma(\tau)$. We therefore intuitively identify the $(p+1)$-dimensional world volume, which is mapped to M, with arbitrary dimensional tubular neighbourhood, which can be matched with (46). However, the assumption of the reference geodesic Ψ must be changed to completeness and unboundness. Here, slowly but surely, the connection between the topological objects in string theory, with the essence of Theorem 9 begins to come to light.

Since the p-brane mapped the $(p+1)$-dimensional world volume to M, then the general form of action (70) represents the area of this hypersurface, which is embedded in M. Up to the tension T_p [57]. In the context of the Cheeger problem of *curved tubes*, where the d-dimensional Lebesgue measure denotes the volume of generalized *curved tube* Ω_a and the assuming that the boundary $\partial\Omega_a$ is smooth, or Lipshitz, the $(d-1)$-dimensional Hausdorff measure represents the perimeter of Ω_a [45]. The same consideration can obviously be made in the context of the p-brane. However, the problem lies in the correct mathematical identification of the volume of the *curved tube* and the generalized p-brane action (70), which represents the $(p+1)$-dimensional world volume.

In the case of a restriction from an *unbounded tube* to closed *curved tube*, which is defined as tubular neighbourhood of smooth and closed geodesic Ψ, the volume and perimeter of Ω_a depend on local parametrization of the reference geodesic, as $\gamma : I \to \mathbb{R}^d$, where I is an open interval and $\gamma(s) \in \Psi$ for all $s \in I$. Then $\gamma(s)$ is a unit speed parametrization. The local parametrization of Ψ itself, is not as important as the local parametrization of the tube Ω_a [45]. Relation for d-dimensional Frenet-Serret frame [53] gives us great insight into the properties of the metric $G := \nabla\phi \cdot (\nabla\phi)^T = diag(J^2, 1, \dots, 1)$, where J represents the Jacobian of the map (52) [45]. Jacobian can be identified with the Jacobi field around the geodesic Ψ. Under these conditions, we can express the volume of Ω_a by the relation (59) and the perimeter can be defined by (60). In fact, it is a local parametrization by general curvilinear Fermi coordinates [51]. Therefore, the question arises as to whether the volume of $(p+1)$-dimensional world volume can be described by identical parametrization according to Fermi coordinates.

If this option exists, then the formula of the action (70) can be directly identified with the volume (59) of Ω_a, where Ψ si complete and unbounded. For this purpose, it is necessary to construct Fermi coordinates for specific spacetimes and then find the Jacobi field equation in these spacetimes, which corresponds to equation (57).

9.4.1 Fermi coordinates for specific spacetimes

Reassurance to us in this situation can be provide by elegant theorem [86], which states that under general conditions, around an imaginary time-like geodesic $\Sigma(\tau)$ exist a neighbourhood on which a Fermi coordinates can be properly defined. The transformation by Fermi coordinates can also be smoothly generalized to general spacetimes [87]. However, there is a question about the possible extension of coordinate charts for the system of Fermi coordinates. So we need to find particular examples where this coordinate transformation satisfies the mentioned extension. These considerations further help define the Jacobi field of general submanifold of the spacetime manifold, which measure the difference between the desired Fermi coordinates and space-like geodesics.

Let M is a spacetime manifold and denotes the mentioned imaginary time-like geodesic. Fermi coordinates are therefore given by the Definition 5. In general we assume that M is a D-dimensional spacetime. However, without the loss of generality and simplicity, it is possible to choose tetrad of vectors $u_0(\tau), u_1(\tau), u_2(\tau), u_3(\tau)$, which is paralle along $\Sigma(\tau)$. Then the fermi coordinates can be defined as the following system of equations

$$x^0\left(exp_{\Sigma(\tau)}\big(\vartheta^n\epsilon_n(\tau)\big)\right) = \tau\,, x^l\left(exp_{\Sigma(\tau)}\big(\vartheta^n\epsilon_n(\tau)\big)\right) = \vartheta^l, \tag{90}$$

where $n = \{0,1,2,3\}$ and $l = \{1,2,3\}$ [51, 86]. The essence of the exponential map in (90), we have already clarified in the section 7.1.

The infinitesimal line element, is therefore given as

$$ds^2 = -\big(1 - c(x,y,z)\big)dt^2 + dx^2 + dy^2 + dz^2 + [(1 - kr^2)^{-1} - 1]dr^2, \tag{91}$$

where r^2 represents ball, $c(x,y,z)$ is a smooth function, which vanishes with its partial derivatives at $x = y = z = 0$ and k is a constant [86, 87]. When we set $c(x,y,z) \equiv 0 = k$, then the relation (91) corresponds to the Minkowski metric. This case reflects the choice $M = \boldsymbol{R}^{1,D-1}$. Which would mean that at the end of the consideration of parametrization by Fermi coordinates, we would consider only actions describing the closed strings or *p*-branes in $\boldsymbol{R}^{1,D-1}$. We will also get to that, but for now, let’s proceed in general.

