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GENEROVANÉ FUZZY IMPLIKÁTORY VE FUZZY ROZHODOVÁNÍ

GENERATED FUZZY IMPLICATIONS IN FUZZY DECISION MAKING

DISERTAČNÍ PRÁCE
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Abstract

The Ph.D. thesis is focused on implications in many-valued logic. The main topic of interest are fuzzy implications generated by one-variable functions. This approach is known to-day mainly in case of t-norms, which are used to model a conjunction in many-valued logics. Several possibilities of construction of fuzzy implications via one-variable functions are given. Properties of these classes of generated implications and their intersections with known classes of $(S, N)-$ and $R-$ implications are studied.

The second topic of interest is a construction of fuzzy preference structures (FPS for short) using these generated fuzzy implications. Fuzzy preference structures present one of well-known apparatuses to model preference when working with vague notions. Our approach to construction of FPS utilizes the connection between fuzzy preference relations and fuzzy implications.

The last part is focused on a many-valued case of the modus ponens rule. Modus ponens is the most frequent rule of inference and it is used for example in artificial intelligence. There are two possible definitions of modus ponens, one with implicative rules and other with clause-based rules. In the case of many-valued logic, it is necessary to distinct between these two definitions, therefore we study them separately. One possible approach to define many-valued discrete case of modus ponens rule is also studied.

Keywords

Triangular norm, fuzzy decision making, fuzzy implication, generator function, fuzzy preference structures, many-valued modus ponens

Abstrakt

Dizertačná práca sa zaoberá implikátormi vo viachodnotovej logike. Hlavným objektom záujmu sú implikátory vytvorené pomocou funkcie jednej premennej, čo je prístup známy hlavne v prípade t-noriem, ktoré modelujú konjunkciu vo fuzzy logike. Opísaných je niekoľko možností konštrukcie fuzzy implikátora pomocou funkcie jednej premennej. Skúmané sú vlastnosti takto vygenerovaných implikátorov a takisto prienik tried generovaných implikátorov so známymi triedami $(S, N)-$ a $R-$ implikátorov.

Ďalej sa práca zaoberá možnostou konštrukcie fuzzy preferenčných štruktúr s pomocou uvedených implikátorov. Fuzzy preferenčné štruktúry sú jedným z využívaných nástrojov pri modelovaní preferencie v práci s vágnymi pojmami. Prezentovaný prístup konštrukcie využíva vzťah medzi reláciou preferencie a fuzzy implikátormi.

V poslednej časti sa zaoberáme viachodnotovou podobou pravidla modus ponens. Modus ponens je najčastejšie využívaným pravidlom odvodzovania a nachádza využitie napr. v umelej inteligencii. Modus ponens je možné definovať s využitím implikatívnych alebo klauzálnych pravidiel. V prípade viachodnotovej logiky musíme tieto dve možnosti rozlišovať, skúmané sú preto obidve. Takisto je ukázaný jeden z možných prístupov pri definovaní viachodnotovej diskrétnej podoby tohoto pravidla.

Kľúčové slová

Triangulárna norma, fuzzy rozhodovanie, fuzzy implikátory, vytvárajúca funkcia, fuzzy preferenčné štruktúry, viachodnotový modus ponens

Used symbols

$[a, b]$ closed interval

$]a, b[$ open interval

μ_M membership function of fuzzy subset M

$\mu_M(x)$ grade of membership of x in fuzzy subset M

$\mathcal{F}(X)$ system of fuzzy sets

$A \times B$ Cartesian product of sets A and B

$\cap_T$ intersection of fuzzy sets based on t-norm T

$\cup_S$ union of fuzzy sets based on t-conorm S

$\neg$ logical negation

$\vee$ logical disjunction

$\Rightarrow$ logical implication

$\equiv$ equivalence

$\vee_D$ fuzzy disjunction based on D

$\lceil x \rceil$ ceiling function

$\lfloor x \rfloor$ floor function

Declaration

I declare that I have elaborate my doctoral thesis on Generated fuzzy implications in fuzzy decision making independently under supervision of the doctoral thesis supervisor and using literature and other information sources, which are all cited in the work and listed in the bibliography at the end of work.

I furthermore declare that, concerning the creation of this doctoral thesis, I did not infringe the copyrights of third parties, in particular, I have not infringed any copyright. In particular, I have not unlawfully encroached on anyone's personal copyright and I am fully aware of the consequences in the case of breaking Regulation 11 and the following of the Copyright Act No 121/2000 Vol., including the possible consequences of criminal law resulted from Regulation 152 of Criminal Act No 140/1961 Vol.

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Contents

1 Introduction

Fuzzy logic and fuzzy sets are basic framework when working with vague notions. In classical logic all assertions are either true or false (i.e have truth values 1 or 0 respectively). In case of fuzzy logic the truth value may be any value in the interval $[0, 1]$. Connected with fuzzy logic is the notion of fuzzy sets. Classical set is given by it's characteristic function with values 0 and 1. Likewise, a fuzzy set is given by it's membership function with values from interval $[0, 1]$. The advantage of this approach is illustrated in the simple example:

Let's turn out attention to the assertion "He is a tall man". Suppose that we want to construct a set $\mathcal{T}$ of all "tall men". Obviously, this decision depends on one's personal experience. (For example a professional basketball player and regular people probably have a different notion of "being tall".) Moreover, if we want to evaluate this assertions only by "true" or "false", we get the following paradox: a $180cm$ tall man may be considered "tall" (i.e is in the set $\mathcal{T}$) but a $179cm$ one is considered "not tall at all" (and belong to the set $\mathcal{T}'$).

In this example we are working with vague notions. It is therefore better to consider the characteristic function with all values from interval $[0, 1]$ not only two values 0 and 1. For example a $190cm$ tall man can be considered "tall", while $170cm$ one is "not tall at all", a $185cm$ tall man can be considered "tall" in the degree 0.75, etc.

This approach was introduced by Lotfi A. Zadeh in 1965 in the article *Fuzzy sets*. Fuzzy sets were at first used in control theory and fuzzy regulation and later it expanded to other sectors where the informations are incomplete or imprecise, such as economy, bioinformatics, medicine, genealogical research etc.

The truth value of some assertion can not be decided in the classical two-valued (Aristotle) logic. Such assertions are known as logical paradoxes. Recall the well-known liar's paradox, which is sometimes credited to Epimenides. One of the versions of this paradox is a statement "This statement is false." Hypothesis that previous sentence is true leads to the conclusion that it is false, which is a contradiction. On the other hand, hypothesis that the statement is false also lead to contradiction.

The need of working with the vague notions is evident in so-called "paradox of the heap": One grain of sand is not a heap. If you don't have a heap and add just one grain of sand, then you won't get a heap. Both these assertions are obvious, but using them one can conclude that no number of grains will make a heap, which is in a contradiction with our experience.

These limitations of classical logic was known long ago, however, multivalued logics were not proposed until the beginning of 20th century. The three-valued logic was proposed by polish mathematician and philosopher Jan Łukasiewicz around 1920. Later, Łukasiewicz together with Alfred Tarski extended this logic for $n \geq 2$. In 1932, Hans Reichenbach formulated a logic with infinitely many values.

2 Preliminaries

2.1 Fuzzy logic connectives

In this paragraph we briefly introduce basic definitions, properties and examples of fuzzy logic connectives. First we turn our attention to the fuzzy negations, which are monotonic extensions of classical negations.

Definition 2.1 (see, e.g., Fodor and Roubens [22]) *A decreasing function* $N : [0,1] \to [0,1]$ *is called a* fuzzy negation *if* $N(0) = 1, N(1) = 0$. *A negation* N *is called*

1. strict *if it is strictly decreasing and continuous for arbitrary* $x \in [0,1]$,
2. strong *if it is an involution, i.e., if* $N(N(x)) = x$ *for all* $x \in [0,1]$.

A dual negation based on a negation N is given by

$$N^d(x) = 1 - N(1 - x).$$

Some examples of strict and/or strong negations are included in the following example. More examples of negations can be found in [22].

Example 2.2 *The following are some examples of fuzzy negations:*

- $N_s(x) = 1 - x$ *strong negation, standard negation,*
- $N(x) = 1 - x^2$ *strict, not strong negation,*
- $N(x) = \sqrt{1 - x^2}$ *strong negation,*
- $N_{G_1}(1) = 0, N_{G_1}(x) = 1$ *if* $x < 1$ *non-continuous, greatest, Gödel negation,*
- $N_{G_2}(0) = 1, N_{G_2}(x) = 0$ *if* $x > 0$ *non-continuous, smallest, dual Gödel negation.*

Lemma 2.3 *Let* $N : [0,1] \to [0,1]$ *be a strict negation. Then its dual negation,* N^d, *is also strict.*

Monotonic extension of the classical conjunction is called a *fuzzy conjunction*.

Definition 2.4 *An increasing mapping* $C : [0,1]^2 \to [0,1]$ *is called a* fuzzy conjunction *if, for any* $x, y \in [0,1]$, *it holds*

- $C(x, y) = 0$ *whenever* $x = 0$ *or* $y = 0$,
- $C(1, 1) = 1$.

We define fuzzy conjunction as a monotonic extension of classical two-valued conjunction. In general, fuzzy conjunction don't posses additional properties such as commutativity or associativity. The special fuzzy conjunctions called *triangular norms* are widely used in applications to model a conjunction in multivalued logic or an intersection of fuzzy sets. Triangular norms were introduced by Schweizer and Sklar in [41] as a generalization of triangular inequality to probabilistic metric spaces.

Definition 2.5 (Klement, Mesiar and Pap [35]) *A* triangular norm *(t*-norm *for short) is a binary operation on the unit interval* $[0,1]$*, i.e., a function* $T : [0,1]^2 \to [0,1]$ *such that for all* $x, y, z \in [0,1]$*, the following four axioms are satisfied:*
(T1) Commutativity $T(x, y) = T(y, x)$,
(T2) Associativity $T(x, T(y, z)) = T(T(x, y), z)$,
(T3) Monotonicity $T(x, y) \leq T(x, z)$ *whenever* $y \leq z$,
(T4) Boundary Condition $T(x, 1) = x$.

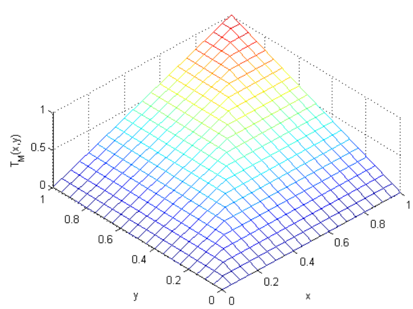
Example 2.6 *Four most common t-norms are:*

- *Minimum t-norm*
  $T_M(x, y) = \min(x, y)$,

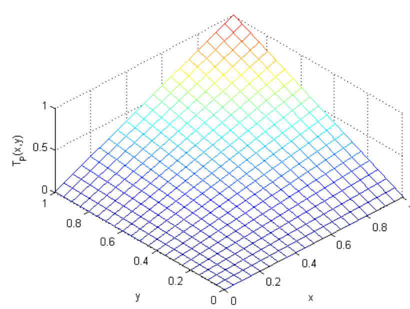
- *Product t-norm*
  $T_P(x, y) = x \cdot y$,

- *Łukasiewicz t-norm*
  $T_L(x, y) = \max(0, x + y - 1)$,

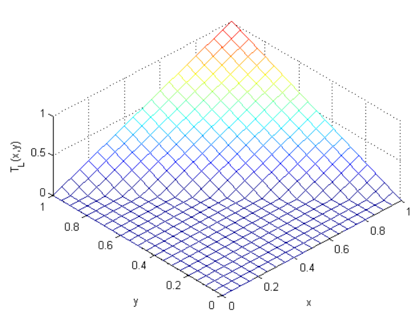
- *Drastic t-norm*
  $$T_D(x, y) = \begin{cases} \min(x, y) & \textit{if } \max(x, y) = 1, \\ 0 & \textit{otherwise.} \end{cases}$$



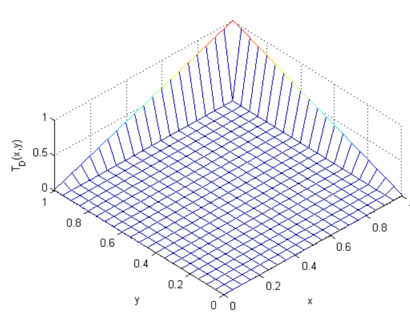
(a) Minimum t-norm



(b) Product t-norm



(c) Łukasiewicz t-norm



(d) Drastic t-norm

Figure 1: Basic t-norms

Remark 2.7 *Another interesting t-norms are given by*

$$T^s(x,y) = \log_s \left(1 + \frac{(s^x - 1)(s^y - 1)}{s - 1}\right),$$

where $s \in]0, \infty[- \{1\}$ *and the limit cases are* $T^0 = T_M$, $T^1 = T_P$ *and* $T^\infty = T_L$. *The functions* $T^s : [0,1]^2 \to [0,1]$ *are called* Frank t-norms.

Since t-norms are two-variable functions, it is possible to compare them in the following way:

Definition 2.8

- *If, for two t-norms* T_1 *and* T_2, *the inequality* $T_1(x,y) \leq T_2(x,y)$ *holds for all* $(x,y) \in [0,1]^2$, *then we say that* T_1 *is* weaker *than* T_2, *or equivalently that* T_2 *is* stronger *than* T_1, *and we write* $T_1 \leq T_2$.

- *We shall write* $T_1 < T_2$ *whenever* $T_1 \leq T_2$ *and* $T_1 \neq T_2$ *(i.e. there exists* $(x_0, y_0) \in [0,1]^2$ *such that* $T_1(x_0,y_0) < T_2(x_0,y_0)$*).*

Remark 2.9

- *Note that, for any t-norm* T *and for any* $a \in [0,1]$ *it holds that* $T(a,a) \leq a$.

- *Element* $a \in [0,1]$ *that satisfies equality* $T(a,a) = a$ *is called an* idempotent element of t-norm T. *Any t-norm has at least two idempotent elements:* 0 *and* 1, *these are called* trivial idempotent elements.

- *Using axioms (T3) and (T4) one can show that minimum t-norm is the strongest one and drastic t-norm is the weakest one, i.e. for any t-norm* T *it holds that*

$$T_D \leq T \leq T_M.$$

- *Basic t-norms are ordered in following way:*

$$T_D < T_L < T_P < T_M.$$

Because t-norms are associative, they can be uniquely extended to n-nary operation on the unit interval:

Definition 2.10 *Let* T *be a t-norm,* $x \in [0,1]$ *and* $n \in N$, *by* $x^{(n)}$ *we denote:*

$$x^{(n)} = \begin{cases} x & \text{if } n = 1, \\ T(x, x_T^{(n-1)}) & \text{if } n > 1. \end{cases}$$

One of the most important properties of functions is a continuity. Since t-norms are special case of two-variable functions, continuity of t-norms is defined as:

Definition 2.11 *Let* $T : [0,1]^2 \to [0,1]$ *be a t-norm. We say that t-norm* T *is* continuous *if function* T *is continuous in any point* $(x,y) \in [0,1]^2$.

Theorem 2.12 (Klement, Mesiar and Pap [35]) *A t-norm T is continuous if it is continuous in the first variable, i.e. if for any $y \in [0,1]$, one-variable function*

$$T(.,y) : [0,1] \to [0,1]$$

is continuous.

The weaker form of continuity frequently used in case of t-norms is left- (or right-) continuity:

Definition 2.13 *We say that t-norm T is* left-continuous (right-continuous *respectively*), *if for any $y \in [0,1]$ and any increasing (decreasing) sequence $(x_n)_{n\in N}$ it holds that*

$$\lim_{n\to\infty} T(x_n,y) = T(\lim_{n\to\infty} x_n, y).$$

Theorem 2.14 (Klement, Mesiar and Pap [35]) *A triangular norm T is left-continuous (right-continuous respectively), if and only if it is left-continuous (right-continuous) in the first variable, i.e. if for any $y \in [0,1]$ and for any sequence $(x_n)_{n\to\infty}$ it holds that*

$$\sup T(x_n,y) = T(\sup x_n, y) \quad (\inf T(x_n,y) = T(\inf x_n, y)) .$$

Remark 2.15 *Triangular norms T_M, T_P and T_L are continuous and T_D is right-continuous.*

Definition 2.16 (Klement, Mesiar and Pap [35])

- *A t-norm T is said to be* strictly monotone *if it is strictly increasing on $]0,1]^2$ as a function from $[0,1]^2$ into $[0,1]$ or, equivalently, if*

$$T(x,y) < T(x,z) \text{ whenever } x \in]0,1[\text{ and } y < z.$$

- *A t-norm T is called* strict *if it is continuous and strictly monotone.*

Example 2.17 (Klement, Mesiar and Pap [35])

- *The Minimum T_M and the Łukasiewicz t-norm T_L are continuous but not strictly monotone.*
- *The t-norm T defined by*

$$T(x,y) = \begin{cases} \frac{xy}{2} & \text{if } \max(x,y) < 1, \\ xy & \text{otherwise,} \end{cases}$$

is strictly monotone but not continuous.

- *Among the four basic t-norms, only the Product T_P is a strict t-norm.*

Theorem 2.18 (Klement, Mesiar and Pap [35]) *A t-norm T is strictly monotone if and only if the cancellation law holds, i.e., if $T(x,y) = T(x,z)$ and $x > 0$ imply $y = z$.*

For any two numbers from the interval $]0,1[$ there exists a natural number n with the property $x^n \leq y$. This fact is well-known as Archimedean property on the interval $]0,1[$. In the case of t-norms, the Archimedean property is defined similarly:

Definition 2.19 (Klement, Mesiar and Pap [35]) *A t-norm T is called* Archimedean *if for all $(x,y) \in]0,1[^2$ there is an integer $n \in N$ such that*

$$x_T^{(n)} < y.$$

Theorem 2.20 (Klement, Mesiar and Pap [35]) *A t-norm T is Archimedean if and only if for each $x \in]0,1[$ we have*

$$\lim_{x \to \infty} x_T^{(n)} = 0.$$

At least for continuous t-norms it is possible to characterize the Archimedean property by their diagonal mapping:

Theorem 2.21 (Klement, Mesiar and Pap [35])

- *If T is an Archimedean t-norm, then for each $x \in]0,1[$ we have*

$$T(x,x) < x.$$

- *If T is right-continuous, then it is Archimedean if and only if for all $x \in]0,1[$ it holds that $T(x,x) < x$.*

Note that each strict t-norm T is Archimedean.

Definition 2.22 (Klement, Mesiar and Pap [35]) *A t-norm T is called* nilpotent *if it is continuous and if each element $a \in]0,1[$ is nilpotent, i.e., if there exists some $n \in N$ such that $a_T^{(n)} = 0$.*

Definition 2.23 (Klement, Mesiar and Pap [35]) *An element $x \in]0,1[$ is called* zero divisor *of t-norm T if there exists some $y \in]0,1[$ such that $T(x,y) = 0$.*

Nilpotent t-norms can be completely characterized:

Theorem 2.24 (Klement, Mesiar and Pap [35]) *Let T be a continuous Archimedean t-norm. Then the following are equivalent:*

- *T is nilpotent.*
- *There exists some nilpotent element of T.*
- *T is not strict.*
- *T has zero divisors.*

In the literature we can find many ways of constructing new t-norms. The most common are the following:

- ordinal sum:
  Let $(T_\alpha)_{\alpha\in A}$ be a class of t-norms and let $(]a_\alpha, e_\alpha[)_{\alpha\in A}$ be a system of non-overlapping intervals. Then the mapping $T : [0,1]^2 \to [0,1]$ given by

  $$T(x,y) = \begin{cases} a_\alpha + (e_\alpha - a_\alpha)\cdot T_\alpha\left(\frac{x-a_\alpha}{e_\alpha-a_\alpha}, \frac{y-a_\alpha}{e_\alpha-a_\alpha}\right) & \text{if } (x,y)\in]a_\alpha, e_\alpha[^2, \\ \min(x,y) & \text{otherwise,} \end{cases}$$

  is a t-norm which is called *ordinary sum* of summands $\langle a_\alpha, e_\alpha, T_\alpha\rangle,\ \alpha\in A$.

- additive or multiplicative generating:
  Let $f : [0,1] \to [0,\infty]$ be a continuous decreasing function such that $f(1) = 0$. Then the function $T_{<f>}$ defined as

  $$T_{<f>}(x,y) = f^{-1}\left(\min(f(x)+f(y), f(0))\right)$$

  is a t-norm. Function f is called an *additive generator of t-norm $T_{<f>}$.*
  Let $g : [0,1] \to [0,\infty]$ be a continuous increasing function such that $g(0) = 0$. Then the function $T^{<g>}$ defined as

  $$T^{<g>}(x,y) = g^{-1}\left(\max(g(x)\cdot g(y), g(0))\right)$$

  is a t-norm and function g is called a *multiplicative generator of t-norm $T_{<f>}$.*

- φ-transformation:
  Let φ be an increasing bijection of interval $[0,1]$ and let T be an arbitrary t-norm. Then the mapping T_φ defined as

  $$T_\varphi(x,y) = \varphi^{-1}\left(T(\varphi(x), \varphi(y))\right)$$

  is a t-norm and it is called a *φ-transformation of t-norm T.*

Theorem 2.25 (Ling [38]) *Function $T : [0,1]^2 \to [0,1]$ is a continuous Archimedean t-norm if and only if there exists a function $f : [0,1] \to [0,\infty]$, such that $f(1) = 0$ and for any $x, y \in [0,1]$ we have*

$$T(x,y) = f^{-1}\left(\min(f(x)+f(y), f(0))\right).$$

The function f is called an additive generator *of t-norm T and it is unique up to a positive multiplicative constant.*

Theorem 2.26 (Klement, Mesiar and Pap [35]) *Function $T : [0,1]^2 \to [0,1]$ is a continuous Archimedean t-norm if and only if there exists a function $g : [0,1] \to [0,1]$, such that $g(1) = 1$ and for any $x, y \in [0,1]$ we have*

$$T(x,y) = g^{-1}\left(\max(g(x)\cdot g(y), g(0))\right).$$

The function f is called an multiplicative generator *of t-norm T and it is unique up to a positive exponent.*

Remark 2.27

- *Let* $f : [0,1] \to [0,\infty]$ *be an additive generator of a continuous Archimedean* t*-norm* T*, then the function* $g : [0,1] \to [0,1]$ *given by* $g(x) = e^{-f(x)}$ *is a multiplicative generator of* t*-norm* T*.*

- *Let* $g : [0,1] \to [0,1]$ *be a multiplicative generator of a continuous Archimedean* t*-norm* T*, then the function* $f : [0,1] \to [0,\infty]$ *given by* $f(x) = -\log g(x)$ *is an additive generator of* t*-norm* T*.*

- *Triangular norm* T *is strict if and only if for its additive generator* f *it holds* $f(0) = \infty$ *(and for multiplicative generator it holds* $g(0) = 0$*).*

- *Triangular norm* T *is nilpotent if and only if for its additive generator* f *it holds* $f(0) < \infty$ *(and for multiplicative generator it holds* $g(0) > 0$*).*

From the properties of multiplicative generators and theorem 2.26 it follows that any strict t-norm is isomorphic with product t-norm T_P.