The partial derivatives of the metric elements in (91) evaporates on the time-like geodesic $\Sigma(\tau) = (t, 0,0,0)$. Then the tetrad can be defined as $u_0(\tau) = (1,0,0,0), u_1(\tau) = (0,1,0,0)$, $u_2(\tau) = (0,0,1,0)$, $u_3(\tau) = (0,0,0,1)$. This system is orthonormal and paralle along the time-like geodesic $\Sigma(\tau)$ at $\tau = t$ [86]. Based on this, it is therefore straight forward to create an inverse map from Fermi coordinates to Cartesian and conversely, see [51, 86]. For our purposes, the metric in Fermi coordinates is more important.

Theorem 12 ([86]). *The metric in Fermi coordinates for the imaginary time-like geodesic $\Sigma(\tau)$, is given by*

$$g_{00} = -\left[1 - c\left(x^1\left[\frac{sin(\beta b)}{\beta b}\right], x^2\left[\frac{sin(\beta b)}{\beta b}\right], x^3\left[\frac{sin(\beta b)}{\beta b}\right]\right)\right], \tag{92}$$

$$g_{0i} = 0,$$

$$g_{ij} = \frac{x^i x^j}{\beta^2} + \frac{sin^2(\beta b)}{\beta^2 b^2}\left(\delta_{ij} - \frac{x^i x^j}{\beta^2}\right),$$

when $k > 0$. If $k < 0$, then

$$g_{00} = -\left[1 - c\left(x^1\left[\frac{sinh(\beta b)}{\beta b}\right], x^2\left[\frac{sinh(\beta b)}{\beta b}\right], x^3\left[\frac{sinh(\beta b)}{\beta b}\right]\right)\right], \tag{93}$$

$$g_{0i} = 0,$$

$$g_{ij} = \frac{x^i x^j}{\beta^2} + \frac{sinh^2(\beta b)}{\beta^2 b^2}\left(\delta_{ij} - \frac{x^i x^j}{\beta^2}\right),$$

where $\{x^0, x^1, x^2, x^3\}$ represents the Fermi coordinates in the spacetime, with spatial restriction $l = \{1,2,3\}$ in (90). And β^2 denotes the ball in Fermi coordinates. Thanks to Theorem 12, the transformation from Fermi coordinates into a system $\{t, x, y, z\}$ can be derived. For more details, see [86, 88].

An interesting consequences of Theorem 12, is a change of spatial coordinates to spherical coordinates as $x^1 = \beta\, sin(\theta) cos(\varphi)$, $x^2 = \beta\, sin(\theta) sin(\varphi)$ and $x^3 = \beta\, cos(\theta)$. Since the metric is generally given by Theorem 12, then

$$ds^2 = g_{00} dt^2 + d\beta^2 + \frac{sin^2(\beta b)}{\beta^2 b^2}(d\theta^2 + sin^2(\theta) d\varphi^2), \tag{94}$$

when $k > 0$, and

$$ds^2 = g_{00} dt^2 + d\beta^2 + \frac{sinh^2(\beta b)}{\beta^2 b^2}(d\theta^2 + sin^2(\theta) d\varphi^2), \tag{95}$$

when $k < 0$. The term g_{00} is defined by (92) for $k > 0$ and by (93) for $k < 0$ [86, 87]. From the relations (94) and (95), the transformation from Fermi coordinates to $\{t, x, y, z\}$ can be derived, and also the inverse can be provided, see [86]. Proof of this transformation also shows that $c(x, y, z)$ is independent of the coordinate transformation.

The following theorem reflects the mechanism of the Jacobi field, along the time-like geodesic, which is located in the curved spacetime M. In the general case, M would be a D-dimensional pseudo-Riemannian manifold with a time component in the metric, but due to our restriction of Fermi coordinates according to equation (90), we will provide a Jacobi field for the congruence of spatial, infinitesimally close geodesics, which are orthogonal to the Fermi coordinates of the time-like geodesic $\Sigma(\tau)$. In the context of string theory, therefore, it is the case, where $p = 0$, and $\Sigma(\tau)$ corresponds to the world line of a point particle. The proof of generalization of the mentioned mechanism is almost straightforward. As we can see in [87], there exist a consistent system of Fermi coordinates in general D-dimensional spacetime. We will take this fact as proven, so we will continue to use it automatically in generalizations.

Theorem 13 ([86]). *Let M be a spacetime manifold with metric defined by* (94) *or* (95). *Let Π be a two-dimensional submanifold, which is generated by the Fermi coordinates t and β. The angular coordinates are fixed. Then the induced metric on the submanifold $\Pi \subset M$ is given by*

$$ds^2 = g_{00} dt^2 + d\beta^2, \tag{96}$$

and the Gaussian curvature Λ of Π is provided by the following relation

$$\Lambda^{\Pi} = -\frac{1}{\sqrt{-g_{00}}} \frac{\partial^2}{\partial \beta^2} \sqrt{-g_{00}} \tag{97}$$

It is obvious that g_{00} is the function of β only, then it is straightforward that Λ is totally geodesic in M. Which implies that Λ^{Π} directly corresponds to the sectional curvature of Λ. In other words, the intrinsic geometry of the submanifold is in all aspects the same as its extrinsic geometry.

Let $g_{00} = g_{00}(\beta)$. Let also the vector field $J := \partial/\partial t$ be a variation vector field for the geodesic variation of space-like geodesic $\iota_t(\beta) = (t, \beta)$, which is parametrized in Λ by t. Then the Jacobi field equation is given by [50, 51, 54, 86]

$$\nabla_{\frac{\partial}{\partial \beta}} \nabla_{\frac{\partial}{\partial \beta}} J = R_{\frac{\partial}{\partial t} \frac{\partial}{\partial \beta}} (\partial/\partial \beta). \tag{98}$$

Clearly this relation is identical to the original formula of the Jacobi field equation (55). We have thus explicitly shown that within the system of Fermi coordinates in a curved spacetime M, it is possible to derive an equivalent expression of Jacobi equation (55).

Next, let's specify the conditions to get as close as possible to the geometric view of the Cheeger problem of curvet tube Ω_a. Let the Riemannian curvature tensor be described in the terms of sectional curvature, so $R_{\frac{\partial}{\partial t}\frac{\partial}{\partial \beta}}(\partial/\partial\beta) = -\Lambda^{\Pi}J$. It folllows that this equation is identical to the relation (56). As can be seen from the Theorem 9, it is necessary to consider the sectional curvature of the underlying manifold $\mathbb{R}^d$ constant and zero, within the geometry of *curved tube* Ω_a. Then the Riemannian curvature tensor transform to $-\Lambda^{\Pi}J$. Since Λ^{Π} represents the sectional curvaturre, then the equation (97) takes the form of the original formula (57).

In this way, we have explicitly proved that it is possible to provide the same parametrization to the *curved tube* $\Omega_a \subset M$, as in the classical case of the Cheeger problem (1). Naturally, without any detriment, this procedure can also be considered adequate in the case of *unbounded tubes* [45]. To define the Jacobi field of general submanifold, which represents the arbitrary dimensional *curved tube*, one can see [50, 51, 54]. We will also use this fact in analogy with the action on *p*-branes (70).

9.4.2 The Cheeger constant of curved tubes and a topological nature of p-brane, world sheet and Polyakov action.

The previous simplified case of a typical curved spacetime M in which a two-dimensional submanifold Π is nested, corresponds, within the Cheeger problem and Theorem 9, to an two dimensional *unbounded tube*, which is defined as the neighbourhood of a complete and unbounded reference geodesic Ψ [45, 51]. If we perform an operation where we fix two reference times $\tau_1 = t_1$ and $\tau_2 = t_2$, this delimiting the given submanifold Π, with respect to the time-like reference geodesic $\Sigma(\tau)$. Then we continuously and smoothly close the geodesic $\Sigma(\tau)$ to the choosen fixed times $\tau_1 = \tau_2$, this will provide the exactly sma situation, as in the case of classical *curved tubes*, where Ψ is closed and smooth. In other words, the situation is equivalent to Theorem 9, when $d = 2$ [45]. It clearly follows that the *curved tube* Ω_a can be locally parametrized by the map (52), in the case of constant sectional curvature manifold $M = \mathbb{R}^d$ and in the same time, if M is classical curved spacetime, the tube Ω_a can be also parametrized by system of Fermi coordinates. Both cases, are of course slightly different, because the pseudo-Riemannian metric, in the case of curved spacetime M. However, it is just a matter of adding a time component.