Theorem 2.28 (Klement, Mesiar and Pap [35]) *A function* $T : [0,1]^2 \to [0,1]$ *is a strict* t*-norm if and only if there exists a strictly increasing automorphism* $\varphi : [0,1] \to [0,1]$*, such that* $T = (T_P)_\varphi$*, i.e.*

$$T(x,y) = \varphi^{-1}\left(T_P(\varphi(x), \varphi(y))\right) \qquad \forall x,y \in [0,1].$$

Similar theorem holds for all nilpotent t-norms and Łukasiewicz t-norm T_L.

Theorem 2.29 (Klement, Mesiar and Pap [35]) *A function* $T : [0,1]^2 \to [0,1]$ *is a nilpotent* t*-norm if and only if there exists a strictly increasing automorphism* $\varphi : [0,1] \to [0,1]$*, such that* $T = (T_L)_\varphi$*, i.e.*

$$T(x,y) = \varphi^{-1}\left(T_L(\varphi(x), \varphi(y))\right) \qquad \forall x,y \in [0,1].$$

For the construction of continuous Archimedean triangular norms with additive or multiplicative generator we deal with functions which have inverse functions. But it is possible to construct the non-continuous t-norms via generator. For example, function $t : [0,1] \to [1,2]$, which is given by

$$t(x) = \begin{cases} 2 - x, & \text{if } x \in [0,1[, \\ 0, & \text{if } x = 1, \end{cases}$$

is additive generator of Drastic product (Ling [38]). It means that generators of non-continuous t-norms do not have to be bijections. In the construction of fuzzy operators we will use a generalization of inverse function. The reason for this is following: Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing function. If f is either not continuous or bounded, then the inverse function f^{-1} is defined only on a subset of interval $[0,\infty]$. Because in our construction we need function defined on whole interval $[0,\infty]$, we use a monotonic extension of f^{-1} which is called a *pseudo-inverse*:

Definition 2.30 *Let $f : [0,1] \to [0,\infty]$ be a decreasing and non-constant function. The function $f^{(-1)} : [0,\infty] \to [0,1]$ which is defined by*

$$f^{(-1)}(x) = \sup\{z \in \ [0,1]; f(z) > \ x\},$$

is called the pseudo-inverse *of the function f, with the convention* $\sup \emptyset = 0$.

By this definition, f can be any decreasing function, but only strictly decreasing functions are important for us. Lets turn our attention to the following example:

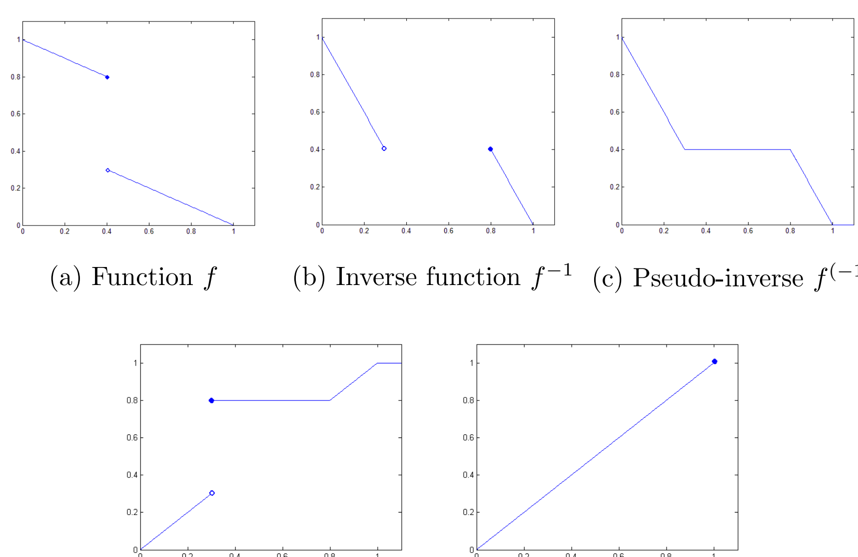
Example 2.31 *Let $f_1, f_2 : [0,1] \to [0,\infty]$ be defined as follows:*

$$f_1(x) = -\ln x \qquad\qquad f_2(x) = \begin{cases} 1 - \frac{x}{2} & x \in [0, 0.4], \\ \frac{1-x}{2} & x \in]0.4, 1], \end{cases}$$

Obviously $f_1^{(-1)}(x) = f_1^{-1}(x) = e^{-x}$, while for f_2^{-1} and $f_2^{(-1)}$ we get:

$$f_2^{-1}(x) = \begin{cases} 1 - 2x & x \in [0, 0.3[, \\ 2 - 2x & x \in [0.8, 1], \end{cases} \qquad\qquad f_2^{(-1)}(x) = \begin{cases} 1 - 2x & x \in [0, 0.3[, \\ 0.4 & x \in [0.3, 0.8[, \\ 2 - 2x & x \in [0.8, 1], \\ 0 & x \in]1, \infty]. \end{cases}$$

Note, that inverse f_2^{-1} is strictly decreasing and pseudo-inverse $f_2^{(-1)}$ is decreasing. Also note, that $f_2^{(-1)}$ is continuous and domains of f_2^{-1} and $f_2^{(-1)}$ are different.



(a) Function f (b) Inverse function f^{-1} (c) Pseudo-inverse $f^{(-1)}$

(d) Function $f \circ f^{(-1)}$ (e) Function $f^{(-1)} \circ f$

Figure 2: Functions f, f^{-1} and $f^{(-1)}$ and their compositions

A pseudo-inverse can be defined also for increasing functions:

Definition 2.32 *Let $\varphi : [0,1] \to [0,\infty]$ be an increasing and non-constant function. The function $\varphi^{(-1)} : [0,\infty] \to [0,1]$ which is defined by*

$$\varphi^{(-1)}(x) = \sup\{z \in [0,1]; \varphi(z) < x\},$$

is called the pseudo-inverse of the function φ, with the convention $\sup \emptyset = 0$.

Some properties of pseudo-inverse are mentioned in the previous example. In general, following properties holds for pseudo-inverse of an increasing function:

Remark 2.33 *Let $f : [a,b] \to [c,d]$ be an increasing and non-constant function, then*

- *pseudo-inverse function $f^{(-1)}$ is increasing and left-continuous,*
- $(f^{(-1)})^{(-1)}(x) = f(x)$ *if and only if f is left-continuous and $f(a) = c$,*
- *if f is a bijection, then $f^{(-1)}(x) = f^{-1}(x)$,*
- *if f is strictly increasing, then its pseudo-inverse is a continuous function,*
- *for any $x \in [a,b]$ it holds that $f^{(-1)}(f(x)) \leq x$,*
- *if f is strictly increasing, then $f^{(-1)}(f(x)) = x$,*
- *if f is a surjection, then $f(f^{(-1)}(x)) = x$.*

For decreasing function f there are similar properties. Note that some properties differ mainly in the type of continuity:

Remark 2.34 *Let $f : [a,b] \to [c,d]$ be a decreasing and non-constant function, then*

- *pseudo-inverse function $f^{(-1)}$ is decreasing and right-continuous,*
- $(f^{(-1)})^{(-1)}(x) = f(x)$ *if and only if f is right-continuous and $f(a) = d$,*
- *if f is a bijection, then $f^{(-1)}(x) = f^{-1}(x)$,*
- *if f is strictly decreasing, then its pseudo-inverse is a continuous function,*
- *for any $x \in [a,b]$ it holds that $f^{(-1)}(f(x)) \leq x$,*
- *if f is strictly decreasing, then $f^{(-1)}(f(x)) = x$,*
- *if f is a surjection, then $f(f^{(-1)}(x)) = x$.*

Dual operator to a fuzzy conjunction is called a *fuzzy disjunction*. A fuzzy disjunction is the monotonic extension of classical disjunction:

Definition 2.35 *An increasing mapping $D : [0,1]^2 \to [0,1]$ is called a* fuzzy disjunction *if, for any $x, y \in [0,1]$, it holds*

- $D(x,y) = 1 \quad$ *whenever* $\quad x = 1 \quad$ *or* $\quad y = 1$,
- $D(0,0) = 0$.

The dual mapping to a t-norm is a *triangular conorm* (*t-conorm* for short). The t-conorms are used to model a union of fuzzy sets or as disjunctions in fuzzy logic. One possible definition is axiomatic:

Definition 2.36 *A* triangular conorm *(*t-conorm *for short) is a binary operation on the unit interval* $[0,1]$*, i.e., a function* $S : [0,1]^2 \to [0,1]$ *such that for all* $x, y, z \in [0,1]$*, the following four axioms are satisfied:*
(S1) Commutativity $S(x,y) = S(y,x)$,
(S2) Associativity $S(x, S(y,z)) = S(S(x,y), z)$,
(S3) Monotonicity $S(x,y) \leq S(x,z)$ *whenever* $y \leq z$,
(S4) Boundary Condition $S(x,0) = x$.

The original definition was given by Schweizer and Sklar:

Definition 2.37 (Schweizer and Sklar [41]) *Dual operator to t-norm is called a* triangular conorm (t-conorm)*, defined as* $S(x,y) = 1 - T(1-x, 1-y)$.

Of course, the mentioned definitions are equivalent, both are used in the literature.

Example 2.38 *Dual t-conorms to basic t-norms are (Fig. 3):*

- *Maximum t-conorm*
  $S_M(x,y) = \max(x,y)$,
- *Probabilistic sum*
  $S_P(x,y) = x + y - x \cdot y$,
- *Łukasiewicz t-conorm*
  $S_L(x,y) = \min(x+y, 1)$.
- *Drastic t-conorm*
  $S_D(x,y) = \begin{cases} \max(x,y) & \text{if } \min(x,y) = 0, \\ 1 & \text{otherwise,} \end{cases}$

Remark 2.39 *Let* N *be an arbitrary fuzzy negation and* C *be a fuzzy conjunction. Let* $D_N : [0,1]^2 \to [0,1]$ *be a mapping defined as*

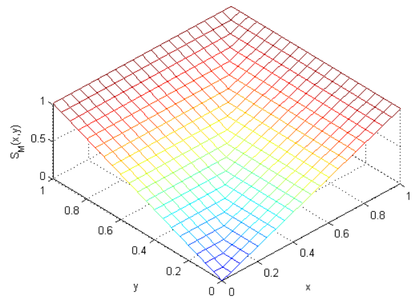
$$D_N(x,y) = N(C(N(x), N(y))).$$

Then D_N *is a fuzzy disjunction. Also, for an arbitrary disjunction* D *and negation* N*, mapping* $C_N : [0,1]^2 \to [0,1]$ *defined as*

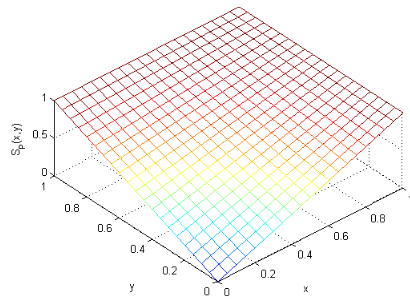
$$C_N(x,y) = N(D(N(x), N(y)))$$

is a fuzzy conjunction. Note that equality $(C_N)_N = C$ *is not true with arbitrary conjunction* C *and negation* N*.*

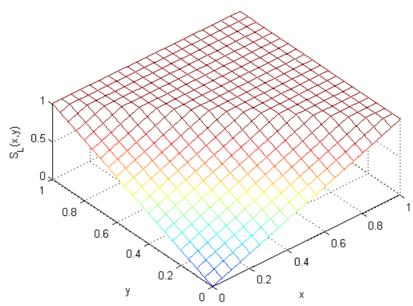
Special classes of fuzzy conjunctions and disjunctions are called t-seminorms C and t-semiconorms D. We use these mappings as the truth functions for conjunctions and disjunctions in some parts of the thesis.



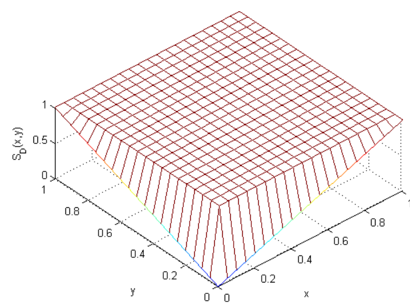
(a) Maximum t-conorm



(b) Probabilistic sum



(c) Łukasiewicz t-conorm



(d) Drastic t-conorm

Figure 3: Basic t-conorms

Definition 2.40 (Schweizer and Sklar [41])

(i) A t-seminorm C *is a fuzzy conjunction that satisfied the boundary condition*

$$C(1, x) = C(x, 1) = x \text{ for all } x \in [0, 1].$$

(ii) A t-semiconorm D *is a fuzzy disjunction that satisfied the boundary condition*

$$D(0, x) = D(x, 0) = x \text{ for all } x \in [0, 1].$$

In the literature, we can find several different definitions of fuzzy implications. We will use the following one, which is equivalent to the definition introduced by Fodor and Roubens in [22]. More information on this topic can be found in [3] and [39].

Definition 2.41 *A function* $I : [0, 1]^2 \to [0, 1]$ *is called a* fuzzy implication *if it satisfies the following conditions:*

(I1) I *is decreasing in its first variable,*

(I2) I *is increasing in its second variable,*

(I3) $I(1, 0) = 0$, $I(0, 0) = I(1, 1) = 1$.

Several classes of fuzzy implications are well-known. First one is based on a tautology $B \Rightarrow H \equiv \neg B \vee H$:

Definition 2.42 (Baczyński and Jayaram [2]) *A function $I : [0,1]^2 \to [0,1]$ is called an* (S,N)-implication *if there exists a t-conorm S and a fuzzy negation N such that*

$$I(x,y) = S(N(x), y), \qquad \forall x,y \in [0,1].$$

If N is a strong negation, then I is called strong implication.

Example 2.43 *Implications obtained using three basic continuous t-conorms and standard negation N_s are:*

- *Kleene-Dienes implication*

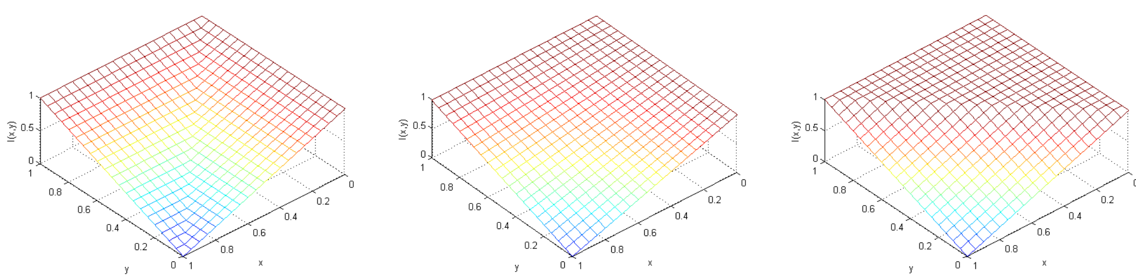
$$I_{S_M}(x,y) = \max(1-x, y),$$

- *Reichenbach implication*

$$I_{S_P}(x,y) = 1 - x + x \cdot y,$$

- *Łukasiewicz implication*

$$I_{S_L}(x,y) = \min(1-x+y, 1).$$



(a) Kleene-Dienes implication (b) Reichenbach implication (c) Łukasiewicz implication

Figure 4: (S,N)-implications

Other well-known approach to obtain a fuzzy implication uses residuation with respect to t-norm:

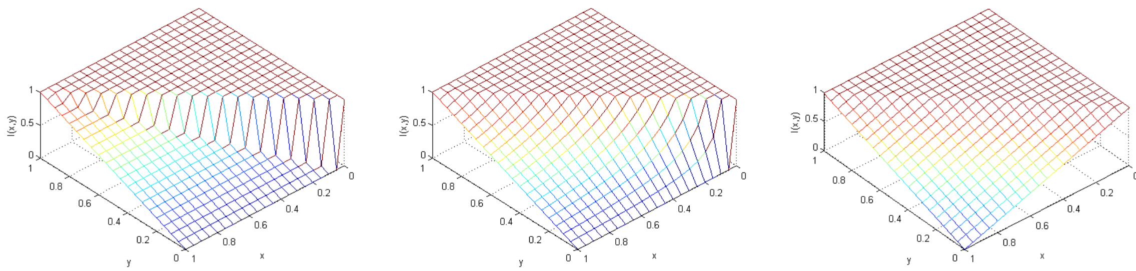
Definition 2.44 (Fodor and Roubens [22]) *A function $I : [0,1]^2 \to [0,1]$ is called an* R-implication *if there exists a t-norm T such that*

$$R_T(x,y) = \sup\{t \in [0,1]; T(x,t) \le y\}, \qquad \forall x,y \in [0,1].$$

Example 2.45 *For three basic continuous t-norms we get the following residual implications (Fig 5):*

- *Gödel implication*

$$R_{T_M}(x,y) = \begin{cases} 1 & \text{if } x \le y, \\ y & \text{otherwise,} \end{cases}$$



(a) Gödel implication (b) Goguen implication (c) Łukasiewicz implication

Figure 5: R-implications

- *Goguen implication*

$$R_{T_P}(x,y) = \min(\frac{y}{x}, 1),$$

- *Łukasiewicz implication*

$$R_{T_L}(x,y) = \min(1 - x + y, 1).$$

In a classical logic there is no difference between (S,N)-implications and R-implications. In the previous examples it was shown that this property doesn't hold in fuzzy case in general. Observe that the Łukasiewicz implication belongs to both classes, while the rest of mentioned implications are either (S,N)- or R-implications.

The third well-known class of implications is the class of Q-implications (Q is short for quantum logic). This class is based on the tautology $(A \Rightarrow B) \equiv (\neg A \vee (A \wedge B))$. The Q-implication is therefore defined as

$$I^Q_{S,T}(x,y) = S(N(x), T(x,y)) \qquad \forall x,y \in [0,1].$$

Example 2.46 *Using a t-norm T_M, t-conorm S_M and negation N_S we get so-called* Zadeh implication *I_Z:*

$$I_Z(x,y) = I^Q_{T_M,S_M}(x,y) = \max(1 - x, \min(x,y)).$$

For a t-norm T_L, t-conorm S_L and a negation N_S we get Kleene-Dienes implication I_{S_M}:

$$I^Q_{T_L,S_L}(x,y) = \min(1 - x + \max(x + y - 1, 0), 1) = \max(1 - x, y),$$

Note that I_Z is not monotone and therefore does not meet the criteria in Definition 2.41. L. A. Zadeh used the mapping I_Z as fuzzy implication before the Definition 2.41 was proposed.

Fuzzy implications may possess several important properties. Note that some of these properties (namely (EP), (CP) and (LI)) are well-known tautologies in classical two-valued logic.