Physical interpretation, which is reflected by both Theorem 9 and Theorem 13, lies in the agreement between their geometric nature. We know, that in the case of closed string, there exist a map from the corresponding two-dimensional world sheet to the D-dimensional spacetime [64]. From a topological point of view, the structure of the two-dimensional world sheet is identical to the structure of the two dimensional tube Ω_a,i.e. *curved strip*, which is embedded in M. If we make the following restriction, that the underlying manifold corresponds directly to $M = \mathbb{R}^d$, in the case of two dimensional submanifold Ω_a, then the same situation appears, when to the case, when $M = \boldsymbol{R}^{1,D-1}$. We have to take these considerations from the point of view of physical interpretation, within the action of a closed string. Then the following theorem is naturally valid.

Theorem 14. *Let the two dimensional world sheet of closed string and the two dimensional tubular neighbourhood Ω_a are parametrized by Fermi coordinates. Let M be a constant sectional curvature manifold. Then there exist a Jacobi field in relation to the time-like geodesic $\Sigma(\tau)$ and also in the relation to the reference geodesic Ψ. Since there exist a Jacobi fields, then the following quantities can be considered equivalent*

$$|\partial\Omega_r^I| = \frac{d}{dr}\left(\int_I \int_{B_r} J(s,t)dt\, ds\right) \approx S_{NG} = -T\int d\sigma d\tau \sqrt{\left(\dot{X}\cdot\dot{X'^2}\right) - \dot{X}^2 X'^2}, \qquad (99)$$

up to the proportionality constant T. And it is satisfied that $d = 2$, so $J(s,t) = 1 - \kappa(s)t$. Also the tube is locally parametrized by $\Omega_a^I := \phi(I \times B_a)$, where every $r \in (0, a]$ [45].

In compliance with the previous assumptions, we identify the partial derivative of the volume $|\Omega_r^I|$ of *curved tube* Ω_a with classical Nambu-Goto action S_{NG}, which represents the area of the two dimensional world sheet up to the proportionality constant [57]. The situation is topologically identical, since Ω_a, with $d = 2$, corresponds to the two dimensional world sheet, which is embedded by the morphism structure $X^\mu(\sigma,\tau)$ in M.

The Nambu-Goto action describes a string propagating in a flat background geometry and is the special case of the more general p-brane action (70) [57, 63, 64, 84]. The flatness corresponds to the zero sectional curvature of the underlying manifold M, so $K_{sec}^M = 0$, both in the case $M = \mathbb{R}^d$ or $M = \boldsymbol{R}^{1,D-1}$. Then the Riemannian curvature tensor takes the form (55) and the Jacobi field equation is given by (57). The equation is in both cases identical, in the sense of the vector field.

As we have already mentioned in the section 9.3.1, the Nambu-Goto action is complicated for obtaining the equations of motion, due to the presence of the square root in

(75). Therefore, we perform the given equivalence with the Polyakov action. By analogy, therefore the following statement is satisfied

$$|\partial\Omega_a^I| \cong S_\sigma = -\frac{1}{2}T\int d^2\sigma\sqrt{-m}m^{\alpha\beta}\partial_\alpha X\cdot\partial_\beta X, \tag{100}$$

where we made a generalization from the section $r \in (0, a]$ to whole tube of the width a [45]. To support this argument exactly, look back to the definition of Ω_a [45, 51].

We assume that the *curved tube* does not overlap itself, which in term of the action of closed string, evokes causality within the world sheet structure. Because the previously mentioned map of the reference geodesic $\Psi \times (0, a) \ni (q, t) \to q + tN(q)$ induces a smooth diffeomorphism for any smooth vector field N along Ψ. Then the geodesic Ψ is compact. This condition holds for sufficiently small a. Now if we assume that a represents the width of the two dimensional world sheet, than the causality is directly implied by the mentioned smooth diffeomorphism. Under these conditions, as we have already mention, Ω_a can be globally or locally parametrized by Fermi coordinates. Both in the context of flat background geometry of D-dimensional Minkowski spacetime and in the context of the constan sectional curvature manifold M. Then the associated metric has the form

$$dl^2 = J(s,t)^2 ds^2 + dt^2, \tag{101}$$

where $J(s,t)$ is the exact solution of the Jacobi field equation (57), when the Riemannian curvature tensor satisfies (55) and $K_{sec}^M = 0$ [50]. When comparing the relations (96) and (101), we see a striking similarity. Due to the constant zero sectional curvature, the equation (56) is then identically zero. This provides that equivalent formula for the metric (101).