Definition 2.47 *A fuzzy implication $I : [0,1]^2 \to [0,1]$ satisfies:*

(NP) *the* left neutrality property*, or is called* left neutral*, if*

$$I(1, y) = y; \quad y \in [0,1],$$

(EP) *the* exchange principle *if*

$$I(x, I(y, z)) = I(y, I(x, z)) \text{ for all } x, y, z \in [0,1],$$

(IP) *the* identity principle *if*

$$I(x, x) = 1; \quad x \in [0,1],$$

(OP) *the* ordering property *if*

$$x \leq y \iff I(x, y) = 1; \quad x, y \in [0,1],$$

(CP) *the* contrapositive symmetry with respect to a given negation N *if*

$$I(x, y) = I(N(y), N(x)); \quad x, y \in [0,1].$$

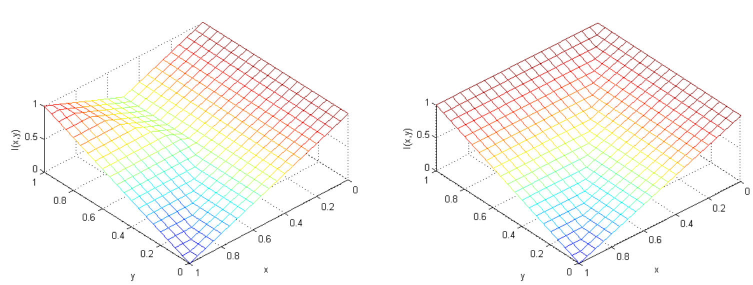
(LI) *the* law of importation with respect to a t-norm T *if*

$$I(T(x, y), z) = I(x, I(y, z)); \quad x, y, z \in [0,1].$$

(WLI) *the* weak law of importation with respect to a commutative and increasing function $F : [0,1]^2 \to [0,1]$ *if*

$$I(F(x, y), z) = I(x, I(y, z)); \quad x, y, z \in [0,1].$$

Definition 2.48 *Let $I : [0,1]^2 \to [0,1]$ be a fuzzy implication. The function $N_I : [0,1] \to [0,1]$ defined by $N_I(x) = I(x, 0)$ for all $x \in [0,1]$, is called the* natural negation *of I.*



(a) Zadeh implication (b) Kleene-Dienes implication

Figure 6: Q-implications

2.2 Fuzzy relations

Let Ω be a classical set and A be a subset of Ω. It is well-known that A is given by its characteristic function $\chi_A : \Omega \to \{0,1\}$ with the property that $\chi_A(x) = 1$ if and only if $x \in A$. In similar way, a fuzzy subset F of the set Ω is given by it's *membership function*, which is a mapping $\mu_F : \Omega \to [0,1]$. Membership function is illustrated in the following example:

Example 2.49 *Let F be the fuzzy set of real numbers that are approximately equal to 5. Membership function of F could be given by*

$$\mu_F(x) = \begin{cases} 1 - |5 - x| & \text{if } x \in [4,6], \\ 0 & \text{otherwise.} \end{cases}$$

A classical binary relation is a set of ordered pairs of elements. Fuzzy relation is therefore defined as a special case of fuzzy set:

Definition 2.50 *Let X and Y are two classical sets. A binary* fuzzy relation *from X to Y is any fuzzy subset of the set $X \times Y$, i.e. $R \in \mathcal{F}(X \times Y)$. A fuzzy relation is defined by a membership function $\mu_R : X \times Y \to [0,1]$.*

Basic operations such as intersection, union and complement of fuzzy relations P and R gives the relations with membership functions as follows:

$$\mu_{P \cap_T R}(x,y) = T(\mu_P(x,y), \mu_R(x,y)),$$

$$\mu_{P \cup_T R}(x,y) = S(\mu_P(x,y), \mu_R(x,y)),$$

$$\mu_{\overline{P^n}}(x,y) = N(\mu_P(x,y)).$$

The standard operations are based on minimum t-norm T_M, maximum t-conorm S_M and the standard negation N_s.

Definition 2.51 *Let X, Y, U are the classic sets, let P and R are fuzzy relations $P \in \mathcal{F}(X \times Y)$ and $R \in \mathcal{F}(Y \times U)$ and let T be a t-norm. Then the relation $P \circ_T R \in \mathcal{F}(X \times U)$ with membership function*

$$\mu_{P \circ_T R}(x,z) = \sup_{y \in Y} T(\mu_P(x,y), \mu_R(y,z))$$

is called sup-T-composition *of fuzzy relations P and R.*

Important properties of fuzzy relations are derived from the properties of crisp relations. Some of the properties (like asymmetry) depends on used t-norm:

Definition 2.52 (Zadeh [52]) *Let R be a binary fuzzy relation on the set X. Then the relation R is called:*

- reflexive*, if* $\forall x \in X; \mu_R(x,x) = 1$,
- irreflexive*, if* $\forall x \in X; \mu_R(x,x) = 0$,

- antireflexive, *if* $\forall x \in X; \mu_R(x,x) \neq 1$,
- symmetric, *if* $\forall x, y \in X; \mu_R(x,y) = \mu_R(y,x)$,
- T-antisymmetric, *if* $\forall x, y \in X; x \neq y \Rightarrow T(\mu_R(x,y), \mu_R(y,x)) = 0$,
- T-asymmetric, *if* $\forall x, y \in X; T(\mu_R(x,y), \mu_R(y,x)) = 0$,
- T-transitive, *if* $\forall x, y, z \in X; T(\mu_R(x,y), \mu_R(y,z)) \leq \mu_R(x,z)$.

If relation R is T_1-transitive and $T_2 \leq T_1$, then R is also T_2-transitive. Because of this fact, any min-transitive relation R is also T-transitive with any t-norm.

Definition 2.53 *A fuzzy relation R is called*

- T-equivalence, *if it is reflexive, symmetric and T-transitive,*
- T-partially ordered, *if it is reflexive, T-antisymmetric and T-transitive.*

A preference structure is a basic concept of preference modelling. In a classical preference structure (PS), a decision-maker makes one of three decisions for each pair (a, b) from the set $\boldsymbol{A}$ of all alternatives. His decision defines a triplet P, I, J of crisp binary relations on $\boldsymbol{A}$:

1) a is preferred to $b \Leftrightarrow (a, b) \in P$ (strict preference).

2) a and b are indifferent $\Leftrightarrow (a, b) \in I$ (indifference).

3) a and b are incomparable $\Leftrightarrow (a, b) \in J$ (incomparability).

A *preference structure (PS)* on a set $\boldsymbol{A}$ is a triplet (P, I, J) of binary relations on $\boldsymbol{A}$ such that

(ps1) I is reflexive, while P and J are irreflexive,

(ps2) P is asymmetric, while I and J are symmetric,

(ps3) $P \cap I = P \cap J = I \cap J = \emptyset$,

(ps4) $P \cup I \cup J \cup P^t = \mathbf{A} \times \mathbf{A}$ where $P^t(x, y) = P(y, x)$.

Using characteristic mappings [50] a minimal definition of (PS) can be formulated as a triplet (P, I, J) of binary relations on $\boldsymbol{A}$ such that

- I is reflexive and symmetric.
- $P(a, b) + P^t(a, b) + I(a, b) + J(a, b) = 1$ for all $(a, b) \in \boldsymbol{A}^{\mathbf{2}}$.

A preference structure can be characterized by the reflexive relation $R = P \cup I$ called the *large preference relation.* The relation R can be interpreted as

$$(a, b) \in R \Leftrightarrow a \text{ is prefered to } b \text{ or } a \text{ and } b \text{ are indifferent.}$$

It can be easily proved that

$$co(R) = P^t \cup J,$$

where $co(R)$ is the complement of R and

$$P = R \cap co(R^t), \ \ I = R \cap R^t, \ \ J = co(R) \cap co(R^t).$$

This allows us to construct a preference structure (P, I, J) from a reflexive binary operation R only.

We shall consider a continuous De Morgan triplet (T, S, N) consisting of a continuous t-norm T, continuous t-conorm S and a strong fuzzy negation N such that $T(x, y) = N(S(N(x), N(y)))$. The main problem lies in the fact that the completeness condition (ps4) can be written in many forms, e.g.:

$$co(P \cup P^t) = I \cup J, P = co(P^t \cup I \cup J), P \cup I = co(P^t \cup J).$$

Note that it was proved in [22, 50] that reasonable constructions of fuzzy preference structure (FPS) should use a nilpotent t-norm only. Since any nilpotent t-norm (t-conorm) is isomorphic to the Łukasiewicz t-norm (t-conorm), it is enough to restrict our attention to De Morgan triplet $(T_L, S_L, 1 - x)$. Then we can define (FPS) as the triplet of binary fuzzy relations (P, I, J) on the set of alternatives $\boldsymbol{A}$ satisfying:

- I is reflexive and symmetric.
- $\forall (a, b) \in \boldsymbol{A^2}, \ \ P(a, b) + P^t(a, b) + I(a, b) + J(a, b) = 1.$

It has been mentioned, that it is possible to construct preference structure from a large preference relation R in the classical case, however, in fuzzy case this is not possible. This fact was proved by Alsina in [1] and later by Fodor and Roubens in [22]:

Proposition 2.54 (Fodor and Roubens [22], Proposition 3.1) *There is no continuous de Morgan triplet (T, S, N) such that $R = P \cup_S I$ holds with $P(a, b) = T(R(a, b), N(R(b, a)))$ and $I(a, b) = T(R(a, b), R(b, a))$.*

Because of this negative result, Fodor and Roubens (among others) proposed axiomatic construction. Assume that we deal with the Łukasiewicz triplet $(T_L, S_L, 1 - x)$.

(R1) Independence of Irrelevant Alternatives:
For any two alternatives a, b the values of $P(a, b), I(a, b), J(a, b)$ depend only on the values $R(a, b), R(b, a)$. I.e., there exist functions $p, i, j : [0, 1]^2 \to [0, 1]$ such that, for any $a, b \in \mathbf{A}$,

$$P(a, b) = p(R(a, b), R(b, a)),$$
$$I(a, b) = i(R(a, b), R(b, a)),$$
$$J(a, b) = j(R(a, b), R(b, a)).$$

(R2) Positive Association Principle:
Functions $p(x, 1-y),\ i(x, y),\ j(1-x, 1-y)$ are increasing in x and y.

(R3) Symmetry:
$i(x, y)$ and $j(x, y)$ are symmetric functions.

(R4) $(P,\ I,\ J)$ is (FPS) for any reflexive relation R on a set $\boldsymbol{A}$ such that

$$S_L(P, I) = R, \quad S_L(P, J) = 1 - R^t.$$

It was proved ([22], Theorem 3.1) that for all $x, y \in [0, 1]$ it holds:

$$T_L(x, y) \ \leq\ p(x, 1-y), i(x, y), j(1-x, 1-y) \ \leq\ T_M(x, y).$$

The mentioned triplet (p, i, j) is called *the monotone generator triplet*. Summarizing, the monotone generator triplet is a triplet (p, i, j) of mappings $[0, 1]^2 \to [0, 1]$ such that

(gt1) $p(x, 1-y),\ i(x, y),\ j(1-x, 1-y)$ are increasing in both coordinates,

(gt2) $T_L(x, y) \ \leq\ p(x, 1-y), i(x, y), j(1-x, 1-y) \ \leq\ T_M(x, y)$,

(gt3) $i(x, y) = i(y, x)$,

(gt4) $p(x, y) + p(y, x) + i(x, y) + j(x, y) = 1$,

(gt5) $p(x, y) + i(x, y) = x$.

Using these properties, one may show that also $j(x, y) = j(y, x)$ and $p(x, y) + j(x, y) = 1 - y$. Therefore the axiom (R4) can be expressed as a system of functional equations:

(R4')

$$p(x, y) + i(x, y) = x,$$

$$p(x, y) + j(x, y) = 1 - y.$$

Definition 2.55 *Let* $\varphi : [0, 1] \to [0, 1]$ *be an order-automorphism. Then*

$$T_\varphi(x, y) = \varphi^{-1}(T(\varphi(x), \varphi(y))),$$

$$S_\varphi(x, y) = \varphi^{-1}(S(\varphi(x), \varphi(y))),$$

$$(N_s)_\varphi(x) = \varphi^{-1}(1 - \varphi(x)),$$

are called φ-transformations *of* T, S, *and* N_s, *respectively.*

Remark 2.56 *It is possible to formulate similar axioms in the framework of more general De-Morgan triplet* $(T_L)_\varphi, (S_L)_\varphi, (N_s)_\varphi)$, *which is a* $\varphi-$*transformation of* $(T_L, S_L, 1-x)$. *The solution is then expressed as* $(p, i, j)_\varphi$.

2.3 Aggregation deficit and its properties

In [45] there was introduced a new operator, called an *aggregation deficit* R_D*, which is based on a disjunction* D. We recall its definition and important properties; their proofs can be found in [45]. The motivation is following. Assume the truth value $TV(\boldsymbol{A}) = a$. We would like to know conditions on truth values $TV(\boldsymbol{B}) = b$ and $TV(\boldsymbol{C}) = c$ such that they aggregate together with a or $1-a$ to have $D(c,a) \geq x$ and $D(b, 1-a) \geq y$. In order to obtain this aggregation deficit, R_D is defined by the next inequalities:

$$x \leq D(c,a) \quad \text{and} \quad y \leq D(b, 1-a).$$

$$c \geq R_D(a,x) \quad \text{and} \quad b \geq R_D(1-a, y).$$

This leads naturally to the following definition.

Definition 2.57 (Smutná-Hliněná and Vojtáš [45]) *Let D be a fuzzy disjunction. The aggregation deficit is defined by*

$$R_D(x,y) = \inf\{z \in [0,1]; D(z,x) \geq y\}.$$

Example 2.58 (Smutná-Hliněná and Vojtáš [45]) *For the basic t-conorms S_M, S_P and S_L we obtain the following aggregation deficits:*

$$R_{S_M}(x,y) = \begin{cases} 0 & \text{if } x \geq y, \\ y & \text{otherwise,} \end{cases} \qquad R_{S_P}(x,y) = \begin{cases} 0 & \text{if } x \geq y, \\ \frac{y-x}{1-x} & \text{otherwise,} \end{cases}$$

$$R_{S_L}(x,y) = \begin{cases} 0 & \text{if } x \geq y, \\ y - x & \text{otherwise.} \end{cases}$$

Remark 2.59 *Note that one easily verifies the hybrid monotonicity of the aggregation deficit R_D. Let D_1 and D_2 be the disjunctions such that $\forall x, y \in [0,1]; D_1(x,y) \leq D_2(x,y)$. Then $R_{D_1}(x,y) \geq R_{D_2}(x,y)$ for every x, y. This follows from the fact that the aggregation deficit R_D is decreasing in its first argument.*

Let $D : [0,1]^2 \to [0,1]$ be a t-semiconorm. Then $R_D(x,y) \leq y$ for $(x,y) \in [0,1]^2$. If $x \geq y$, then $R_D(x,y) = 0$. It means, that for any aggregation deficit R_D it holds that $R_D \leq R_{S_M}$. More, if the partial mappings of disjunction D are infimum-morphism ($\inf_{a \in M} D(x,a) = D(x, \inf_{a \in M} a)$, where M is subset of interval $[0,1]$) then $x \geq y$ if and only if $R_D(x,y) = 0$. It follows from boundary condition and monotonicity of t-semiconorm D. Consider an aggregation deficit R_D, then the partial mapping $R_D(.,1)$ is negation on $[0,1]$. The aggregation deficit R_S of t-conorm S coincides with residual coimplication J_S, which was introduced by Bernard De Baets in [6] for different purpose.

3 Actual progress

We turn our attention to the investigation of properties under which the fuzzy implications are $(S, N)-$implications or $R-$implications. The following characterization of $(S, N)-$implications is from [2].

Theorem 3.1 (Baczyński and Jayaram [2], Theorem 5.1) *For a function* $I : [0, 1]^2 \to [0, 1]$, *the following statements are equivalent:*

- *I is an* (S, N)*-implication generated from some t-conorm and some continuous (strict, strong) fuzzy negation* N.
- *I satisfies (I2), (EP) and* N_I *is a continuous (strict, strong) fuzzy negation.*

For $R-$implications we have the following characterization, which is from [22].

Theorem 3.2 (Fodor and Roubens [22], Theorem 1.14) *For a function* $I : [0, 1]^2 \to [0, 1]$, *the following statements are equivalent:*

- *I is an R-implication based on some left-continuous t-norm* T.
- *I satisfies (I2), (OP), (EP), and* $I(x, .)$ *is a right-continuous for any* $x \in [0, 1]$.

At the moment we know a lot of families of generated fuzzy implications. We recall some classes of generated fuzzy implications which were proposed in various papers. Recently, several possibilities have occurred how to generate implications using appropriate one-variable functions.

We list the well-known of them and their properties and examples. Yager [51] introduced two new families of fuzzy implications, called f-generated and g-generated fuzzy implications, respectively, and discussed their properties as listed in [22] or [21]. Also Jayaram in [33] discussed f-generated fuzzy implications with respect to three classical logic tautologies, such as distributivity, the law of importation and the contrapositive symmetry.

Proposition 3.3 (Yager [51]) *If* $f : [0, 1] \to [0, \infty]$ *is a strictly decreasing and continuous function with* $f(1) = 0$, *then the function* $I : [0, 1]^2 \to [0, 1]$ *defined by*

$$I(x, y) = f^{-1}(x \cdot f(y)), \quad x, y \in [0, 1], \tag{1}$$

with the understanding $0 \cdot \infty = 0$, *is a fuzzy implication.*

The function f is called an $f-$generator and the fuzzy implication represented by (1) is called an $f-$ fuzzy implication. For illustration we present some examples of $f-$fuzzy implications.

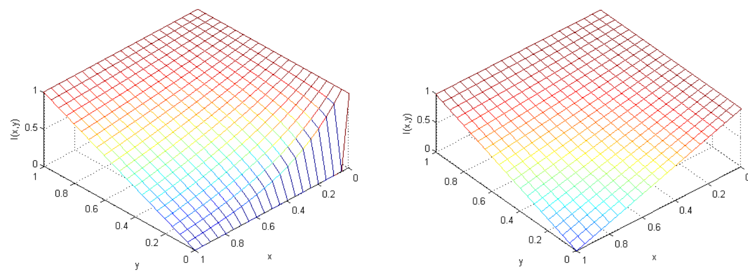
Example 3.4 (Baczyński and Jayaram [4])

- *If we take* $f(x) = -\log x$ *as the* f*-generator which is an additive generator of the product t-norm* T_P, *then we obtain the Yager implication (see Fig. 7):*

$$I_{YG}(x, y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0, \\ y^x, & \text{otherwise,} \end{cases}$$

which is neither an (S,N)-implication nor an R-implication.

- *If we take $f(x) = 1 - x$ as the f-generator which is an additive generator of the Łukasiewicz t-norm $T_L(x, y) = \max(x + y - 1, 0)$, then we obtain the Reichenbach implication I_{S_P}, which is an (S, N)-implication.*



(a) Yager implication (b) Reichenbach implication

Figure 7: f-implications

Baczyński and Jayaram in [4] have shown that the generator from which f-generated fuzzy implication is obtained, is only unique up to a positive multiplicative constant. They also have investigated the natural negations of the mentioned fuzzy implications and their relations with (S, N)- and $R-$implications .

Theorem 3.5 (Baczyński and Jayaram [4]) *The f-generator of an f-generated fuzzy implication is uniquely determined up to a positive multiplicative constant, i. e., if f_1 is an f-generator, then f_2 is an f-generator such that $I_{f_1} = I_{f_2}$ if and only if there exists a constant $c \in (0, \infty)$ such that $f_2(x) = c.f_1(x)$ for all $x \in [0, 1]$.*

Theorem 3.6 (Baczyński and Jayaram [4]) *Let f be an f-generator of an $f-$generated fuzzy implication I_f.*

- *If $f(0) = \infty$, then the natural negation N_{I_f} is the Gdel negation N_{G_1}, which is non-continuous.*

- *The natural negation N_{I_f} is a strict fuzzy negation if and only if $f(0) < \infty$.*

- *I_f is continuous if and only if $f(0) < \infty$.*

Theorem 3.7 (Yager [51], p. 197) *If f is an f-generator of an f-generated implication I_f , then*

- *I_f satisfies (NP) and (EP),*

- *$I_f(x, x) = 1$ if and only if $x = 0$ or $x = 1$, i. e., I_f does not satisfy (IP),*

- *$I_f(x, y) = 1$ if and only if $x = 0$ or $y = 1$, i. e., I_f does not satisfy (OP),*

- *I_f satisfies (CP) with some fuzzy negation N if and only if $f(0) < 1$, f_1 defined by $f_1(x) = \frac{f(x)}{f(0)}$, $x \in [0, 1]$ is a strong negation and $N = N_{I_f}$.*

Theorem 3.8 (Baczyński and Jayaram [4]) *If f is an f-generator, then the following statements are equivalent:*

- *I_f is an (S,N)-implication.*
- *$f(0) < \infty$.*

Theorem 3.9 (Baczyński and Jayaram [4]) *If f is an f-generator, then I_f is not an R-implication.*

Yager [51] has also proposed another class of implications called the g-generated implications. In a similar way as in the part about $f-$fuzzy implications we present their properties.

Proposition 3.10 (Yager [51], p. 197) *If $g : [0,1] \to [0,\infty]$ is a strictly increasing and continuous function with $g(0) = 0$, then the function $I : [0,1]^2 \to [0,1]$ defined by*

$$I(x,y) = g^{(-1)}\left(\frac{1}{x} \cdot g(y)\right), \quad x, y \in [0,1], \tag{2}$$

with the understanding $\frac{1}{0} = \infty$ and $0 \cdot \infty = \infty$, is a fuzzy implication.

The function g is called a g–generator and the fuzzy implication represented by (2) is called a g–implication.

Example 3.11 (Baczyński and Jayaram [4])

- *If we take the g-generator $g(x) = -\log(1-x)$, which is an additive generator of the probabilistic sum t-conorm S_P, then we obtain the following fuzzy implication (see Fig. 8):*

$$I_{YG}(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0, \\ 1-(1-y)^x, & \text{otherwise,} \end{cases}$$

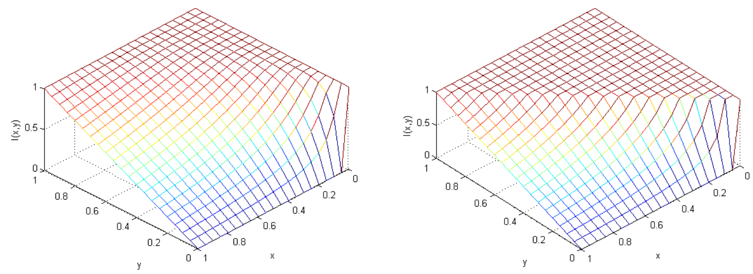
  which is neither an (S,N)-implication nor an R-implication.

- *If we take the g-generator $g(x) = x$, which is a continuous additive generator of the Łukasiewicz t-conorm $S_L(x,y) = \min(x+y,1)$, then we obtain the Goguen implication R_{T_P}, which is an R-implication.*

Now we present results concerning properties of $g-$generators, the natural negations of the mentioned fuzzy implications and their relations with (S,N)- and $R-$implications. More details can be found in [4].