All the foregoing considerations can be straightforwardly generalized to the case where M is a general curved spacetime. Let M be a pseudo-Riemannian manifold, which represents a general curved spacetime. Then there exist a general metric $W_{\mu\nu}$. The generalization of Polyakov action (83) describes a map from the world sheet to M, with respect to $W_{\mu\nu}$. Based on these considerations, the generalized sigma model action, which represents the area of two-dimensional world sheet is equivalent to the partial derivative of (60), with respect to a. Explicitly,

$$S_\sigma = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-m}m^{\alpha\beta}\partial_\alpha X^\mu\cdot\partial_\beta X^\nu W_{\mu\nu}, \tag{102}$$

where $m_{\alpha\beta}$ denotes the auxiliary world-sheet metric, in other words the dynamical metric [57, 84]. According to Theorem 13, there exist the induced metric on the two-dimensional submanifold $\Pi \subset M$, which is given by (96), and there also exist equivalent metric of these two-dimensional submanifold, which is in this case identical to Ω_a and is given by (101).

The dynamical metric is the function of spacetime parametrization $m_{\alpha\beta} = m_{\alpha\beta}(\sigma, \tau)$. We now know, that the two dimensional submanifold $\Pi \subset M$, corresponds to the world sheet of closed string. We constantly assume that M is general curved spacetime. The following map provides embedding of Π, as

$$X^{\mu}(\sigma, \tau) : \Pi \to M.$$

At the same time, it is possible to parametrize the two-dimensional submanifold Π with Fermi coordinates, according to the relation (93), which also provides the explicit formula for the sectional curvature of Π. From the point of view of mathematic consistency, the identification of dynamical metric $m_{\alpha\beta}(\sigma, \tau)$ and metric of tubular neighbourhood (101) is then appropriate, even in the case of general curved spacetime M, where $W_{\mu\nu}$ is the general metric of M. This metric reflects whether the sectional curvature of the manifold M is constant, so

$$K_{sec}^{M} = \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases}, K_{sec}^{M} = const.$$

Then the Riemannian curvature tensor takes the form (55). And the Jacobi field equation is given by the relation (56). Whereby the previous arguments are confirmed and the following theorem can be produced.

Theorem 15. *Let* $K_{sec}^{M} = 0$. *Then* $M = \boldsymbol{R}^{1,D-1}$ *in the case of two-dimensional world sheet and* $M = \mathbb{R}^{d}$, *with* $d \geq 2$. *Both case coincide, up to the time component of the induced metric. Let* Π *denotes the two-dimensional submanifold of* M *and* $X^{\mu}(\sigma, \tau) : \Pi \to M$. *Then the following parametrization of* Π *as a world sheet or curved tube (possibly unbounded) are identical*

$$\xi : (\sigma, \tau) \to X^{\mu}(\sigma, \tau), \tag{103}$$

and

$$(s, t) \to \gamma(s) + t_{\mu} e_{\mu}(s). \tag{104}$$

Due to the validity of Theorem 15, the equation of Polyakov or Nambu-Goto action can be written as an area of world sheet [57, 60], by integration over an existing Jacobi field along the time-like geodesic $\Sigma(\tau)$. The form of Jacobi field for $K_{sec}^{M} = 0$ is given by (58) in the section 7.1. Let's write the following formula for Polyakov action

$$S_{\sigma} = -\frac{1}{2}T\int d^2\sigma\sqrt{-m}m^{\alpha\beta}\partial_{\alpha}X\cdot\partial_{\beta}X = -\frac{1}{2}T\frac{\partial}{\partial a}\left(\int_{I}\int_{B_r}J(s,t)dt\,ds\right), \tag{105}$$

where we integrate over the section $\Omega_a^I := \phi(I\times B_a)$, for every $r\in(0,a]$, within the meaning of Ω_a^I, due to the possibility of unboundness of the tube [45]. Because Π is the two-dimensional world sheet, then (46) is subject to restriction $d = 2$. Without a loss of generality, an identical situation as in (105), would suit to Nambu-Goto action.

The beautiful meaning of the relation (105) requires from us a generalization for the concept of *p*-brane action. The relation (70) in the section 9.3.1, represents the *p*-brane action, which corresponds to the volume of $(p+1)$-dimensional world volume, up to the *p*-tension T_p. The topological structure of *p*-brane itself is sweeping out the mentioned $(p+1)$-dimensional world volume in D-dimensional spacetime, which we denote as M. If we imagine the given situation in the topological point of view, then the projected $(p+1)$-dimensional world volume corresponds to $(D+1)$-dimensional tubular neighbourhood Ω_a along the generalized reference time-like geodesic $\Sigma(\tau)$. However, we must guarantee the equivalence of parametrization using natural generalized spacetime coordinates and Fermi curvilinear coordinates.