Theorem 3.12 (Baczyński and Jayaram [4]) *The g-generator of a g-generated fuzzy implication is uniquely determined up to a positive multiplicative constant, i. e., if g_1 is a g-generator, then g_2 is a g-generator such that $I_{g_1} = I_{g_2}$ if and only if there exists a constant $c \in (0,\infty)$ such that $g_2(x) = c.g_1(x)$ for all $x \in [0,1]$.*

Theorem 3.13 (Yager [51], p. 201) *If g is a g-generator of a g-generated implication I_g, then*



(a) Yager implication (b) Goguen implication

Figure 8: g-implications

- *I_g satisfies (NP) and (EP),*
- *I_g satisfies (IP) if and only if $g(1) < 1$ and $x \leq g_1(x)$ for every $x \in [0,1]$, where g_1 is defined by $g_1(x) = \frac{g(x)}{g(1)}$, $x \in [0,1]$,*
- *if $g(1) = 1$, then $I_g(x,y) = 1$ if and only if $x = 0$ or $y = 1$, i. e., I_g does not satisfy (OP) when $g(1) = 1$,*
- *I_g does not satisfy the contrapositive symmetry (CP) with any fuzzy negation.*

Theorem 3.14 (Baczyński and Jayaram [4]) *Let g be a g-generator.*

- *The natural negation of I_g is the Gödel negation N_{G_1}, which is not continuous.*
- *I_g is continuous except at the point $(0,0)$.*

Theorem 3.15 (Baczyński and Jayaram [4]) *If g is a g-generator, then I_g is not an (S,N)-implication.*

Theorem 3.16 (Baczyński and Jayaram [4]) *If g is a g-generator of I_g, then the following statements are equivalent:*

- *I_g is an R-implication.*
- *There exists a constant $c \in]0,\infty[$ such that $g(x) = c \cdot x$ for all $x \in [0,1]$.*
- *I_g is the Goguen implication R_{T_P}.*

The f- and g-generators can be seen as the continuous additive generators of t-norms and t-conorms, respectively. A new family of fuzzy implications called the h-generated implications has been proposed by Jayaram in [32], where h can be seen as a multiplicative generator of a continuous Archimedean t-conorm. We present its definitions, examples and a few of its properties. More details can be found in [4].

Proposition 3.17 (Jayaram [32]) *If $h : [0,1] \to [0,1]$ is a strictly decreasing and continuous function with $h(0) = 1$, then the function $I : [0,1]^2 \to [0,1]$ defined by*

$$I(x,y) = h^{(-1)}(x \cdot h(y)), \quad x,y \in [0,1], \tag{3}$$

is a fuzzy implication.

The function h is called an h–generator and the fuzzy implication represented by (3) is called an h–generated implication.

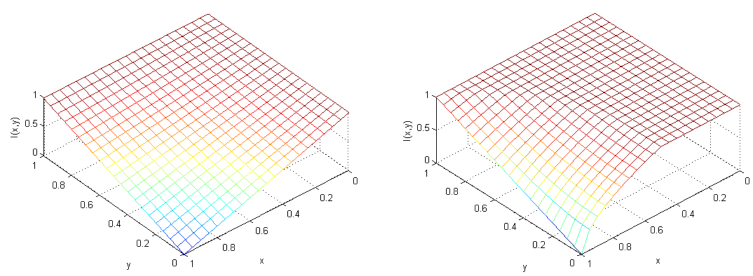
Example 3.18 (Baczyński and Jayaram [4])

- *If we take $h(x) = 1-x$, which is a continuous multiplicative generator of the algebraic sum t-conorm S_P, then we obtain the Reichenbach implication $I_{T_P},$, which is an S-implication.*

- *If we consider the family of h-generators $h_n(x) = 1 - \frac{x^n}{n}, n \in \mathbb{N}$, then we obtain the following fuzzy implications (see Fig. 9):*

$$I_n(x, y) = \min\left((n - n \cdot x + x \cdot y^n)^{\frac{1}{n}}, 1\right),$$

  which are (S, N)-implications.



(a) Reichenbach implication (b) I_2 implication

Figure 9: h-implications

Theorem 3.19 (Baczyński and Jayaram [4]) *The h-generator of an h-generated implication is uniquely determined, i. e., h_1, h_2 are h-generators such that $I_{h_1} = I_{h_2}$ if and only if $h_1 = h_2$.*

Theorem 3.20 (Baczyński and Jayaram [4]) *Let h be an h-generator of I_h.*

- *The natural negation N_{I_h} is a continuous fuzzy negation.*

- *I_h is continuous.*

Theorem 3.21 (Baczyński and Jayaram [4]) *Let h is an h-generator of an h-generated implication I_h, then*

- *I_h satisfies (NP) and (EP),*

- *I_h satisfies (IP) if and only if $h(1) > 0$ and $x.h(x) \leq h(1)$ for every $x \in [0, 1]$,*

- *I_h does not satisfy (OP),*

- *I_h satisfies (CP) with some fuzzy negation N if and only if $h = h^{-1}$ and $N = N_{I_h}$.*

Theorem 3.22 (Baczyński and Jayaram [4]) *If h is an h-generator, then I_h is an (S,N)-implication generated from some t-conorm S and continuous fuzzy negation N.*

Theorem 3.23 (Baczyński and Jayaram [4]) *If h is an h-generator, then I_h is not an R-implication.*

Smutná in [44] introduced generated fuzzy implications I_f, I^g and I^g_N. The implications I_f are generated with using strictly decreasing functions, the implications I^g are generated with using strictly increasing functions.

Proposition 3.24 (Smutná [44]) *Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing function such that $f(1) = 0$. Then the function $I_f : [0,1]^2 \to [0,1]$ which is given by*

$$I_f(x,y) = f^{(-1)}(f(y^+) - f(x)),$$

where $f(y^+) = \lim_{t\to y^+} f(t)$ and $f(1^+) = f(1)$, is a fuzzy implication.

Construction of the fuzzy implications I^g is described in the following proposition:

Proposition 3.25 (Smutná [44]) *Let $g : [0,1] \to [0,\infty]$ be a strictly increasing function such that $g(0) = 0$. Then the function $I^g(x,y) : [0,1]^2 \to [0,1]$ which is given by*

$$I^g(x,y) = g^{(-1)}(g(1-x) + g(y)), \tag{4}$$

is a fuzzy implication.

Implications I^g may be further generalized. This generalization is based on the replacement of the standard negation by an arbitrary fuzzy negation.

Proposition 3.26 (Smutná [44]) *Let $g : [0,1] \to [0,\infty]$ be a strictly increasing function such that $g(0) = 0$ and N be a fuzzy negation. Then I^g_N, defined by*

$$I^g_N(x,y) = g^{(-1)}(g(N(x)) + g(y)),$$

is a fuzzy implication.

Fuzzy implications are closely related to the generators of a strict preference. The following proposition can be found in [22]. Fodor and Roubens supposed general triplet $(T_\varphi, S_\varphi, N_\varphi)$:

Proposition 3.27 (Fodor and Roubens [22], Proposition 3.5) *Let $S : [0,1]^2 \to [0,1]$ be any continuous t-conorm and $N : [0,1] \to [0,1]$ be a strong fuzzy negation. If $(p,i,j)_\varphi$ is a solution of the system*

$$S(p(x,y), i(x,y)) = x,$$

$$S(p(x,y), j(x,y)) = N(y),$$

then $I^\to(x,y) = N_\varphi(p(x,y))$ is a fuzzy implication such that

$$I^\to(1,x) = x \qquad \forall x \in [0,1],$$

$$I^\to(x,0) = N_\varphi(x) \quad \forall x \in [0,1].$$

Since we are dealing with Łukasiewicz triplet $(T_L, S_L, 1-x)$, this proposition can be simplified:

Proposition 3.28 *Let (p, i, j) be a solution of the system in (R4'), then $I^{\rightarrow}(x, y) = 1 - p(x, y)$ is a fuzzy implication and*

$$I^{\rightarrow}(1, x) = x \qquad \forall x \in [0, 1],$$

$$I^{\rightarrow}(x, 0) = 1 - x \quad \forall x \in [0, 1].$$

The generator of indifference i and t-norms has common properties (both are symmetric and increasing mappings from $[0, 1]^2$ to $[0, 1]$). The following theorem shows that we can use some continuous t-norms in defining mapping i:

Theorem 3.29 (Fodor and Roubens [22]) *Assume that $p(x, y) = T_1(x, N_\varphi(y))$ and $i(x, y) = T_2(x, y)$, where T_1 and T_2 are continuous t-norms. Then $(p, i, j)_\varphi$ satisfies (R4) if and only if there exists a number $s \in [0, \infty]$ such that*

$$T_1(x, y) = \varphi^{-1}\left(T^s(\varphi(x), \varphi(y))\right),$$

$$T_2(x, y) = \varphi^{-1}\left(T^{1/s}(\varphi(x), \varphi(y))\right),$$

where T^s and $T^{1/s}$ belong to the Frank family.

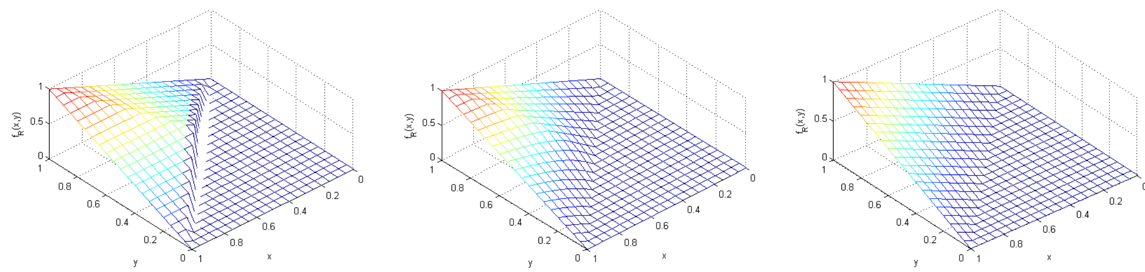
In Pavelka's language of evaluated expressions, we would like to achieve the following: from $(\boldsymbol{C} \vee_D \boldsymbol{A}, x)$ and $(\boldsymbol{B} \vee_D \neg \boldsymbol{A}, y)$ to infer $(\boldsymbol{C} \vee_D \boldsymbol{B}, f_{\vee_D}(x, y))$ where $f_{\vee_D}(x, y)$ should be the best promise, we can give the truth function of disjunction $\vee_D$ and x and y. In the previous section we have mentioned the construction of new fuzzy operator, which is called the aggregation deficit. The formulation of a result on sound and complete full resolution is based on the aggregation operators. Smutná - Hliněná and Vojtáš in [45] investigated the *resolution truth function* $f_{R_D} : [0, 1]^2 \rightarrow [0, 1]$, which is defined by

$$f_{R_D}(x, y) = \inf_{a \in [0,1]} \{D(R_D(a, x), R_D(1 - a, y))\}.$$

Example 3.30 (Smutná-Hliněná and Vojtáš [45]) *For the aggregation deficits R_{S_M}, R_{S_P} and R_{S_L}, which are corresponded with the basic t-conorms, we obtain the following functions (see Fig. 10):*

$$f_{R_{S_M}}(x, y) = \begin{cases} 0 & \text{if } x + y \leq 1, \\ \min(x, y) & \text{otherwise,} \end{cases} \qquad f_{R_{S_P}}(x, y) = \begin{cases} 0 & \text{if } x + y \leq 1, \\ \frac{x+y-1}{\max(x,y)} & \text{otherwise,} \end{cases}$$

$$f_{R_{S_L}}(x, y) = \begin{cases} 0 & \text{if } x + y \leq 1, \\ x + y - 1 & \text{otherwise.} \end{cases}$$



(a) $f_{R_{S_M}}$ (b) $f_{R_{S_P}}$ (c) $f_{R_{S_L}}$

Figure 10: resolution truth functions

Theorem 3.31 (Smutná-Hlinĕná and Vojtáš [45]) *Assume the truth evaluation of proposition variables is a model of* $(\boldsymbol{C} \vee_D \boldsymbol{A}, x)$ *and* $(\boldsymbol{B} \vee_D \neg \boldsymbol{A}, y)$*. Then*

$$TV(\boldsymbol{C} \vee_D \boldsymbol{B}) \geq f_{R_D}(x, y).$$

4 The doctoral thesis objectives

The topic of the thesis is the study of generated fuzzy implications and their applications. There are two well-known families of implications - (S, N)–implications and R–implications. The main part of my research is studying connections between several classes generated fuzzy implications and families of (S, N)–implications and R–implications. Another hot topic of my research is the investigation of generators of fuzzy preference structures. This leads to search for the special conditions of the mentioned generated fuzzy implications. And the last direction of my thesis is devoted to fuzzy resolution, particularly to modelling of fuzzy modus ponens.

At this point my research can continue in several directions. Unless unforeseen circumstances occur, it is quite probable that the thesis will explore some of the following research directions:

- Investigation of some classes of generated implications
- Generated fuzzy implications as the generators of fuzzy preference structures
- Generated fuzzy implications in fuzzy resolution

5 Generated fuzzy implications

5.1 Generated implications I_f^* and their properties

In [44] Smutná introduced generated implications I_f (for original description see the Theorem 3.24). This class of implications were later studied by Hliněná and Biba in [28].

In the article [9] the original description of I_f was slightly modified (and notation was changed to I_f^*). However, for continuous functions f both the definitions are equivalent. This later article presents new results concerning I_f^* implications as well as stronger versions of some previous results.

In this section we will focus on I_f^* implications, their properties and intersections with classes of $(S, N)-$ and $R-$ implications. Implications I_f^* are described in the following way:

Proposition 5.1 (Biba, Hliněná, Kalina, and Král' [9]) *Let $f : [0, 1] \to [0, \infty]$ be a strictly decreasing function such that $f(1) = 0$. Then the function $I_f^*(x, y) : [0, 1]^2 \to [0, 1]$ which is given by*

$$I_f^*(x, y) = f^{(-1)} \left(\max\{0, f(y) - f(x)\}\right) \tag{5}$$

is a fuzzy implication.

Proof. We proceed by the points of the Definition 2.41.

(I1)
- Let $x_1, x_2, y \in [0, 1]$ and $x_1 \le x_2$ and $x_1 \ge y$. Function f is decreasing and therefore $f(x_1) \ge f(x_2)$ and $f(y)-f(x_1) \le f(y)-f(x_2)$. Pseudoinverse $f^{(-1)}$ of function f is decreasing, too, and $f^{(-1)} (f(y) - f(x_1)) \ge f^{(-1)} (f(y) - f(x_2))$. Therefore $I_f^*(x_1, y) \ge I_f^*(x_2, y)$ and it means that the function I_f is decreasing in its first variable.
- If $x_1 \le y \le x_2$, then $I_f^*(x_1, y) = f^{(-1)}(0) = 1$ and $I_f^*(x_2, y) \le 1$.
- If $x_1 \le x_2 \le y$, then $I_f^*(x_1, y) = I_f^*(x_2, y) = 1$.

(I2)
- Let $x, y_1, y_2 \in [0, 1]$ and $y_1 \le y_2$ and $x \ge y_2$. Function f is decreasing and therefore $f(y_1) \ge f(y_2)$ and $f(y_1) - f(x) \ge f(y_2) - f(x)$. Pseudoinverse $f^{(-1)}$ of function f is decreasing too and $f^{(-1)} (f(y_1) - f(x)) \le f^{(-1)} (f(y_2) - f(x))$. Therefore $I_f^*(x, y_1) \le I_f^*(x, y_2)$ and this means that the function I_f is increasing in its second variable.
- If $y_1 \le x \le y_2$, then $I_f^*(x, y_2) = f^{(-1)}(0) = 1$ and $I_f^*(x, y_1) \le 1$.
- If $x \le y_1 \le y_2$, then $I_f^*(x, y_1) = I_f^*(x, y_2) = 1$.

(I3) From the formula for function I_f^* we get $I_f^*(0, 0) = I_f^*(1, 1) = 1$ and for $I_f^*(1, 0)$ we have

$$I_f^*(1, 0) = f^{(-1)} (f(0) - f(1)) = f^{(-1)}(f(0)) = \sup\{z \in [0, 1]; f(z) > f(0)\} = 0.$$

The I_f^* implications are illustrated by the following examples.

Example 5.2 (Hliněná and Biba [28, 11]) Let $f_1, f_2, f_3 : [0, 1] \to [0, \infty]$ be strictly decreasing functions defined as follows:

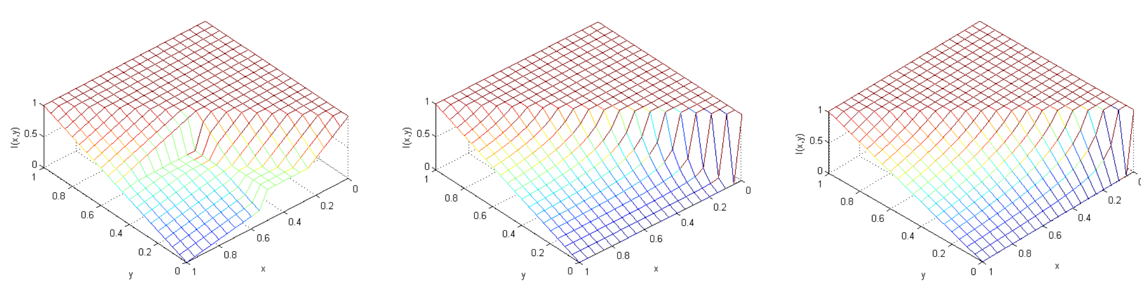
- $f_1(x) = \begin{cases} 1 - x & \text{if } x \leq 0.5, \\ 0.5 - 0.5x & \text{otherwise,} \end{cases}$
- $f_2(x) = \frac{1}{x} - 1,$
- $f_3(x) = -\ln(x).$

Then for $f_1^{(-1)}, f_2^{(-1)}, f_3^{(-1)}$, we get

- $f_1^{(-1)}(x) = \begin{cases} 1 - 2x & \text{if } x \leq 0.25, \\ 0.5 & \text{if } 0.25 < x \leq 0.5, \\ 1 - x & \text{otherwise,} \end{cases}$
- $f_2^{(-1)}(x) = \min\left\{\frac{1}{1+x}, 1\right\},$
- $f_3^{(-1)}(x) = \min\{e^{-x}, 1\},$

and the generated implications are (Fig. 11):

- $I^*_{f_1}(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ 1 - 2x + 2y & \text{if } x \leq 0.5, y < 0.5, x - y \leq 0.25, x > y, \\ 0.5 & \text{if } x \leq 0.5, y < 0.5, x - y > 0.25, \\ 0.5 & \text{if } x > 0.5, y < 0.5, x \leq 2y, \\ 0.5 + y - 0.5x & \text{if } x > 0.5, y < 0.5, x > 2y, \\ 1 - x + y & \text{if } x > 0.5, y \geq 0.5, \end{cases}$
- $I^*_{f_2}(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ \frac{1}{\frac{1}{y} - \frac{1}{x} + 1} & \text{otherwise,} \end{cases}$
- $I^*_{f_3}(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ \frac{y}{x} & \text{otherwise.} \end{cases}$



(a) Implication $I^*_{f_1}$ (b) Implication $I^*_{f_2}$ (c) Implication $I^*_{f_3}$

Figure 11: Fuzzy implications I^*_f

In our investigation we will use the following technical result of pseudo-inverse functions.

Proposition 5.3 (Hliněná and Biba [29]) *Let c be a positive real number. Then the pseudo-inverse of a positive multiple of any monotone function $f : [0,1] \to [0,\infty]$ satisfies*

$$(c \cdot f)^{(-1)}(x) = f^{(-1)}\left(\frac{x}{c}\right).$$

Proof. Let f be a decreasing function. Then

$$f^{(-1)}(x) = \sup\{z \in [0,1]; f(z) > x\}$$

and

$$(c \cdot f)^{(-1)}(x) = \sup\{z \in [0,1]; c \cdot f(z) > x\} = \sup\left\{z \in [0,1]; f(z) > \frac{x}{c}\right\} = f^{(-1)}\left(\frac{x}{c}\right).$$

The proof for the case of increasing function is analogous.

It is well-known that generators of continuous Archimedean t-norms are unique up to a positive multiplicative constant, and this is also valid for the f generators of I_f^* implications.

Proposition 5.4 (Biba, Hliněná, Kalina, and Kráľ [9], Hliněná and Biba [28]) *Let c be a positive constant and $f : [0,1] \to [0,\infty]$ be a strictly decreasing function. Then the implications I_f^* and $I_{c \cdot f}^*$ which are based on functions f and $c \cdot f$, respectively, are identical.*

Proof.

- Let $x, y \in [0,1], x \leq y$ and c be a positive real number. From Proposition 5.1 we get $I_{c \cdot f}^*(x,y) = I_f^*(x,y) = 1$.
- Let $x, y \in [0,1], x > y$ and c be a positive real number. Then from Proposition 5.1 and Lemma 5.3 we get

$$\begin{aligned} I_{c \cdot f}^*(x,y) &= (c \cdot f)^{(-1)}((c \cdot f)(y) - (c \cdot f)(x)) \\ &= f^{(-1)}\left(\frac{(c \cdot f)(y) - (c \cdot f)(x)}{c}\right) = f^{(-1)}(f(y) - f(x)) = I_f^*(x,y). \end{aligned}$$

Corollary 5.5 (Biba, Hliněná, Kalina, and Kráľ [9]) *Let $f : [0,1] \to [0,\infty]$ be bounded and strictly decreasing function such that $f(1) = 0$. Let $f^*(x) = \frac{f(x)}{f(0)}$. Then $I_f^* = I_{f^*}^*$ and also $f^*(0) = 1$. Hence, if f is a bounded function we can always assume that $f(0) = 1$.*

By Definition 2.47 and the following equivalence for a strictly decreasing function f

$$f^{(-1)}(x_0) = 1 \iff x_0 \leq \lim_{x \to 1^-} f(x) = f(1^-). \tag{6}$$

we get directly a condition under which I_f^* satisfies (NP).