As we have already mentioned, the system of Fermi coordinates also exist for the cases of general D-dimensional curved spacetimes, see [87]. Then the following theorem is fully functional.

Theorem 16. *Let $\sigma^0 = \tau$ and σ^i denotes one time-like coordinate and p space-like coordinates, which provides a natural parametrization of p-brane. Then the following map*

$$\xi^i:\ (\sigma^i,\tau)\to X^{\mu}(\sigma^i,\tau) \tag{106}$$

is equivalent with Fermi cordites for general spacetimes [87].

Due to the validity of the Theorem 16, a straightforward generalization of relation (105) can be performed for the *p*-brane action. Since

$$\int d\omega_p. = \int\sqrt{-det(G_{\alpha\beta})}d^{p+1}\sigma = \int\sqrt{-det(g_{\mu\nu}(X)\partial_{\alpha}X^{\mu}\partial_{\beta}X^{\nu})}d^{p+1}\sigma \tag{107}$$

denotes the volume of the $(p+1)$-dimensional world volume, which is embedded in M, then the relation (107) can be matched with the formula (59) for d-dimensional Lebesgue measure, or in other words, with the volume of $(D+1)$-dimensional *curved tube* Ω_a. Let us support this argument by the fact that the induced metric $G_{\alpha\beta}$ is the pull-back metric of the flat Minkowski spacetime [57, 63, 84]. Under this condition and within the wording of Theorem 9, consider that $K_{sec}^{M}=0$. Then the induced metric directly corresponds to the metric of flat pseudo-Riemannian manifold, and we can identify the formula (70), with the volume of *curved tube*, up to the *p*-brane tension, as follows

$$-T_p|\Omega_a| = S_p = -T_p \int d\omega_p \tag{108}$$

The existence of Jacobi field along the generalized time-like geodesic $\Sigma(\tau)$, which corresponds to the reference geodesic Ψ, is guaranteed, since the generalization of (90) is fulfilled [87]. And the action (108) is invariant under reparametrization [57], which only adds weight to the mentioned argumentation, given Fermi curvilinear coordinates.

Appendix 1. An exemplary step would be to bring to light the existence of a tubular neighbourhood. From physical point of view, its existence can be considered guaranteed, but from a purely mathematical side, not everything is so straightforward. Which in turn can have a significant impact on physical interpretation in the structures of *p*-branes to world sheets.

The answer is provided by the theorem of the existence of a tubular neighbourhood. The local existence of tubular neighbourhood is ensured by the properties of exponential map (section 7.1) [51]. Global existence is ensured if the given manifold M is compact [89]. Which is often not the case. For example, the often used Minkowski spacetime is not a compact manifold. It is also true that the universal cover of a simply connected, unbounded and time-orientable manifold is not compact [90]. Due to these topological obsequies, only the local existence of the tubular neighbourhood, i.e. the fulfillment of the properties of the exponential map, will suffice.

9.4.3 Unexpected topological nature of Hamilton's principle

Classical string motion, in the simplest case, minimizes the world sheet area, with respect to its volume [57]. We work identically with the action in the case of zero-branes, i.e. motion of particle. Classical action of particle minimizes the length of the world line, by moving along a time-like geodesic. In accordance with Theorem 9, let us stick to the generalization of these cases.

The motion of a p-brane extremizes, or rather, it minimizes the $(p+1)$-dimensional world volume, which is mapped to M [57, 63, 84]. Then the p-brane action is given as volume of the $(p+1)$-dimensional submanifold $P \subset M$ by the relation $S_p = -T_p V_p$, where V_p is the corresponding volume. With respect to Theorem 9, the structure of P is identical to the topology of *unbounded curved tube* Ω_a and all particular assumptions and results regarding parametrization of P and Ω_a are also fulfilled after previous analyzes. Then

$$S_p = -T_p V_p = -T_p |\Omega_a|. \qquad (109)$$

An interesting consequences of the combination of Polyakov and Nambu-Goto action with the p-brane structure is the generalization of these actions. Since the Polyakov and Nambu-Goto action are equivalent, then they corresponds to the area of two dimensional world sheet, in other words to the area of Π, with respect to its volume. Then, the whole example corresponds to the *unbounded curved tube* Ω_a, with $d = 2$ and M being general curved spacetime. The agreement with Theorem 9 is obtained by setting $K_{sec}^M = 0$.