Proposition 5.6 (Biba, Hliněná, Kalina, and Král' [9]) *Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing function such that $f(1) = 0$. Then I_f^* satisfies (IP) and (NP). Moreover, f is continuous at $x = 1$ if and only if I_f^* satisfies (OP).*

The meaning of continuity of the function f in $x = 1$ is introduced in the next example.

Example 5.7 *Let function $f : [0,1] \to [0,1]$ be given by*

$$f(x) = \begin{cases} 1 - \frac{x}{2} & x \in [0,1[, \\ 0 & x = 1. \end{cases}$$

Pseudoinverse $f^{(-1)} : [0,1] \to [0,1]$ will be given by

$$f^{(-1)}(x) = \begin{cases} 1 & x \leq 0.5, \\ 2 - 2x & x \in]0.5, 1]. \end{cases}$$

implication $I_f* : [0,1]^2 \to [0,1]$ will be given by

$$I_f^*(x,y) = \begin{cases} y & x = 1, \\ 1 & \text{otherwise.} \end{cases}$$

For this implication it holds $I_f^*(0.5, 0.4) = 1$. Therefore I_f^* doesn't have (OP). It is due to the fact that $f^{(-1)}(x) = 1$ for some $x > 0$, which is a consequence of violation of continuity of f at $x = 1$. From continuity in $x = 1$ we have $f^{(-1)}(x) = 1$ only for $x = 0$ and from strictly decreasing function f we have $f(y) - f(x) = 0$ only for $x = y$, where $x, y \in [0,1]$. It means that continuity in $x = 1$ is equivalent with (OP) for implication I_f^*.

The continuity of a strictly decreasing function f implies that $f\left(f^{(-1)}(x)\right) = x$. Therefore we can formulate the following proposition.

Proposition 5.8 (Biba, Hliněná, Kalina, and Král' [9]) *Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing function continuous at 1 and 0, such that $f(1) = 0$. Then the implication I_f^* satisfies (EP) if and only if f is continuous.*

Proof. The fact that continuity of f implies (EP) of the corresponding I_f^* implication, was proved in [28].
In order to prove the converse statement we show that if f is continuous at 0 and 1 and there exists at least one discontinuity point in $]0,1[$ then I_f^* does not satisfy (EP). We need to show that there exists a triple $(x_0, y_0, z_0) \in (0,1)^3$ such that

$$I_f^*(x_0, I_f^*(y_0, z_0)) \neq I_f^*(y_0, I_f^*(x_0, z_0)), \tag{7}$$

i.e.

$$f^{(-1)}(\max\{0, f(I_f^*(y_0, z_0)) - f(x_0)\}) \neq f^{(-1)}(\max\{0, f(I_f^*(x_0, z_0)) - f(y_0)\}). \tag{8}$$

Let us assume $x_0 < y_0$. Straightforwardly from (5) in Proposition 5.1 we have that if $z_0 \geq x_0$ and $z_0 \geq y_0$ then $I_f^*(x_0, z_0) = I_f^*(y_0, z_0) = 1$ and (EP) is not violated. So, we will assume $z_0 < x_0 < y_0$.

Let c be a discontinuity point. Because f is strictly decreasing, inequality (8) holds, e.g., if

$$I_f^*(y_0, z_0) = I_f^*(x_0, z_0) \tag{9}$$

and

$$f(I^*(x_0, z_0)) - f(y_0), \quad f(I^*(y_0, z_0)) - f(x_0) \tag{10}$$

do not belong to the same interval of constantness of $f^{(-1)}$.

Roughly speaking, in order to satisfy condition (9) we look for a triple (x_0, y_0, z_0) such that $f(x_0)$ and $f(y_0)$ are 'small enough' to ensure that values of $f(z_0) - f(x_0)$ and of $f(z_0) - f(y_0)$ belong to the interval of constantness of $f^{(-1)}$ corresponding to c. In this case we get $I_f^*(x_0, z_0) = I_f^*(y_0, z_0) = c$. So, we can rewrite the differences in condition (10) as follows

$$f(c) - f(y_0), \quad f(c) - f(x_0).$$

Then, in order to satisfy condition (10), the difference between $f(x_0)$ and $f(y_0)$ must ensure that $f(c) - f(x_0)$ and $f(c) - f(y_0)$ lead to different values of $f^{(-1)}$.

The detailed description of the choice procedure for the triple (x_0, y_0, z_0) follows. Since f is continuous at 1, for arbitrarily chosen $\varepsilon > 0$ there exists $\delta > 0$ for which we can find $x_0, y_0 \in]1 - \delta, 1[$ fulfilling

$$f(y_0) < f(x_0) < \varepsilon. \tag{11}$$

We take ε mentioned above fulfilling $\varepsilon < \min\{f(c^-) - f(c^+),\ f(c^+)\}$.

By Definition 2.30 and from the fact that f is strictly decreasing it follows that $f^{(-1)}$ is a continuous function. Using continuity of $f^{(-1)}$ we get the existence of values t_1 and t_2 for which

$$0 < f(c^+) - \varepsilon < t_2 < t_1 < f(c^+)$$

and $f^{(-1)}(t_1) < f^{(-1)}(t_2)$. Fix x_0, y_0 fulfilling inequality (11) and moreover

$$0 < f(y_0) \le f(c) - t_1 < f(c) - t_2 \le f(x_0). \tag{12}$$

Further, let $z_0 \in]0, c[$ be such that $f(z_0) \in (f(c^-),\ f(c^-) + f(y_0))$. Since $f(c^-)$ is the left-hand-side limit of f at c we can find such a value z_0. Therefore, we get that

$$f(z_0) - f(y_0) \in (f(c^+),\ f(c^-)), \quad f(z_0) - f(x_0) \in (f(c^+),\ f(c^-))$$

which implies

$$\begin{aligned} I_f^*(y_0, z_0) &= f^{(-1)}\left(f(z_0) - f(y_0)\right) = c, \\ I_f^*(x_0, z_0) &= f^{(-1)}\left(f(z_0) - f(x_0)\right) = c, \end{aligned}$$

i.e., condition (9) is fulfilled.

Inequality (12) gives $f(c) - f(x_0) \le t_2$ and $f(c) - f(y_0) \ge t_1$. This implies that $f^{(-1)}\left(f(c) - f(x_0)\right) \ne f^{(-1)}\left(f(c) - f(y_0)\right)$ and, for the triplet (x_0, y_0, z_0), I_f^* violates (EP).

In Proposition 5.8 we have considered just the case when f is continuous at 0 and 1. Now, we will deal with functions f being non-continuous at 0 and/or at 1.

Proposition 5.9 (Biba, Hliněná, Kalina, and Král' [9]) *Let* $f : [0, 1] \to [0, \infty]$ *be a strictly decreasing function with* $f(1) = 0$ *discontinuous at* 1. *Then*

(a) *If* $2f(1^-) \geq f(0)$ *then* I_f^* *satisfies (EP). Moreover,*

$$I_f^*(x,y) = \begin{cases} y, & \text{if } x = 1, \\ 1, & \text{otherwise.} \end{cases} \tag{13}$$

(b) *If* $2f(1^-) \geq f(0^+)$ *and* $2f(0^+) \leq f(0)$*, then* I_f^* *satisfies (EP). Moreover,*

$$I_f^*(x,y) = \begin{cases} y, & \text{if } x = 1, \\ 0, & \text{if } x \neq 0 \text{ and } y = 0, \\ 1, & \text{otherwise.} \end{cases} \tag{14}$$

Proof.(a) First we show that formula (13) is true. If $x \neq 1$ and $y \in [0,1]$ is arbitrary then $f(y) - f(x) \leq f(1^-)$ and this implies $I_f^*(x,y) = 1$. If $x = 1$ then $I_f^*(x,y) = y$ follows directly from strict decreasingness of f. Formula (13) implies the following

$$\begin{aligned} I_f^*(x, I_f^*(y,z)) &= \begin{cases} I_f^*(y,z), & \text{if } x = 1, \\ 1, & \text{if } x \neq 1, \end{cases} \\ I_f^*(y, I_f^*(x,z)) &= \begin{cases} I_f^*(y,z), & \text{if } x = 1, \\ I_f^*(y,1), & \text{if } x \neq 1, \end{cases} \end{aligned}$$

and since $I_f^*(y,1) = 1$, we get that (EP) is satisfied for I_f^*.
(b) Also in this case we show first formula (14). The fact that $I_f^*(1,y) = y$ is due to strict decreasingness of f. Let $x \neq 0$. Then $f(0^+) \leq 2f(1^-) \leq 2f(0^+) \leq f(0)$ and this implies $f(0) - f(x) \geq f(0) - f(0^+) \geq f(0^+)$. This means

$$I_f^*(x,0) = f^{(-1)}(\max\{0, f(0) - f(x)\}) = 0.$$

If $0 < y < x < 1$ then $f(y) - f(x) \leq f(0^+) - f(1^-) \leq f(1^-)$ and hence

$$I_f^*(x,y) = f^{(-1)}(\max\{0, f(y) - f(x)\}) = 1.$$

Formula (14) implies the following

$$\begin{aligned} I_f^*(x, I_f^*(y,z)) &= \begin{cases} I_f^*(y,z), & \text{if } x = 1, \\ I_f^*(x,z), & \text{if } y = 1, \\ 0, & \text{if } z = 0 \text{ and } x, y \neq 0, \\ 1, & \text{if } z \neq 0 \text{ and } x, y \neq \{0,1\}, \end{cases} \\ I_f^*(y, I_f^*(x,z)) &= \begin{cases} I_f^*(y,z), & \text{if } x = 1, \\ I_f^*(x,z), & \text{if } y = 1, \\ 0, & \text{if } z = 0 \text{ and } x, y \neq 0, \\ 1, & \text{if } z \neq 0 \text{ and } x, y \neq \{0,1\}, \end{cases} \end{aligned}$$

and this implies that, in this case, (EP) is fulfilled for I_f^*.

Proposition 5.10 (Biba, Hliněná, Kalina, and Král' [9]) *Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing function discontinuous at 0 and continuous at 1 with $f(1) = 0$. Then I_f^* satisfies (EP) if and only if f is continuous in $]0,1]$ and fulfils the inequality $2f(0^+) \leq f(0)$. Moreover, in this case $N_{I_f^*} = N_{G_1}$.*

Proof. Let us assume that $2f(0^+) \leq f(0)$ and f is continuous in $]0,1]$. First we prove that the natural negation related to I_f^* is N_{G_1}. For all $x \in]0,1]$,

$$N_{I_f^*} = I_f^*(x,0) = f^{(-1)}(f(0) - f(x)) \leq f^{(-1)}(f(0) - f(0^+)) \leq f^{(-1)}(f(0^+)) = 0.$$

From Proposition 5.8 it follows $I_f^*(x, I_f^*(y,z)) = I_f^*(y, I_f^*(x,z))$ for $x,y,z \neq 0$. Therefore we will consider only the case when at least one out of x,y,z is equal to 0. Assume $z = 0$. Then

$$I_f^*(x, I_f^*(y,z)) = I_f^*(x,0) = 0, \quad I_f^*(y, I_f^*(x,z)) = I_f^*(y,0) = 0.$$

Assume $x = 0$. Then

$$I_f^*(x, I_f^*(y,z)) = 1, \quad I_f^*(y, I_f^*(x,z)) = I_f^*(y,1) = 1.$$

The case when $y = 0$ could be treated similarly, i.e. I_f^* satisfies (EP).

On the other hand, assume that I_{f^*} satisfies (EP) and $2f(0^+) > f(0)$. Then from Proposition 5.8 it follows that f is continuous in $]0,1]$. Because of continuity of f in $]0,1]$, there exist $x,y \in]0,1[$, $x > y$, such that $I_f^*(x,0) = 0$ and $I_f^*(y,0) = c > 0$ for a $c < 1$. From these formulas we get, using Proposition 5.6,

$$\begin{aligned} I_f^*(y, I_f^*(x,0)) &= I_f^*(y,0) = c, \\ I_f^*(x, I_f^*(y,0)) &= I_f^*(x,c) > I_f^*(1,c) = c. \end{aligned}$$

This implies that (EP) is violated, which is a contradiction.

Example 5.11 *Define $f : [0,1] \to [0,\infty[$ by the following*

$$f(x) = \begin{cases} 1 - x^2, & \text{if } x \in]0,1], \\ 1.5, & \text{if } x = 0. \end{cases}$$

Then $f^{(-1)} : [0,\infty[\to [0,1]$ is given by

$$f^{(-1)}(x) = \begin{cases} \sqrt{1-x}, & \text{if } x \in [0,1[, \\ 0, & \text{if } x \geq 1. \end{cases}$$

Let us compute $I_f^*(0.7, I_f^*(0.8,0))$ and $I_f^*(0.8, I_f^*(0.7,0))$.

$$\begin{aligned} I_f^*(0.8,0) &= f^{(-1)}(1.5 - (1 - 0.8^2)) = 0, \\ I_f^*(0.7,0) &= f^{(-1)}(1.5 - \sqrt{1 - 0.7^2}) = f^{(-1)}(0.99) = 0.1, \end{aligned}$$

and

$$\begin{aligned} I_f^*(0.7, I_f^*(0.8,0)) &= I_f^*(0.7,0) = 0.1, \\ I_f^*(0.8, I_f^*(0.7,0)) &= I_f^*(0.8,0.1) = \sqrt{0.63} \doteq 0.79, \end{aligned}$$

i.e (EP) is violated for I_f^*.

We study the properties of implications I_f^* under which they are (S,N)– or R–implications. Because there are relations between (S,N)– implications and (EP) and continuity of $N_{I_f^*}$, Propositions 5.8, 5.9 and 5.10 lead us to dealing with continuous function f. Continuity of a function f implies continuity of the corresponding natural negation based on I_f^*. Moreover, for a continuous and bounded strictly decreasing function f such that $f(1)=0$ and $f(0)=c$ the natural negation $N_{I_f^*}$ is strong.

Proposition 5.12 (Biba, Hliněná, Kalina, and Kráľ [9]) *Let $f:[0,1]\to[0,c]$ be a continuous bounded strictly decreasing function such that $f(1)=0$. Then the implication I_f^* possess (CP) only with respect to its natural negation $N_{I_f^*}(x)=f^{-1}(f(0)-f(x))$.*

Proof. Let $f:[0,1]\to[0,c]$ be a continuous bounded decreasing function, such that $f(1)=0$ and $N_{I_f^*}(x)=f^{-1}(f(0)-f(x))$. Since we deal with classical inverse function, we have

$$\forall x\in[0,1]; f(N_{I_f^*}(x))=f(0)-f(x),$$

and therefore

$$\forall x,y\in[0,1]^2; f(N_{I_f^*}(x))-f(N_{I_f^*}(y))=f(y)-f(x).$$

Using this equality, for $I_f^*(N_{I_f^*}(y),N_{I_f^*}(x))$ we get

$$I_f^*(N_{I_f^*}(y),N_{I_f^*}(x))=f^{(-1)}\left(\max(0,f(y)-f(x))\right).$$

On the other hand, $I_f^*(x,y)=f^{(-1)}\left(\max(0,f(y)-f(x))\right)$, therefore I_f possess (CP). Let I_f^* possess (CP) w.r. to $N(x)$. Then the following holds:

$$I_f^*(x,0)=f^{-1}\left(\max(0,f(0)-f(x))\right)=f^{-1}(f(0)-f(x)),$$

$$I_f^*(1,N(x))=f^{-1}\left(\max(0,(f(N(x)))-0)\right)=f^{-1}(f(N(x))).$$

Pseudo-inverses are replaced by classical inverse functions because we are dealing with continuous function f. (CP) means that $I_f^*(1,N(x))=I_f^*(x,0)$, therefore $f^{-1}(f(N(x)))=f^{-1}(f(0)-f(x))$ and $N(x)=N_{I_f^*}(x)$ for all x.

The continuity of a generator f implies that I_f^* is an R–implication ([22], Theorem 1.16).

Corollary 5.13 (Biba, Hliněná, Kalina, and Kráľ [9]) *Let $f:[0,1]\to[0,c]$ be a strictly decreasing continuous bounded function and $f(1)=0$. Then $I_f^*(x,y)=I_f^*(I_f^*(y,0),I_f^*(x,0))$.*

A strictly decreasing continuous function f can be used as an additive generator of a t-norm T and as a generator of an implication I_f^* at the same time. Therefore the relation between the t-norm T and the implication I_f^*, generated by the same function f, is interesting.

Proposition 5.14 (Biba, Hliněná, Kalina, and Kráľ [9]) *Let $f:[0,1]\to[0,\infty]$ be a strictly decreasing continuous function such that $f(1)=0$. If f is an additive generator of a t-norm T, then I_f^* satisfies (LI) with respect to T.*

Proof. Assume that f is an additive generator of T, i.e., $T(x,y) = f^{(-1)}(f(x)+f(y))$.

- Let $T(x,y) \leq z$. Then, by Proposition 5.6, $I_f^*(T(x,y),z)) = 1$. The inequality $T(x,y) \leq z$ can be rewritten into
$$f^{(-1)}(f(x)+f(y)) \leq z. \tag{15}$$
Regardless of the value of $f(x)+f(y)$, (15) is equivalent to
$$f(x)+f(y) \geq f(z) \quad \Leftrightarrow \quad x \leq f^{(-1)} \max\{0, f(z)-f(y)\},$$
which gives $I_f^*(x, I_f^*(y,z)) = 1$.

- Let $T(x,y) > z$. Then the continuity and strict decreasingness of f implies $f(z) > f(x)+f(y)$ and we have that
$$\begin{aligned} I_f^*(T(x,y),z)) &= f^{(-1)}(f(z)-f(x)-f(y)) = \\ &= f^{(-1)}\left(f\left(f^{(-1)}(f(z)-f(y))\right)-f(x)\right) = I_f^*(x, I_f^*(y,z)). \end{aligned}$$

Proposition 5.15 (Biba, Hliněná, Kalina, and Kráľ [9]) *Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing continuous function and $f(1) = 0$. Let T be a t-norm such that $T(x,y) \leq f^{(-1)}(f(x)+f(y))$, then the following inequalities hold:*

1, $I_f^*(x, I_f^*(y,z)) \leq I_f^*(T(x,y),z)$,

2, $T(I_f^*(x,z), I_f^*(y,z)) \leq I_f^*(T(x,y),z)$,

3, $T(I_f^*(x,y), I_f^*(y,z)) \leq I_f^*(x,z)$,

4, $T(x, I_f^*(x,y)) \leq y$.

Proof. We show just the third inequality because the proofs of remaining inequalities are analogous.
Let $x \leq z$. Then, by Proposition 5.6, $I_f^*(x,z) = 1$ and the discussed inequality is fulfilled. Now we assume $x > z$. We distinguish 3 cases concerning the value of y.

- $y \leq z$. This implies
$$T(I_f^*(x,y), I_f^*(y,z)) = I_f^*(x,y) = f^{(-1)}(f(y)-f(x)).$$
In this case we have that
$$f(y)-f(x) > f(z)-f(x) \quad \Leftrightarrow \quad I_f^*(x,y) < I_f^*(x,z).$$

- $z < y \leq x$. This gives
$$\begin{aligned} T(I_f^*(x,y), I_f^*(y,z)) &\leq f^{(-1)}\left(f((I_f^*(x,y)) + f((I_f^*(y,z))\right) = \\ &= f^{(-1)}\left(f(y)-f(x)+f(z)-f(y)\right) = \\ &= f^{(-1)}(f(z)-f(x)) = I_f^*(x,z). \end{aligned}$$

- $x < y$. In this case
$$T(I_f^*(x,y), I_f^*(y,z)) = I_f^*(y,z) = f^{(-1)}(f(z) - f(y)).$$
Now, $f(z) - f(y) > f(z) - f(x)$. This gives immediately
$$I_f^*(y,z) < I_f^*(x,z).$$

Proposition 5.16 (Biba, Hliněná, Kalina, and Král' [9]) *Let $f : [0,1] \to [0,\infty]$ be a continuous strictly decreasing function such that $f(1) = 0$. Then the implication I_f^* is continuous.*

Proof. Obviously, by Proposition 5.6, I_f^* is continuous for all (x,y) such that $x < y$. Let $x_0 > y_0$ then
$$\lim_{x\to x_0, y\to y_0} I_f^*(x,y) = \lim_{x\to x_0, y\to y_0} f^{-1}(f(y) - f(x)) = f^{-1}(f(y_0) - f(x_0)) = I_f^*(x_0, y_0).$$
Let $x_0 = y_0$ and $1 > \varepsilon > 0$. Then $f(1-\varepsilon) > 0$. Because of the continuity of f at x_0, there exists a $\delta > 0$ such that $f(x_0 - \delta) - f(x_0 + \delta) < f(1-\varepsilon)$. For all $x \in \,]x_0 - \delta, x_0 + \delta[$, $y \in \,]x_0 - \delta, x_0 + \delta[$ we have either $I_f^*(x,y) = 1$ or $0 < f(y) - f(x) < f(1-\varepsilon)$. In both cases $I_f^*(x,y) > 1 - \varepsilon$.