The mentioned generalization corresponds to the following equation [57]

$$\widetilde{S_p} = -\int d^{p+1}\sigma \left\{ \frac{T_p}{2} \sqrt{-m} \left[m^{\alpha\beta} \partial_\alpha X \cdot \partial_\beta X g_{\alpha\beta}(X) \right] - (p-1) + other\ terms \right\}. \qquad (110)$$

Here M is a D-dimensional curved spacetime. Comparing equations (59) and (110), we find that the following relation holds

$$-T_p |\Omega_a| = \widetilde{S_p}, \qquad (111)$$

Because the performed generalization of the Polyakov action is equivalent to the action S_p, since in formula (83) the original metric is replaced by auxiliary field. In other words by dynamical metric $m_{\alpha\beta}$. Which does not detract from the original meaning of the p-brane action.

From the point of view of string theory, we obtain the equations of motion as the minimization of volume of P. Which occurs in the variation of the action (110) with respect to the dynamical metric $m_{\alpha\beta}$. The requirement is that the mentioned variation vanishes as

$$\frac{\delta S}{\delta m_{\alpha\beta}} = 0. \qquad (112)$$

Which leads to

$$\left(m^{\gamma\vartheta}m_{\alpha\beta} - 2m^{\pi\rho}m^{\varphi\omega}\right)G_{\alpha\beta} = (p-1)m^{\gamma\vartheta}, \tag{113}$$

where $G_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X g_{\alpha\beta}(X)$ corresponds to the induced metric on the $(p+1)$-dimensional world volume [57, 84]. If we set $K_{sec}^{M} = 0$, then M is flat spacetime. Then, due to the validity of (111), the variation of action with respect to the metric directly corresponds to minimizing the d-dimensional Legesgue measure of Ω_a, i.e. the volume $|\Omega_a|$. In the general case Ω_a is *unbounded curved tube*.

These arguments suggest that the variation that occurs according to (112) is in fact a minimization within the topology of Ω_a. In general case, Ω_a is *unbounded curved tube*. However, if we want the geometric nature of the $(p+1)$-dimensional world volume to correspond to the original curve tube (46), it is sufficient to intersect the world volume with two independent hypersurfaces. Subsequently continuously and smoothly close the generalized reference time-like geodesic $\Sigma(\tau) = \Sigma(\sigma^0)$. Thus creating a closed $(D+1)$-dimensional *curved tube*. Then, under these conditions, the variation of the action with the respect to the auxiliary field is equivalent to minimizing the volume of Ω_a, so

$$\frac{\delta S}{\delta m_{\alpha\beta}} = 0 \cong min(|\Omega_a|). \tag{114}$$

From a mathematical point of view, a given minimum may not always exist, so let's adjust the equivalence relation (114) to

$$\frac{\delta S}{\delta m_{\alpha\beta}} = 0 \cong inf(|\Omega_a|). \tag{115}$$

Then the equations of motion arise.

From a technical point of view, if we take the trace of (113) and substitute the result into (110), we would get the corresponding Nambu-Goto type action, which is generalized to *p*-branes [57, 60, 63]. Here we can formulate an identical variational problem, as in (115)

From relation (115) it can be deduced that Hamilton principle of least action could in fact possibly depends on topology of the situation itself. Thus the closed strings implies that existence of *unbounded* or closed *curved tubes*, as a $(D+1)$-dimensional world volume. So if we try to use these facts to clarify the nature of automatic validity of Hamilton's variational principle, then the bosonic strings are possibly the key. In particular, the minimization over their topology. Let's illustrate our reasoning with a concrete example.

Let M corresponds to classical Minkowski spacetime. Let us choose a gauge $X^0 \equiv t = Q\tau$ as in the section 9.3.1. Consider a photo shoot of a closed string configuration at a fixed time $\frac{d\vec{x}}{d\tau} = 0$. Then the kinetic energy vanishes and the Nambu-Goto action of classical closed string motion get the form (80) [64, 84]. The action is proportional to the time integral of the potential energy V_s and we obtain the result (81). From a geometrical point of view, the situation corresponds to the fact that the closed string tries to shrink continuously to the smallest possible size. Obviously, when we include quantum effects, this cannot happen. But let's look at the situation more form a topological point of view. The string is forced by the gauge and the form of the action (80) to shrink to the limit zero size, then the corresponding two-dimensional world sheet will smoothly minimize its perimeter. In fact, the situation corresponds to the topological nature of the Theorem 6, i.e. the Cheeger constant of *curved strips* [47]. This approach equivalently conditions the relation (81), for potential energy. Based on these considerations, the following hypothesis can be formulated.