Let $f : [0,1] \to [0,\infty]$ and $\varphi : [0,1] \to [0,1]$ be arbitrary functions. We will denote
$$(f \circ \varphi)(x) = f(\varphi(x)) \quad \text{for all } x \in [0,1].$$

Proposition 5.17 (Biba, Hliněná, Kalina, and Král' [9]) *Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing function such that $f(1) = 0$. Let $\varphi : [0,1] \to [0,1]$ be a strictly increasing automorphism. Then the function $(I_f^*)_\varphi(x,y) = \varphi^{-1}(I_f^*(\varphi(x), \varphi(y)))$ is an implication $I_{f\circ\varphi}^*(x,y)$.*

Proof. We need to prove that $(I_f^*)_\varphi(x,y) = I_{f\circ\varphi}^*(x,y)$. In case $\varphi^{-1} \circ f^{(-1)} = (f\circ\varphi)^{(-1)}$ we have
$$\begin{aligned}\left(I_f^*\right)_\varphi(x,y) &= \left(\varphi^{-1} \circ f^{(-1)}\right)(\max\{0, (f\circ\varphi)(y) - (f\circ\varphi)(x)\}) = \\ &= (f\circ\varphi)^{(-1)}(\max\{0, (f\circ\varphi)(y) - (f\circ\varphi)(x)\}) = I_{f\circ\varphi}^*(x,y).\end{aligned}$$
We concentrate to proving that $\varphi^{-1} \circ f^{(-1)} = (f\circ\varphi)^{(-1)}$. We have that
$$(f\circ\varphi)^{(-1)}(x) = \sup\{t \in [0,1]; f(\varphi(t)) > x\}. \tag{16}$$
φ is an increasing bijection. Let $\varphi(t) = s$. Denote $\vartheta = (f\circ\varphi)^{(-1)}(x)$. Then (16) is equivalent to
$$\varphi(\vartheta) = f^{(-1)}(x) = \sup\{s \in [0,1]; f(s) > x\}$$
and finally
$$\vartheta = \left(\varphi^{-1} \circ f^{(-1)}\right)(x).$$

Using theorems 3.2, 3.1 and propositions 5.6, 5.8, 5.10 we are able to partially characterize class of I_f^* implications:

Theorem 5.18 (Biba, Hliněná, Kalina, and Král' [9]) *Let $f : [0,1] \to [0,\infty]$ be a continuous strictly decreasing function such that $f(1) = 0$. Then I_f^* is an R–implication given by a continuous t-norm. Moreover, if $f(0) < \infty$ then I_f^* is an (S,N)–implication.*

5.2 Generated fuzzy implications I^g and their generalization

The class of implications I^g was introduced by Smutná in [44] (original description is in the Theorem 3.25). This result was presented without proof, full proof can be found in [10].

In this chapter we study properties of implications I^g. We focus our attention on properties of implications I^g and we also study the intersections between implications I^g and classes of $(S, N)-$ and $R-$ implications.

The generated implications I^g are illustrated in the following examples.

Example 5.19 (Hliněná and Biba [28, 11]) *Let* $g_1, g_2 : [0,1] \to [0,\infty]$ *be given by*

- $g_1(x) = \begin{cases} x & \text{if } x \le 0.5, \\ 0.5 + 0.5x & \text{otherwise,} \end{cases}$

- $g_2(x) = -\ln(1-x)$.

Note that both functions g_1 *and* g_2 *are strictly increasing. For functions* $g_1^{(-1)}$ *and* $g_2^{(-1)}$ *we get*

- $g_1^{(-1)}(x) = \begin{cases} x & \text{if } x \le 0,5, \\ 0,5 & \text{if } 0,5 < x \le 0,75, \\ 2x-1 & \text{if } 0,75 < x \le 1, \\ 1 & \text{if } 1 < x, \end{cases}$

- $g_2^{(-1)}(x) = 1 - e^{-x}$ *for* $x \in [0,\infty]$.

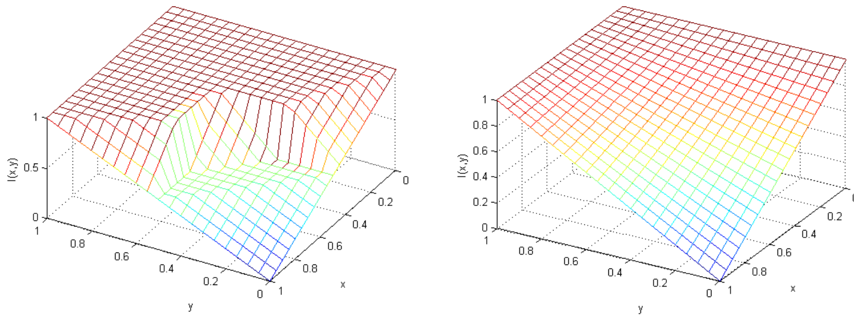
For our functions g_1 *and* g_2 *we have the following (Fig. 12)*

- $I^{g_1}(x,y) = \begin{cases} 1-x+y & \text{if } x \ge 0.5, y \le 0.5, x-y \ge 0.5, \\ 0.5 & \text{if } x \ge 0.5, y \le 0.5, 0.25 \le x-y < 0.5, \\ 1-2x+2y & \text{if } x \ge 0.5, y \le 0.5, x-y < 0.25, \\ \min(1-x+2y, 1) & \text{if } x < 0.5, y \le 0.5, \\ \min(2-2x+y, 1) & \text{if } x \ge 0.5, y > 0.5, \\ 1 & \text{if } x < 0.5, y > 0.5, \end{cases}$

- $I^{g_2}(x,y) = 1 - e^{\ln(x(1-y))} = 1 - x + xy.$

In the case of implications I_f^*, functions f and $(c \cdot f)$ give the same implication. This is also true for the g generators of implications I^g, since Lemma 5.3 holds for all monotone functions.

Proposition 5.20 *Let* c *be a positive constant and* $g : [0,1] \to [0,\infty]$ *be a strictly increasing function. Then the implications* I^g *and* $I^{c \cdot g}$ *which are based on functions* g *and* $c \cdot g$*, respectively, are identical.*



(a) Implication I^{g_1} (b) Implication I^{g_2}

Figure 12: Fuzzy implications I^g

Proof. Let $x, y \in [0,1]$ and c be a positive real number. Then from Proposition 3.25 and Lemma 5.3 we get

$$\begin{aligned} I^{c\cdot g}(x,y) &= (c\cdot g)^{(-1)}\left((c\cdot g)(1-x) + (c\cdot g)(y)\right) \\ &= g^{(-1)}\left(\frac{(c\cdot g)(1-x) + (c\cdot g)(y)}{c}\right) = g^{(-1)}\left(g(1-x) + g(y)\right) = I^g(x,y). \end{aligned}$$

Corollary 5.21 *Let $g : [0,1] \to [0,\infty]$ be bounded and strictly increasing function such that $g(0) = 0$. Let $g^*(x) = \frac{g(x)}{g(1)}$. Then $I^g = I^{g^*}$ and also $g^*(1) = 1$. Hence, if g is a bounded function we can always assume that $g(1) = 1$.*

The following lemma and proposition are following from by the fact that $g^{(-1)}(g(x)) = x$ for a strictly monotonous function g.

Lemma 5.22 *Let $g : [0,1] \to [0,\infty]$ be a strictly increasing function such that $g(0) = 0$. Then the natural negation related to I_g is $N_{I^g}(x) = 1 - x$.*

Proposition 5.23 *Let $g : [0,1] \to [0,\infty]$ be a strictly increasing function such that $g(0) = 0$. Then I^g satisfies (NP) and (CP) with respect to N_s.*

If a function g is continuous and strictly increasing, then we have $g \circ g^{(-1)}(x) = x$ for all $x \in [0,1]$. We use this fact in the proof of the following result:

Proposition 5.24 *Let $g : [0,1] \to [0,\infty]$ be a continuous and strictly increasing function such that $g(0) = 0$. Then I^g satisfies the (EP).*

Proof. Let g be the function as in proposition. For continuous increasing function g using (4) we get

$$I^g(y,z) = g^{(-1)}(g(1-y) + g(z)) = \begin{cases} g^{-1}(g(1-y) + g(z)) & g(1-y) + g(z) < g(1), \\ 1 & \text{otherwise.} \end{cases}$$

Repeated application (4) we get

$$I^g(x, I^g(y,z)) = \begin{cases} g^{-1}(g(1-x)+g(1-y)+g(z)) & g(1-x)+g(1-y)+g(z) < g(1), \\ 1 & \text{otherwise.} \end{cases}$$

For the mapping $I^g(y, I^g(x,z))$ we give the same formula and therefore (EP) is satisfied.

There exist also non-continuous functions g such that I^g satisfy (EP). It is illustrated by the following example:

Example 5.25 *Let $g : [0,1] \to [0,1]$ be given by*

$$g(x) = \begin{cases} 0 & x = 0, \\ \frac{1}{2}(x+1) & \text{otherwise.} \end{cases}$$

For its pseudo-inverse we get

$$g^{(-1)}(x) = \begin{cases} 0 & x \le \frac{1}{2}, \\ 2x-1 & x > \frac{1}{2}. \end{cases}$$

Because g is continuous in $]0,1]$ and strictly monotone, we have to show that (EP) holds for I^g only for triples (x,y,z) such that $x = 1$ or $y = 1$ or $z = 0$, and it follows from strict monotonicity of g and equality $g^{(-1)} \circ g(x) = x$.

Any $R-$implication must satisfy an ordering property (OP). For implications I^g we get the following proposition concerning (OP):

Proposition 5.26 *Let $g : [0,1] \to [0,c]$ be a strictly increasing bounded function such that $g(0) = 0$. If for all $x \in [0,1]$, it is $g(1-x) = g(1^-) - g(x)$, then I^g posses (OP).*

Proof. Let $g : [0,1] \to [0,\infty]$ be a strictly increasing function such that $g(0) = 0$ and let $g(1-x) = g(1^-) - g(x)$ holds for all $x \in [0,1]$. The proof is divided to three parts:

- First we show that $I^g(x,x) = 1$ for all $x \in [0,1]$:

  $$I^g(x,x) = g^{(-1)}(g(1-x)+g(x)) = g^{(-1)}\left(g(1^-)\right) =$$
  $$= \sup\{z \in [0,1]; g(z) < g(1^-)\} = 1.$$

  The last equality follows from the fact that g is a strictly increasing function, which means that $\forall z \in [0,1[; g(z) < g(1^-)$. (Recall that $g(1^-) = \lim_{t \to 1^-} g(t)$.)

- Let $y \ge x$, since I^g is a fuzzy implication, it is increasing in the second argument and therefore $I^g(x,y) \ge I^g(x,x) = 1$, i.e. $I^g(x,y) = 1$.

- In the last part we need to show that $I^g(x,y) < 1$ whenever $x > y$: Let $x > y$, since g is a strictly increasing function and $g(1-y) + g(y) = g(1^-)$, it holds that $g(1-x) + g(y) < g(1^-)$. Take

  $$\varepsilon = g(1^-) - (g(1-x) + g(y)),$$

  then from the definition of limit we know that there exists $t < 1$, such that $g(1^-) - g(t) < \varepsilon$. Now it holds that $g(t) > g(1-x) + g(y)$ and therefore $\sup\{z \in [0,1]; g(z) < g(1-x) + g(y)\} \le t$. It means that $I^g(x,y) \le t < 1$.

From the Theorem 3.1 and previous propositions we get the following relation between I^g implications and $(S,N)-$implications:

Theorem 5.27 *Let $g : [0,1] \to [0,\infty]$ be a strictly increasing function continuous on $]0,1]$ such that $g(0) = 0$. Then I^g is an (S,N)-implication which is strong.*

Analogically, using the Theorem 3.2 and previous propositions we get the following relation between implications I^g and $R-$implications:

Theorem 5.28 *Let $g : [0,1] \to [0,c]$ be a continuous and strictly increasing bounded function such that $g(0) = 0$. If $\forall x \in [0,1]$; $g(1-x) = g(1^-) - g(x)$, then I^g is an R-implication based on some left-continuous t-norm T.*

The implications I^g can be generalized by substituting the standard negation N_s by the arbitrary one (see the Theorem 3.26). This class was also introduced by Smutná in [44] and it was studied by Biba and Hliněná in [11] and [28]. Several results obtained for I^g implications are valid also for I^g_N implications.

For illustration we introduce a following example:

Example 5.29 (Hliněná and Biba [28, 11]) *Let N be a fuzzy negation given by $N(x) = 1-x^2$ and functions $g_1, g_2 : [0,1] \to [0,\infty]$ are given by*

- $g_1(x) = \begin{cases} x & \text{if } x \le 0.5, \\ 0.5 + 0.5x & \text{otherwise,} \end{cases}$
- $g_2(x) = -\ln(1-x)$.

Then I^g_N implications are given by (Fig. 13)

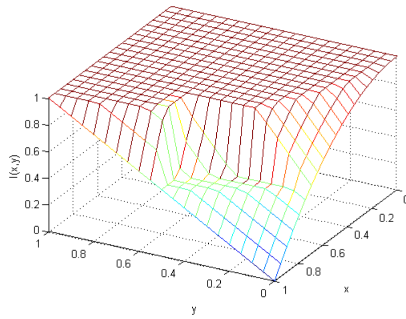
- $I^{g_1}_N(x,y) = \begin{cases} 1 - x^2 + y & \text{if } x \ge \frac{1}{\sqrt{2}}, y \le 0.5, x^2 - y \ge 0.5, \\ 0.5 & \text{if } x \ge \frac{1}{\sqrt{2}}, y \le 0.5, 0.25 \le x^2 - y < 0.5, \\ 1 - 2x^2 + 2y & \text{if } x \ge \frac{1}{\sqrt{2}}, y \le 0.5, x^2 - y < 0.25, \\ \min(1 - x^2 + 2y, 1) & \text{if } x < \frac{1}{\sqrt{2}}, y \le 0.5, \\ \min(2 - 2x^2 + y, 1) & \text{if } x \ge \frac{1}{\sqrt{2}}, y > 0.5, \\ 1 & \text{if } x < \frac{1}{\sqrt{2}}, y > 0.5, \end{cases}$
- $I^{g_2}_N(x,y) = 1 - x^2 + x^2 y$.

From Definition 2.48 we get for the natural negation related I^g_N the following result:

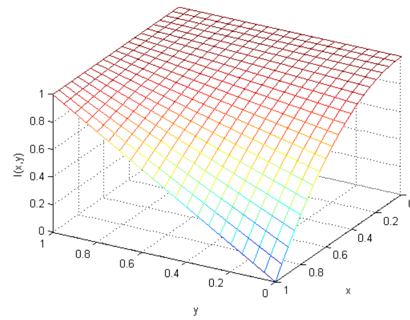
Proposition 5.30 (Biba and Hliněná [11]) *Let $N : [0,1] \to [0,1]$ be a negation and $g : [0,1] \to [0,\infty]$ be a strictly increasing function such that $g(0) = 0$. Then the natural negation is $N_{I^g_N}(x) = N(x)$.*

Remark 5.31 *Note that this follows from the fact that $(g^{(-1)} \circ g)(x) = x$.*

Proof of the following result is analogous to the proof of Proposition 5.24, therefore it is omitted.



(a) Implication $I_N^{g_1}$



(b) Implication $I_N^{g_2}$

Figure 13: Fuzzy implications I_N^g

Proposition 5.32 (Biba and Hliněná [11]) *Let $N : [0,1] \to [0,1]$ be a negation and $g : [0,1] \to [0,\infty]$ be a continuous and strictly increasing function such that $g(0) = 0$. Then I_N^g satisfies the (EP).*

Proposition 5.33 *Let $N : [0,1] \to [0,1]$ be a fuzzy negation and $g : [0,1] \to [0,\infty]$ be a strictly increasing function such that $g(0) = 0$. Then the implication I_N^g satisfies (NP). Moreover, if N is a strong negation then the implication I_N^g satisfies (CP) with respect to the negation N.*

The part concerning (NP) follows from the fact that $(g^{(-1)} \circ g)(x) = x$. The second part can be proved using the fact that $N(N(x)) = x$ for any strong fuzzy negation.

Proposition 5.34 *Let $N : [0,1] \to [0,1]$ be a negation and $g : [0,1] \to [0,\infty]$ be a strictly increasing function such that $g(0) = 0$. The implication I_N^g satisfies (IP) if and only if, for all $x \in [0,1]$, it holds $g(N(x)) \geq g(1^-) - g(x)$.*

Proof. On the one hand, for all $x \in [0,1]$, we have that $g(N(x)) + g(x) \geq g(1^-)$, which implies, for any $t < 1$, $g(t) < g(N(x)) + g(x)$. Therefore $I_N^g(x,x) = g^{(-1)}(g(N(x)) + g(x)) = 1$. On the other hand, let $x \in [0,1]$ and $g(N(x)) + g(x) < g(1^-)$. Let $\varepsilon > 0$ and $g(N(x)) + g(x) < g(1^-) - \varepsilon$. Since $g(1^-) = \lim\limits_{x \to 1^-} g(x)$, we obtain that there exists $t_0 < 1$ such that $g(1^-) - g(t_0) < \varepsilon$. It implies $g(N(x)) + g(x) < g(t_0)$, which means $g^{(-1)}(g(N(x)) + g(x)) \leq t_0 < 1$.

The necessary condition for an implication to be an $R-$implication is the ordering property (OP). In [11] we can found the following proposition (proof is analogous to the proof of Proposition 5.26):

Proposition 5.35 (Biba and Hliněná [11]) *Let $N : [0,1] \to [0,1]$ be a negation and $g : [0,1] \to [0,\infty]$ be an strictly increasing and bounded function such that $g(0) = 0$. If for all $x \in [0,1]$ it is $g(N(x)) = g(1^-) - g(x)$, then I_N^g possesses (OP).*

The following two theorems follow from Theorems 3.2 and 3.1 and above mentioned properties of I_N^g.

Theorem 5.36 (Biba and Hliněná [11]) *Let $g : [0,1] \to [0,c]$ be a continuous and strictly increasing function such that $g(0) = 0$, and negation N be right-continuous. If for all $x \in [0,1]$; $g(N(x)) = g(1^-) - g(x)$, then I_N^g is an R-implication given by some left-continuous t-norm T.*

Theorem 5.37 (Biba and Hliněná [11]) *Let $g : [0,1] \to [0,\infty]$ be a continuous and strictly increasing function such that $g(0) = 0$ and $N : [0,1] \to [0,1]$ be a continuous fuzzy negation. Then I_N^g is an (S,N)-implication where S is the t-conorm generated by g.*

5.3 Generalized generated implications

A new class of generated fuzzy implications can be obtained by combining the previous approaches. We use a strictly decreasing function and a formula similar to Formula (4).

If we compose a strictly decreasing function f with a fuzzy negation N then $g(x) = f(N(x))$ is again an increasing function (though not necessarily strictly increasing). This allows us to generalize the fuzzy implications I^g. This class was introduced in [8] and it was studied in [12].

Theorem 5.38 (Biba, Hliněná, Kalina and Král' [8] without proof, Biba and Hliněná [12]) *Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing function with $f(1) = 0$ and $N : [0,1] \to [0,1]$ be a fuzzy negation. Then the function $I_f^N : [0,1]^2 \to [0,1]$ defined by*

$$I_f^N(x,y) = N\left(f^{(-1)}\left(f(x) + f(N(y))\right)\right), \tag{17}$$

is a fuzzy implication.

Proof. Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing function such that $f(1) = 0$ and let N be a fuzzy negation. We will proceed by points of Definition 2.41.

- Let $x_1 < x_2$, then $f(x_1) + f(N(y)) \geq f(x_2) + f(N(y))$. Pseudo-inverse $f^{(-1)}$ is a decreasing function (not necessarily strictly decreasing), which means that $f^{(-1)}\left(f(x_1) + f(N(y))\right) \leq f^{(-1)}\left(f(x_2) + f(N(y))\right)$. Since N is a fuzzy negation, it is a decreasing function and therefore $I_f^N(x_1,y) \geq I_f^N(x_2,y)$.
- Let $y_1 < y_2$, then $N(y_1) \geq N(y_2)$ and $f(x) + f(N(y_1)) \leq f(x) + f(N(y_2))$. Since $f^{(-1)}$ is decreasing, it holds that $f^{(-1)}\left(f(x) + f(N(y_1))\right) \geq f^{(-1)}\left(f(x) + f(N(y_2))\right)$ and consequently $I_f^N(x,y_1) \leq I_f^N(x,y_2)$.
- $$\begin{aligned} I_f^N(1,0) &= N\left(f^{(-1)}\left(f(1) + f(N(0))\right)\right) = N\left(f^{(-1)}(0+0)\right) = N(1) = 0, \\ I_f^N(0,0) &= N\left(f^{(-1)}\left(f(0) + f(N(0))\right)\right) = N\left(f^{(-1)}(f(0))\right) = N(0) = 1, \\ I_f^N(1,1) &= N\left(f^{(-1)}\left(f(1) + f(N(1))\right)\right) = N\left(f^{(-1)}(f(0))\right) = N(0) = 1. \end{aligned}$$

This concludes the proof.

In this part we investigate the properties of generated fuzzy implications which are mentioned in Theorem 5.38. For illustration we introduce the following examples of fuzzy implication I_f^N.

Example 5.39 (Biba and Hliněná [12]) *Let* $f_1(x) = 1 - x$, $f_2(x) = -\ln x$, *and* $N_1(x) = 1 - x$, $N_2(x) = \sqrt{1 - x^2}$. *Then the functions* $f_1^{(-1)}$ *and* $f_2^{(-1)}$ *are given by* $f_1^{(-1)}(x) = \max(1 - x, 0)$ *and* $f_2^{(-1)}(x) = e^{-x}$. *The fuzzy implications* I_f^N *are given by*

$$I_{f_1}^{N_1}(x, y) = \min(1 - x + y, 1),$$
$$I_{f_2}^{N_1}(x, y) = 1 - x + x \cdot y,$$
$$I_{f_2}^{N_2}(x, y) = \sqrt{1 - x^2 + x^2 \cdot y^2}.$$

Note, that $I_{f_1}^{N_1}$ *and* $I_{f_2}^{N_1}$ *are the well-known Łukasiewicz and Reichenbach implication, respectively. Also note, that for all fuzzy implications it holds that* $I(x, 0) = N(x)$.

We are able to generalize the property from Example 5.39 for all I_f^N implications and $N_{I_f^N}(x)$ negations.

Proposition 5.40 (Biba and Hliněná [12]) *Let* $f : [0, 1] \to [0, \infty]$ *be a strictly decreasing function such that* $f(1) = 0$ *and* N *be an arbitrary fuzzy negation. Then the natural negation* N_I *given by* I_f^N *is* $N_{I_f^N}(x) = N(x)$.

Proof. Let $f : [0, 1] \to [0, \infty]$ be a strictly decreasing function such that $f(1) = 0$ and N be an arbitrary fuzzy negation. For $N_{I_f^N}(x)$ we have

$$N_{I_f^N}(x) = I_f^N(x, 0) = N\left(f^{(-1)}\left(f(x) + f(N(0))\right)\right) = N\left(f^{(-1)}\left(f(x) + f(1)\right)\right) = N\left(f^{(-1)}\left(f(x)\right)\right).$$

Since the function f is strictly decreasing, its pseudo-inverse is continuous, and therefore $f^{(-1)} \circ f(x) = x$. And for natural negation we get

$$N_{I_f^N}(x) = N\left(f^{(-1)}\left(f(x)\right)\right) = N(x).$$

It is well-known that generators of continuous Archimedean t-norms are unique up to a positive multiplicative constant, and this is also valid for the f generators of I_N^f implications. This follows from Lemma 5.3.

Proposition 5.41 (Biba and Hliněná [12]) *Let* c *be a positive constant and* $f : [0, 1] \to [0, \infty]$ *be a strictly decreasing function such that* $f(1) = 0$. *Then the implications* I_N^f *and* $I_N^{c \cdot f}$ *which are based on functions* f *and* $c \cdot f$*, respectively, are identical.*

The above mentioned property of a strictly decreasing function and its pseudo-inverse is again important for fulfilment of (NP). Therefore the proof is similar to the previous and we can omit it.

Proposition 5.42 (Biba and Hliněná [12]) *Let* $f : [0, 1] \to [0, \infty]$ *be a strictly decreasing function such that* $f(1) = 0$. *Then the fuzzy implication* I_f^N *satisfies (NP) if and only if* N *is an involutive negation.*

Proposition 5.43 (Biba and Hliněná [12]) *Let* $f : [0, 1] \to [0, \infty]$ *be a strictly decreasing function such that* $f(1) = 0$. *Then the fuzzy implication* I_f^N *satisfies (CP) with respect to* N *if and only if* N *is an involutive negation.*

Proof. Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing function such that $f(1) = 0$. For fuzzy implications $I_f^N(x,y)$ and $I_f^N(N(y),N(x))$ we get

$$I_f^N(x,y) = N\left(f^{(-1)}\left(f(x) + f(N(y))\right)\right),$$

$$I_f^N(N(y),N(x)) = N\left(f^{(-1)}\left(f(N(y)) + f(N(N(x)))\right)\right).$$

It is obvious that $I_f^N(x,y) = I_f^N(N(y),N(x))$ if and only if $N(N(x)) = x$ for all $x \in [0,1]$.

Proposition 5.44 (Biba and Hliněná [12]) *Let $f : [0,1] \to [0,\infty]$ be a continuous and bounded strictly decreasing function such that $f(1) = 0$ and $N(x) = f^{-1}(f(0) - f(x))$. Then the fuzzy implication I_f^N satisfies (OP).*

Proof. Let $f : [0,1] \to [0,c]$ be a function described in the proposition, and c be a positive real number, it is obvious that $N(x) = f^{-1}(f(0) - f(x))$ is fuzzy negation. Since f is a strictly decreasing and continuous function, it holds

$$I_f^N(x,y) = N\left(f^{(-1)}\left(f(x) + f(N(y))\right)\right) = N\left(f^{(-1)}\left(f(0) + f(x) - f(y)\right)\right).$$

Now we need to distinguish two cases:

- Let $x \leq y$, then $f(x) - f(y) \geq 0$ and $f(0) + f(x) - f(y) \geq f(0)$, i.e
$$I_f^N(x,y) = N\left(f^{(-1)}(f(0))\right) = N(0) = 1.$$
- Let $x > y$, then $f(0)+f(x)-f(y) < f(0)$ and consequently $f^{(-1)}\left(f(0) + f(x) - f(y)\right) > 0$, i.e
$$I_f^N(x,y) < N(0) = 1.$$

Summarizing the previous facts we get that $I_f^N(x,y) = 1$ if and only if $x \leq y$.

Remark 5.45 (Biba and Hliněná [12]) *Let $f : [0,1] \to [0,\infty]$ be a not bounded function. Then the fuzzy implication I_f^N does not hold (OP). This follows from the fact that for all $x, y \in]0,1[$ we get $f(x) + f(N(y)) < f(0)$ and consequently $I_f^N(x,y) < 1$.*

Proposition 5.46 (Biba and Hliněná [12]) *Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing and continuous function such that $f(1) = 0$. Let $N : [0,1] \to [0,1]$ be a strong negation. Then the fuzzy implication I_f^N satisfies (EP).*

Proof. Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing continuous function such that $f(1) = 0$ and N be a strong negation. Then

$$I_f^N(x, I_f^N(y,z)) = I_f^N\left(x, N\left(f^{(-1)}\left(f(y) + f(N(z))\right)\right)\right).$$

Since f is a strictly decreasing and continuous function, the following equality is satisfied

$$f^{(-1)}\left(f(y) + f(N(z))\right) = \begin{cases} 0 & f(y) + f(N(z)) \geq f(0), \\ f^{-1}\left(f(y) + f(N(z))\right) & \text{otherwise.} \end{cases}$$

Now we apply the fact that N is a strong negation and we get

$$I_f^N(x, I_f^N(y,z)) = \begin{cases} N\left(f^{(-1)}\left(f(x)+f(0)\right)\right) & \text{if } f(y)+f(N(z)) \geq f(0), \\ N\left(f^{(-1)}\left(f(x)+f(y)+f(N(z))\right)\right) & \text{otherwise.} \end{cases}$$

And for $I_f^N(y, I_f^N(x,z))$ we have

$$I_f^N(y, I_f^N(x,z)) = \begin{cases} N\left(f^{(-1)}\left(f(y)+f(0)\right)\right) & \text{if } f(x)+f(N(z)) \geq f(0), \\ N\left(f^{(-1)}\left(f(x)+f(y)+f(N(z))\right)\right) & \text{otherwise.} \end{cases}$$

Since $N\left(f^{(-1)}\left(f(x)+f(0)\right)\right) = N\left(f^{(-1)}\left(f(y)+f(0)\right)\right) = 1$, we can write

$$I_f^N(x, I_f^N(y,z)) = \begin{cases} 1 & \text{if } f(y)+f(N(z)) \geq f(0), \\ N\left(f^{(-1)}\left(f(x)+f(y)+f(N(z))\right)\right) & \text{otherwise.} \end{cases}$$

$$I_f^N(y, I_f^N(x,z)) = \begin{cases} 1 & \text{if } f(x)+f(N(z)) \geq f(0), \\ N\left(f^{(-1)}\left(f(x)+f(y)+f(N(z))\right)\right) & \text{otherwise.} \end{cases}$$

If $f(y)+f(N(z)) \geq f(0)$, then also $f(x)+f(y)+f(N(z)) \geq f(0)$, which means that $I_f^N(y, I_f^N(x,z)) = 1$. And, on the contrary, if $f(x)+f(N(z)) \geq f(0)$, then $I_f^N(x, I_f^N(y,z)) = 1$.

The following theorem describes the relationship between the generated fuzzy implications I_N^f and (S,N)– or R–implications.

Theorem 5.47 (Biba and Hliněná [12]) *Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing and continuous function such that $f(1) = 0$. Let $N : [0,1] \to [0,1]$ be a strong negation. Then I_f^N is (S,N)–implication. Moreover, if f is bounded function and $N(x) = f^{-1}(f(0) - f(x))$, then I_f^N is an R–implication as well.*

Some relation between these generated implications and t-norms is described in the next proposition.

Proposition 5.48 (Biba and Hliněná [12]) *Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing and continuous function such that $f(1) = 0$. Let $N : [0,1] \to [0,1]$ be a strong negation. Then the fuzzy implication I_f^N satisfies (LI) with a t-norm $T(x,y) = f^{(-1)}(f(x)+f(y))$.*

Proof. Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing continuous function such that $f(1) = 0$, N be a strong negation, and $T : [0,1]^2 \to [0,1]$ be a t-norm given by $T(x,y) = f^{(-1)}(f(x)+f(y))$. Then

$$I_f^N(T(x,y), z) = \begin{cases} N\left(f^{(-1)}\left(f(0)+f(N(z))\right)\right) & \text{if } f(x)+f(y) \geq f(0), \\ N\left(f^{(-1)}\left(f(x)+f(y)+f(N(z))\right)\right) & \text{otherwise,} \end{cases}$$

and from the previous proof we get for $I_f^N(x, I_f^N(y,z))$ the following formula

$$I_f^N(x, I_f^N(y,z)) = \begin{cases} 1 & \text{if } f(y)+f(N(z)) \geq f(0), \\ N\left(f^{(-1)}\left(f(x)+f(y)+f(N(z))\right)\right) & \text{otherwise.} \end{cases}$$

It is obvious that $N\left(f^{(-1)}\left(f(0)+f(N(z))\right)\right)=1$ and by similar method as we have used in previous proof we get that $I_f^N(T(x,y),z)=I_f^N(x,I_f^N(y,z))$.

Note, that mentioned fuzzy implications are not the only generalizations of fuzzy implications I_N^g. Considering Formula (17) and Lemma 5.49, we can see that N might be replaced by $N^{(-1)}$ if it is a fuzzy negation. Still, there are at least two fuzzy negations (in general different from N) which are related to N. Namely, $N^{(-1)}$ and N^d. Hence we have the following two additional possibilities how to generate fuzzy implications.

If we apply the pseudo-inverse to a negation N we get the following assertion.

Lemma 5.49 (Biba, Hliněná, Kalina, and Král' [8]) *Let $N:[0,1]\to[0,1]$ be a fuzzy negation. Then $N^{(-1)}$ is a fuzzy negation if and only if*

$$N(x)=0 \quad \Leftrightarrow \quad x=1. \tag{18}$$

Proof. Of course, if $x=1$ then $N(x)=0$ obviously holds. Let us assume that the equality $N(x)=0$ holds for an $x<1$. Then we have the following formula

$$N^{(-1)}(0)=\sup\{z\in\ [0,1];N^{(-1)}(z)>\ 0\}\leq x<1,$$

and we get that $N^{(-1)}$ is not a negation.

Theorem 5.50 (Biba, Hliněná, Kalina and Král' [8]) *Let $f:[0,1]\to[0,\infty]$ be a strictly decreasing function with $f(1)=0$, and $N:[0,1]\to[0,1]$ be a fuzzy negation such that* (18) *is fulfilled for N. Then the function $I_f^{(N,N^{(-1)})}:[0,1]^2\to[0,1]$ defined by*

$$I_f^{(N,N^{(-1)})}(x,y)=N^{(-1)}\left(f^{(-1)}\left(f(x)+f(N(y))\right)\right), \tag{19}$$

is a fuzzy implication.

Theorem 5.51 (Biba, Hliněná, Kalina and Král' [8]) *Let $f:[0,1]\to[0,\infty]$ be a strictly decreasing function with $f(1)=0$ and $N:[0,1]\to[0,1]$ be a fuzzy negation. Then function $I_f^{(N,N^d)}:[0,1]^2\to[0,1]$ defined by*

$$I_f^{(N,N^d)}(x,y)=N^d\left(f^{(-1)}\left(f(x)+f(N(y))\right)\right), \tag{20}$$

is a fuzzy implication.

6 Preference structures given by generated fuzzy implications

In this section we study the construction of fuzzy preference structures from fuzzy implications. We use the fuzzy implications I_f and I^g mentioned in the previous chapter. The following results can be found in the article [10] by Biba and Hliněná. The inspiration for this investigation was the article [48] by Šabo and Strežo.

First we turn our attention to the fuzzy implications I_f. In the next example, we deal with the Łukasiewicz triplet $(T_L, S_L, 1-x)$.

Example 6.1 (Biba and Hliněná [10]) *Let* $f(x) = N_s(x)$. *Note that fuzzy negation* N_s *satisfies assumptions of Proposition 5.1. We obtain fuzzy implication* $I_{N_s}(x,y) = \min(1-x+y, 1)$. *For function* p *we have*

$$p(x,y) = 1 - I_{N_s}(x,y) = \max(x-y, 0).$$

In order to satisfy (R4'), mappings i, j *must be* $i(x,y) = \min(x,y)$ *and* $j(x,y) = \min(1-x, 1-y)$. *Obviously* i *and* j *are symmetric functions. Therefore (R3) is satisfied.*

Now, we turn our attention to the properties (gt1)–(gt5). Axioms (R3) and (R4') imply properties (gt3) and (gt5). More, from (R3) and (R4') we have

$$p(x,y) + p(y,x) + i(x,y) + j(x,y) = p(x,y) + i(x,y) + p(y,x) + j(y,x) = x + 1 - x = 1.$$

Therefore property (gt4) again follows from (R3) and (R4').

It is obvious that in this example the properties (gt1) and (gt2) are satisfied, too. Therefore triplet (p,i,j) *is the monotone generator triplet.*

Remark 6.2 *Note that the fuzzy implication* $I_{N_s}(x,y) = \min(1-x+y, 1)$ *from the previous example is the well-known Łukasiewicz implication* I_{T_L}.

The following proposition shows that the fuzzy implications I_{T_L} is the only one we can use:

Proposition 6.3 (Biba and Hliněná [10]) *Let* $f : [0,1] \to [0,\infty]$ *be a strictly decreasing function such that* $f(1) = 0$ *and* $p(x,y) = 1 - I_f(x,y)$. *Then triplet* (p,i,j), *where* $i(x,y) = x - p(x,y)$ *and* $j(x,y) = 1 - y - p(x,y)$, *satisfies (R3) and (R4') if and only if* $I_f(x,y) = I_{T_L}$.

Proof. Let (p,i,j) satisfy (R3) and (R4'). Then by (R3), $i(x,y)$ is symmetric function. Since $p(x,y) = 1 - I_f(x,y)$, from (R4') we get

$$x - 1 + I_f(x,y) = y - 1 + I_f(y,x).$$

From the definition of I_f (see Proposition 5.1), either $I_f(x,y) = 1$ or $I_f(y,x) = 1$. Therefore by previous equality, either $I_f(y,x) = 1 - y + x$, or $I_f(x,y) = 1 - x + y$ in order to satisfy both (R3) and (R4') at the same time. The converse is obvious from previous example.

Remark 6.4 *Note that a fuzzy implication satisfies the ordering property (OP) if the following is true:* $x \leq y$ *if and only if* $I(x, y) = 1$*. The previous proposition can be generalized for all fuzzy implications with (OP).*

Proposition 6.5 (Biba and Hliněná [10]) *Let* $I : [0, 1]^2 \to [0, 1]$ *be a fuzzy implication satisfying (OP), and* $p(x, y) = 1 - I(x, y)$*. Then the triplet* (p, i, j) *satisfies (R3) and (R4') if and only if* $I(x, y) = I_{T_L}$*.*

Proof. Let the triplet (p, i, j) satisfy (R3) and (R4') and $p(x, y) = 1 - I(x, y)$. Using (R4') we get $i(x, y) = x - 1 + I(x, y)$ and from symmetry of $i(x, y)$ we have the equality

$$x + I(x, y) = y + I(y, x).$$

Since $I(x, y)$ satisfies (OP), we have $I(x, y) = 1$ or $I(y, x) = 1$, and therefore we get $I(x, y) = I_{T_L}$. The converse is similar to Example 6.1.

Remark 6.6 (Biba and Hliněná [10]) *Note, that the triplets mentioned in previous propositions satisfy also properties (gt1)–(gt5), this means they are monotone generator triplets.*

Remark 6.7 (Biba and Hliněná [10]) *Note, that it has been proved (see [28]) that continuity of function* f *at* $x = 1$ *is equivalent with (OP) for the fuzzy implication* I_f*.*

In the next example, we will assume de Morgan triplet $((T_L)_\varphi, (S_L)_\varphi, (N_s)_\varphi)$:

Example 6.8 (Biba and Hliněná [10]) *Let* φ *be an order-automorphism and* $f(x) = 1 - \varphi(x)$*, then*

$$I_f(x, y) = \begin{cases} 1 & x \leq y, \\ \varphi^{-1}(1 - \varphi(x) + \varphi(y)) & x > y. \end{cases}$$

The triplet $(p, i, j)_\varphi$ *such that* $p(x, y) = (N_s)_\varphi(I_f(x, y))$*,* $i(x, y) = \varphi^{-1}(\varphi(x) - 1 + \varphi(I_f(x, y)))$*, and* $j(x, y) = \varphi^{-1}(\varphi(I_f(x, y)) - \varphi(y))$*, satisfies axioms (R3) and (R4'): After plugging in* $I_f(x, y)$*, we get*

$$p(x, y) = \varphi^{-1}\left(\max(\varphi(x) - \varphi(y), 0)\right),$$

$$i(x, y) = \varphi^{-1}\left(\min(\varphi(x), \varphi(y))\right),$$

$$j(x, y) = \varphi^{-1}\left(\min(1 - \varphi(x), 1 - \varphi(y))\right).$$

As we have mentioned, we assume de Morgan triplet $((T_L)_\varphi, (S_L)_\varphi, (N_s)_\varphi)$ *in this example. In this case, a more general form of (R4') is needed:*

$$(S_L)_\varphi\left(p(x, y), i(x, y)\right) = x, \quad (S_L)_\varphi\left(p(x, y), j(x, y)\right) = (N_s)_\varphi(y).$$

Obviously the mappings i*,* j *are symmetric functions, i.e. (R3) is satisfied. The proof that axiom (R4') is also satisfied is simple, but lengthy.*

For the triplet $((T_L)_\varphi, (S_L)_\varphi, (N_s)_\varphi)$ and fuzzy implications I_f we get a result similar to Proposition 6.3:

Proposition 6.9 (Biba and Hliněná [10]) *Let φ be an order-automorphism. Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing function such that $f(1) = 0$, and*

$$I_f(x,y) = \begin{cases} 1 & x \leq y, \\ f^{(-1)}(f(y^+) - f(x)) & x > y. \end{cases}$$

Then the system $(p,i,j)_\varphi$ where $p(x,y) = (N_s)_\varphi(I_f(x,y))$ satisfies (R3), and (R4') if and only if $I_f(x,y) = \min(\varphi^{-1}(1-\varphi(x)+\varphi(y)),1)$.

A proof of this fact is similar to the proofs of previous propositions.

Now we turn our attention to the fuzzy implications I^g and I^g_N. The partial mapping of $I^g(x,0)$ is $I^g(x,0) = 1-x$, and for an arbitrary fuzzy negation N we have $I^g_N(x,0) = N(x)$. On the other hand, Proposition 3.28 gives that $I^{\rightarrow}(x,0) = 1-x$, therefore we will investigate function $p(x,y) = 1 - I^g(x,y)$. Using (R4'), we get $i(x,y) = I^g(x,y) + x - 1$ and $j(x,y) = I^g(x,y) - y$. From (R3), the function i is symmetric, which leads to the equality

$$I^g(x,y) - I^g(y,x) = y - x \qquad \forall x,y \in [0,1]. \tag{21}$$

If this equality is fulfilled for some fuzzy implication I, then the described triplet (p,i,j) is a generator triplet.

We are looking for functions g, such that fuzzy implications I^g satisfy the equality (21). Several appropriate functions are given in the following examples.

Example 6.10 (Biba and Hliněná [10]) *Let $g_1(x) = -\ln(1-x)$, then its pseudo-inverse function is $g_1^{(-1)}(x) = 1 - e^{-x}$. The fuzzy implication I^{g_1} is given by*

$$I^{g_1}(x,y) = 1 - x + xy.$$

For the mentioned difference we get

$$I^{g_1}(x,y) - I^{g_1}(y,x) = (1-x+xy) - (1-y+xy) = y - x.$$

Equality (21) holds, and triplet (p,i,j), where $p(x,y) = x(1-y)$, $i(x,y) = xy$, $j(x,y) = (1-x)(1-y)$, satisfies axioms (R3)–(R4') and properties (gt1)–(gt5). Note that fuzzy implication I^{g_1} is the well-known Reichenbach implication which is not isomorphic with I_{T_L}.

Example 6.11 (Biba and Hliněná [10]) *Let $g_2(x) = x$. The pseudo-inverse of function g_2 is given by $g_2^{(-1)}(x) = \min(x,1)$ and therefore the fuzzy implication I^{g_2} is given by*

$$I^{g_2}(x,y) = \min(1-x+y,1) = I_{T_L}(x,y).$$

As we know from example 6.1, the triplet

$$p(x,y) = 1 - I^{g_2}(x,y) = \max(x-y,0),$$

$$i(x,y) = \min(x,y), \quad j(x,y) = \min(1-x,1-y),$$

satisfies axioms (R3)–(R4') and properties (gt1)–(gt5). Equality (21) again holds.

The last example presents fuzzy implication which is related to mentioned Frank t-norms.

Example 6.12 (Biba and Hliněná [10]) *Let* $g_3 = \ln \frac{2}{3^{1-x}-1}$, *then the fuzzy implication* I^{g_3} *is given by*

$$I^{g_3}(x,y) = 1 - \log_3 \left(\frac{(3^x - 1) \cdot (3^{1-y} - 1)}{2} + 1 \right).$$

Note that the function g_3 *is generator of Frank t-conorm and this fuzzy implication* I^{g_3} *is not isomorphic with* I_{T_L}*. For the mentioned difference we get*

$$I^{g_3}(x,y) - I^{g_3}(y,x) = \log_s \frac{(3^{1-x} - 1) \cdot (3^y - 1) + 2}{(3^x - 1) \cdot (3^{1-y} - 1) + 2} =$$

$$= \log_3 \frac{3^{y-x+1} - 3^{1-x} - 3^y + 3}{3^{x-y+1} - 3^{1-y} - 3^x + 3} = \log_3 \frac{\frac{3^{y+1} - 3 - 3^{y+x} + 3^{x+1}}{3^x}}{\frac{3^{x+1} - 3 - 3^{x+y} + 3^{y+1}}{3^y}} = \log_3 \frac{3^y}{3^x} = y - x.$$

Since the equality is satisfied, related triplet (p,i,j) is a generator triplet.