Conjecture 2. *Let M corresponds to classical Minkowski spacetime. Let there exist a closed string, which propagates freely through M. Minimization of the corresponding two-dimensional world sheet conditions fixation of gauge and the subsequent transformation of the actio*n (77) *into a form* (80) *for obtaining the potential energy.*

Simply put, if Π is the two dimensional submanifold, which corresponds to Ω_a, with $d = 2$, then,

$$\text{if there exist } inf\left\{\frac{|\partial S|}{|S|} : S \subset \Omega_a, |S| > 0\right\} \rightarrow S_{NG} = -T \int l_S dt, \quad (116)$$

which implies that

$$V_S = T l_S. \quad (117)$$

We then believe that the deeper principle behind the mechanism of Conjecture 2, is given by the existence of an isoperimetric variation constant $h(\Omega_a)$ [45]. The essence of that mechanism could therefore by possibly clarified by a deeper variational topological principle.

Given the analysis we have done in the sections 9.4.2 and 9.4.3, the Conjecture 2, can be generalized to mechanism for computing the equations of motion of a given closed string within equivalent types of action. The reason is that the variation of the action with respect to the dynamical metric (auxiliary field) coincides with the process of minimizing the area of the two-dimensional world sheet with respect to its volume.

Conjecture 3. *Let Π be a two-dimensional submanifold of flat spacetime M. The submanifold Π directly corresponds to the unbounded or closed curved tube Ω_a (depends on our choice). Then the nature of Hamilton's variational principle lies in the existence of an isoperimetric constant* (1), *which minimizes the area of Π with respect to its volume, where $\Pi := \Omega_a, d = 2$. Let also $K_{sec}^M = 0$. Technically*

$$\exists!\ h(\Omega_a) = inf\left\{\frac{|\partial S|}{|S|} : S \subset \Omega_a, |S| > 0\right\} \rightarrow\ \delta S = 0 \rightarrow equations\ of\ motion, \qquad (118)$$

where $h(\Omega_a) = 1/a$ [45, 47].

A topological approach to bosonic string theory and closed strings actions in general may indirectly clarify the essence of Hamilton's principle of least action.
Approach within the Conjecture 3, can be upgraded to *p*-brane actions as follows.

Conjecture 4 (Generalization of Conjecture 3). *Let P be a $(p+1)$-dimensional world volume, which is mapped to D-dimensional flat spacetime M by p-brane. Let the action be described by relations* (70), (75) *or* (83). *Let also $K_{sec}^M = 0$. Then*

$$\exists!\, h(\Omega_a) = inf\left\{\frac{|\partial S|}{|S|} : S \subset \Omega_a, |S| > 0\right\} \rightarrow\ \delta S_p = 0 \rightarrow equations\ of\ motion, \qquad (119)$$

where $h(\Omega_a) = (d-1)/a$ [45].

Therefore, in general, we believe that a potential explanation for the validity of Hamilton's variational principle could be the existence of a deeper principle that respects only the topology of the situation, not its quantitative description.

9.4.4 Critical dimension as a possible answer

A possible hint of support for our argument in the previous conjectures could be hidden in the critical dimension, where the bosonic string theory is fully functional. In the transition from the Nambu-Goto action (75) to the Polakov action (83), an alternative to a rough definition of the dynamical metric is offered. It is possible to use the Feynman path integral to deal with local gauge fixing and other things, which provide that the process is carried out correctly, we encounter on interesting anomaly [60, 62, 63]. Thus anomaly cannot be removed until the spacetime dimension of the underlying manifold M is $D = 26$ [57]. In general, the procedure can be performed in almost all structures of string theory, as many of them allow the occurrence of both closed and open strings. We mentioned the example of bosonic string theory for its straightforwardness, as it contained only closed strings.

This fixed value of dimension of M, which determines the consistenccy of equations, i.e. different types of action, is a fundamental axiom for the internal consistency of the quantitative apparatus of theory. However, the fixed value of D has no obvious effect on the topology of M itself. Therefore, if the deeper principle sought, which clarifies the validity of the principle of least action, is of a topological and variational nature, then the given principle is hierarchically higher in term of generality than Hamilton's principle. From this point of view, therefore Hamilton's principle should form a phenomenological subunit of that deeper principle, which may possibly consist in the existence of a minimizer of (1). This is pointed out in relation to the variation of the previously mentioned types of action of closed strings.

Previous considerations and conjectures could also suggest that a potential explanation of the validity and nature of Hamilton's principle in string theory could be contained in topological gravity [91] or in the emergent nature of the gravitational interaction as an entropic force (see, [92, 93]).

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