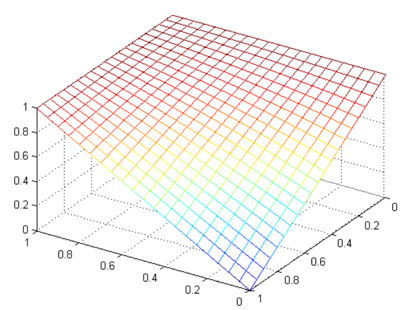


Figure 14: Fuzzy implication I^{g_3}

The following proposition is a generalization of the previous example. We present special class of fuzzy implications with the equality (21). This class of fuzzy implications is not isomorphic with I_{T_L} for arbitrary $s \in]0, \infty[- \{1\}$.

Proposition 6.13 (Biba and Hliněná [10]) *Let* $s \in]0, \infty[- \{1\}$ *and* $g_s(x) = \ln \frac{s-1}{s^{1-x}-1}$. *Then the fuzzy implication* I^{g_s} *satisfies equality* $I(x,y) - I(y,x) = y - x$.

Proof. Let g be the function as described in the proposition. After substituting out $I^{g_s}(x,y)$, $I^{g_s}(y,x)$ and rearranging the terms, we get

$$I^{g_s}(x,y) - I^{g_s}(y,x) = \log_s \frac{(s^{1-x} - 1) \cdot (s^y - 1) + (s - 1)}{(s^x - 1) \cdot (s^{1-y} - 1) + (s - 1)} =$$

$$= \log_s \frac{s^{y-x+1} - s^{1-x} - s^y + s}{s^{x-y+1} - s^{1-y} - s^x + s} = \log_s \frac{\frac{s^{y+1} - s - s^{y+x} + s^{x+1}}{s^x}}{\frac{s^{x+1} - s - s^{x+y} + s^{y+1}}{s^y}} = \log_s \frac{s^y}{s^x} = y - x.$$

Corollary 6.14 (Biba and Hliněná [10]) *Let* $s \in]0, \infty[- \{1\}$. *If*

$$I^{g_s}(x, y) = 1 - \log_s \left(\frac{(s^x - 1) \cdot (s^{1-y} - 1)}{s - 1} + 1 \right),$$

then there exists a triplet of generators (p, i, j), *such that* $p(x, y) = 1 - I^{g_s}(x, y)$.

Remark 6.15 *Note that described triplet of generators is same as triplet in Theorem 3.29 in case when* $\varphi(x) = x$.

We have investigated the case, when $p(x, y) = 1 - I^g(x, y)$. A more general formula is $p(x, y) = N^{(-1)}(I^g_N(x, y))$. In this case, the condition for the generator of triplet is

$$N^{(-1)}(I^g_N(y, x)) - N^{(-1)}(I^g_N(x, y)) = y - x.$$

In this chapter we described construction methods of monotone generators for fuzzy preference structures with use of generated fuzzy implications. It is possible that there exists other solutions of the equality (21). We plan to describe this solutions in the future.

7 Modus ponens

In the last chapter of the thesis we study a many-valued case of modus ponens with clause-based rules and we compare the results with estimations of modus ponens via implicative rules. This part is based on results of work [45]. In the second part we propose a discrete case of many-valued modus ponens. Results presented in this section are found in [27].

For implicative rules, the following estimation of modus ponens is in [23] and [25]

$$\frac{(\boldsymbol{B}, b), (\boldsymbol{B} \to \boldsymbol{H}, r)}{\boldsymbol{H}, f_{\to}(b, r)}.$$

We know that the implication $(\boldsymbol{B} \to \boldsymbol{H})$ is true to degree r (at least). Therefore $\boldsymbol{H}$ must be true to some degree h such that $I(b, h) \geq r$. We need to find the least value h with this property in order to guarantee that $TV(\boldsymbol{H}) \geq h$. Let I be the truth function of implication $\to$, then truth function $f_{\to}$ is residual conjunction of implication I (note mnemonic body-head-rule notation of variables)

$$f_{\to}(b, r) = C_I(b, r) = \inf\{h \in [0, 1]; I(b, h) \geq r\}.$$

7.1 Modus ponens for clause based rules

To be consistent with body-head-rule notation of [37], we will use it also here for clausal rules.

Example 7.1 (Hliněná and Biba [27]) *The following are the logical operators of material implication which are corresponding to basic t-conorms: maximum S_M, probabilistic sum S_P, and Łukasiewicz t-conorm S_L and standard negation N_s.*

$$I_{S_M}(b, h) = \max(1 - b, h), \quad I_{S_P}(b, h) = 1 - b + b \cdot h,$$

$$I_{S_L}(b, h) = \min(1 - b + h, 1).$$

Note, that $I_S(b, h) = S(N(b), h)$, where N is a fuzzy negation and S is a t-conorm. For an arbitrary disjunction D and the standard negation N_s we get $I_D(b, h) = D(1 - b, h)$.

First idea to mimic implicative rules, is to take residua to material implications. The residual conjunctions of previous implications are:

$$C_{I_{S_M}}(b, r) = \begin{cases} 0 & \text{if } b + r \leq 1, \\ r & \text{otherwise,} \end{cases} \qquad C_{I_{S_P}}(b, r) = \begin{cases} 0 & \text{if } b + r \leq 1, \\ \frac{b+r-1}{b} & \text{otherwise,} \end{cases}$$

$$C_{I_{S_L}}(b, r) = \max(0, b + r - 1).$$

Note that all residua to material implication in previous example are zero in the triangle $b + r \leq 1$, where $b, r \in [0, 1]$.

Another possibility is to calculate the lower bound on the truth value of $\boldsymbol{H}$ using aggregation deficit.

Example 7.2 (Hliněná and Biba [27]) *To have a sound clause based modus ponens, we make following observation. Let* $D : [0,1]^2 \to [0,1]$ *be a commutative disjunction. If for all* $b, r \in [0,1]$

$$(\boldsymbol{B}, b) \text{ and } (\neg \boldsymbol{B} \vee_D \boldsymbol{H}, r) \text{ should imply } (\boldsymbol{H}, g_D(b,r)),$$

then using Theorem 2.2

$$r \le D(1-b, h) \implies r \le D(h, 1-b) \implies h \ge R_D(1-b, r).$$

Hence the best possible estimate for h *is*

$$g_D(b,r) = \inf_{b' \ge b} R_D(1-b', r).$$

Since the aggregation deficit R_D *is decreasing in the first argument, hence* $\inf\limits_{b' \ge b} R_D(1-b', r) = R_D(1-b, r)$*, it means that*

$$g_D(b,r) = R_D(1-b, r).$$

Remark 7.3 *Note that the truth value of* $\boldsymbol{H}$ *depends on the truth functions of disjunction and negation. Therefore, on a very formal level, one would write* $g_{\vee_D \neg_N}$*. To make the notation shorter we omit the symbols of disjunction and negation, since it they do not bear any additional information. Because we deal only with the standard negation* N_s *in this section, symbol* N *is omitted as well. We thus use* g_D*.*

For commutative disjunctions we get:

Theorem 7.4 (Hliněná and Biba [27])

1. *Let* $D_1 \le D_2$*, then* $g_{D_1} \ge g_{D_2}$*.*
2. *Let* D *be a t-semiconorm, then* $g_D \le g_{S_M}$*.*
3. *Function* g_D *is increasing in both arguments.*
4. *Let* $D : [0,1]^2 \to [0,1]$ *be a commutative* $t-$*semiconorm. For function* g_D *we get* $g_D(1,1) = 1, g_D(0,x) = g_D(x,0) = 0$*. It means, the function* g_D *is the fuzzy conjunction.*

Proof. The parts 1.-2. directly follow from Remark 2.59. The part 3. is implied from Remark 2.59 and from equality $g_D(b,r) = R_D(1-b,r)$. In the last part we deal with a commutative $t-$semiconorm D. For $t-$semiconorm we have $D(x,0) = x$, therefore we get:

$$g_D(1,1) = R_D(0,1) = \inf\{z \in [0,1]; D(z,0) \ge 1\} = 1.$$

Since $D(x,1) = 1$ we have:

$$g_D(0,x) = R_D(1,x) = \inf\{z \in [0,1]; D(z,1) \ge x\} = 0.$$

Since $D(x,y) \ge 0$ we get:

$$g_D(x,0) = R_D(1-x,0) = \inf\{z \in [0,1]; D(z,1-x) \ge 0\} = 0.$$

Since function g_D is increasing in both arguments (part 3.), g_D is a fuzzy conjunction.

Remark 7.5 (Hliněná and Biba [27]) *If a commutative t−semiconorm D possesses the properties*

$$D(x, y) = 1 \text{ for all } x, y \in [0, 1], \text{ such that } x + y = 1$$

$$D(x, y) < 1 \text{ for all } x, y \in [0, 1], \text{ such that } x + y < 1$$

then g_D is a t−seminorm. These properties guarantee that the boundary condition $g_D(x, 1) = x$ is satisfied for all $x \in [0, 1]$. The second boundary condition, $g_D(1, x) = x$, is satisfied for arbitrary commutative t−semiconorm D. Note that, for example, t−conorm S_L possesses these properties.

Estimation for clause rules and implicative rules are in some cases identical:

Theorem 7.6 (Hliněná and Biba [27]) *Let $g_D : [0, 1]^2 \to [0, 1]$ be truth function based on R_D, where D is a commutative disjunction and $C_I : [0, 1]^2 \to [0, 1]$ be a truth function based on I, where $I(b, h) = D(h, 1 - b)$. Then*

$$C_I(b, r) = g_D(b, r)$$

for all $b, r \in [0, 1]$.

Proof. Let D be a disjunction and $I(b, h) = D(h, 1-b)$. Equality $R_D(1-b, r) = C_I(b, r)$ follows directly from definitions of R_D and C_I. According to Example 7.2, $g_D(b, r) = R_D(1 - b, r)$, and therefore also $g_D(b, r) = C_I(b, r)$.

7.2 Discrete many valued modus ponens

Assume users will evaluate preference on attributes X and Y with fuzzy or linguistic values x and y. In this part we will estimate modus ponens via discrete connectives. It is known ([49] and `http://en.wikipedia.org/wiki/Likert _scale`), that people are not able to sort according to quality to more than 7 ± 2 categories. In accordance with this fact we use coefficients k, l as follows:

$$k \in \{5, 6, 7, 8, 9\} \text{ and } l \in \{5, 6, 7, 8, 9\}.$$

And for m (the number of roundings) we take $m = k * l$, which provides us with good ordering of results. The meaning of these coefficients will be come obvious in the next definition of a discrete fuzzy conjunction:

Definition 7.7 *Let $C : [0, 1]^2 \to [0, 1]$ be a fuzzy conjunction, $k \in \{5, 6, 7, 8, 9\}, l \in \{5, 6, 7, 8, 9\}$ and $m = k * l$. Mapping $C^m_{k,l} : [0, 1]^2 \to [0, 1]$ which is defined as follows*

$$C^m_{k,l}(x, y) = \frac{\left\lceil m \cdot C\left(\frac{\lceil k\cdot x\rceil}{k}, \frac{\lceil l\cdot y\rceil}{l}\right)\right\rceil}{m}$$

is called a discrete fuzzy conjunction.

Obviously this mapping is a fuzzy conjunction. However it is not a t-seminorm. Commutative or associative conjunction C may lead to $C^m_{k,l}$ without these properties. Note, that if a conjunction C is commutative, then the discrete conjunction $C^m_{k,k}$ is commutative, too. Dual mapping to the discrete conjunction is given by a similar equality.

Theorem 7.8 (Hliněná and Biba [27]) *Let $C : [0,1]^2 \to [0,1]$ and $D : [0,1]^2 \to [0,1]$ be the dual conjunction and disjunction which are continuous, $k \in \{5,6,7,8,9\}, l \in \{5,6,7,8,9\}$ and $m = k * l$. Then the dual discrete fuzzy disjunction to $C^m_{k,l}$ is the mapping $D^m_{k,l} : [0,1]^2 \to [0,1]$ such that*

$$D^m_{k,l}(x,y) = \frac{\left\lfloor m \cdot D\left(\frac{\lfloor k\cdot x\rfloor}{k}, \frac{\lfloor l\cdot y\rfloor}{l}\right)\right\rfloor}{m}. \tag{22}$$

Proof. The dual disjunctions to conjunctions C and $C^m_{k,l}$ are given by $D(x,y) = 1 - C(1-x, 1-y)$ and $D^m_{k,l}(x,y) = 1 - C^m_{k,l}(1-x,1-y)$, respectively. For any $k \in \mathbb{N}$ and $t \in [0,1]$ it holds that $\lceil k - k\cdot t\rceil = k - \lfloor k \cdot t\rfloor$ and $k - \lceil k\cdot t\rceil = \lfloor k - k\cdot t\rfloor$. Using these two facts, the rest of the proof is straightforward:

$$D^m_{k,l}(x,y) = 1 - \frac{\left\lceil m \cdot C\left(\frac{\lceil k-k\cdot x\rceil}{k}, \frac{\lceil l-l\cdot y\rceil}{l}\right)\right\rceil}{m} =$$

$$= \frac{\left\lfloor m - m\cdot C\left(1 - \frac{\lfloor k\cdot x\rfloor}{k}, 1 - \frac{\lfloor l\cdot y\rfloor}{l}\right)\right\rfloor}{m} = \frac{\left\lfloor m \cdot D\left(\frac{\lfloor k\cdot x\rfloor}{k}, \frac{\lfloor l\cdot y\rfloor}{l}\right)\right\rfloor}{m}.$$

For an illustration we introduce the following example:

Example 7.9 (Hliněná and Biba [27]) *Let C be a product t-norm T_P. We, for example, calculate the value $C^{25}_{5,5}(\frac{1}{3}, \frac{2}{3})$:*

$$C^{25}_{5,5}(\frac{1}{3},\frac{2}{3}) = \frac{\left\lceil 25 \cdot C\left(\frac{\lceil 5\cdot\frac{1}{3}\rceil}{5}, \frac{\lceil 5\cdot\frac{2}{3}\rceil}{5}\right)\right\rceil}{25} = \frac{\lceil 25\cdot C\left(\frac{2}{5},\frac{4}{5}\right)\rceil}{25} = \frac{\lceil 25\cdot\frac{8}{25}\rceil}{25} = \frac{8}{25}.$$

Conjunction $C^{25}_{5,5}(x,y)$ and its dual disjunction $D^{25}_{5,5}$ are given in Tables 1 and 2.

$y \setminus x$	0	$]0,\frac{1}{5}]$	$]\frac{1}{5},\frac{2}{5}]$	$]\frac{2}{5},\frac{3}{5}]$	$]\frac{3}{5},\frac{4}{5}]$	$]\frac{4}{5},1]$
0	0	0	0	0	0	0
$]0,\frac{1}{5}]$	0	$\frac{1}{25}$	$\frac{2}{25}$	$\frac{3}{25}$	$\frac{4}{25}$	$\frac{1}{5}$
$]\frac{1}{5},\frac{2}{5}]$	0	$\frac{2}{25}$	$\frac{4}{25}$	$\frac{6}{25}$	$\frac{8}{25}$	$\frac{2}{5}$
$]\frac{2}{5},\frac{3}{5}]$	0	$\frac{3}{25}$	$\frac{6}{25}$	$\frac{9}{25}$	$\frac{12}{25}$	$\frac{3}{5}$
$]\frac{3}{5},\frac{4}{5}]$	0	$\frac{4}{25}$	$\frac{8}{25}$	$\frac{12}{25}$	$\frac{16}{25}$	$\frac{4}{5}$
$]\frac{4}{5},1]$	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1

Table 1: conjunction $(T_P)^{25}_{5,5}$

We can see that the conjunction $C^{25}_{5,5}$ in the example is left-continuous. Since the functions $\lceil x\rceil$ and $\lfloor x\rfloor$ are left- and right-continuous, respectively, we are able to generalise this fact:

Theorem 7.10 (Hliněná and Biba [27]) *Let $C : [0,1]^2 \to [0,1]$ be a continuous fuzzy conjunction. Then the discrete fuzzy conjunction $C^m_{k,l}$ is left-continuous and the discrete fuzzy disjunction $D^m_{k,l}$ is right-continuous.*

$y \setminus x$	$[0, \frac{1}{5}[$	$[\frac{1}{5}, \frac{2}{5}[$	$[\frac{2}{5}, \frac{3}{5}[$	$[\frac{3}{5}, \frac{4}{5}[$	$[\frac{4}{5}, 1[$	1
$[0, \frac{1}{5}[$	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1
$[\frac{1}{5}, \frac{2}{5}[$	$\frac{1}{5}$	$\frac{9}{25}$	$\frac{13}{25}$	$\frac{17}{25}$	$\frac{21}{25}$	1
$[\frac{2}{5}, \frac{3}{5}[$	$\frac{2}{5}$	$\frac{13}{25}$	$\frac{16}{25}$	$\frac{19}{25}$	$\frac{22}{25}$	1
$[\frac{3}{5}, \frac{4}{5}[$	$\frac{3}{5}$	$\frac{17}{25}$	$\frac{19}{25}$	$\frac{21}{25}$	$\frac{23}{25}$	1
$[\frac{4}{5}, 1[$	$\frac{4}{5}$	$\frac{21}{25}$	$\frac{22}{25}$	$\frac{23}{25}$	$\frac{24}{25}$	1
1	1	1	1	1	1	1

Table 2: disjunction $(S_P)^{25}_{5,5}$

Remark 7.11 (Hliněná and Biba [27]) *Let $C : [0,1]^2 \to [0,1]$ be a fuzzy conjunction and $D : [0,1]^2 \to [0,1]$ be a fuzzy disjunction. Then the following inequalities hold:*

- $C \leq C^m_{k,l}$,
- $D \geq D^m_{k,l}$.

The first fact follows from inequality $x \leq \frac{\lceil k \cdot x \rceil}{k}$ and monotonicity of a conjunction. The second one follows from inequality $x \geq \frac{\lfloor k \cdot x \rfloor}{k}$ and monotonicity of a disjunction.

Formula similar to equation (22) holds also for the aggregation deficit R_D and its discrete counterpart. The discrete aggregation deficit is denoted by R^*_D. By definition, the aggregation deficit R^*_D is given by the formula

$$R^*_D(x,y) = \inf\{z \in [0,1]; \left\lfloor m \cdot D\left(\frac{\lfloor k \cdot z \rfloor}{k}, \frac{\lfloor l \cdot x \rfloor}{l}\right)\right\rfloor \geq m \cdot y\}.$$

Theorem 7.12 (Hliněná and Biba [27]) *Let $D : [0,1]^2 \to [0,1]$ be a continuous fuzzy disjunction and $D^m_{k,l}$ be a discrete fuzzy disjunction. Let $R_D : [0,1]^2 \to [0,1]$ and $R^*_D : [0,1]^2 \to [0,1]$ be the aggregation deficits given by D and $D^m_{k,l}$ respectively. Then the following equality holds:*

$$R^*_D(x,y) = \frac{\left\lceil k \cdot R_D\left(\frac{\lfloor l \cdot x \rfloor}{l}, \frac{\lceil m \cdot y \rceil}{m}\right)\right\rceil}{k}.$$

Proof. From definition we have that

$$R_D\left(\frac{\lfloor l \cdot x \rfloor}{l}, \frac{\lceil m \cdot y \rceil}{m}\right) = \inf\left\{z \in [0,1]; D\left(z, \frac{\lfloor l \cdot x \rfloor}{l}\right) \geq \frac{\lceil m \cdot y \rceil}{m}\right\},$$

$$R^*_D(x,y) = \inf\left\{z \in [0,1]; D\left(\frac{\lfloor k \cdot z \rfloor}{k}, \frac{\lfloor l \cdot x \rfloor}{l}\right) \geq \frac{\lceil m \cdot y \rceil}{m}\right\}.$$

(The second formula is equivalent to the definition of R^*_D.) Take $n \in \mathbb{N}$, such that $D\left(\frac{n}{k}, \frac{\lfloor l \cdot x \rfloor}{l}\right) \geq \frac{\lceil m \cdot y \rceil}{m}$ and n is the smallest number with this property. Such n always exists and $0 \leq n \leq k$. Now we need to distinguish between two cases: $n = 0$ and $n > 0$.

- If $n > 0$ then $D\left(\frac{n}{k}, \frac{\lfloor l\cdot x\rfloor}{l}\right) \geq \frac{\lceil m\cdot y\rceil}{m} > D\left(\frac{n-1}{k}, \frac{\lfloor l\cdot x\rfloor}{l}\right)$, and therefore we have that $R^*_D(x,y) = \frac{n}{k}$. It is obvious that $R_D(x,y) \leq R^*_D(x,y) = \frac{n}{k}$.

  Since D is continuous, $\frac{n-1}{k} < \inf\{z \in [0,1]; D\left(z, \frac{\lfloor l\cdot x\rfloor}{l}\right) \geq \frac{\lceil m\cdot y\rceil}{m}\}$. (In the other case we get that $D\left(\frac{n-1}{k}, \frac{\lfloor l\cdot x\rfloor}{l}\right) \geq \frac{\lceil m\cdot y\rceil}{m}$. That is not possible since $\frac{n}{k}$ is the smallest k-fraction with mentioned property.)

  Summarizing previous two facts we have $\frac{n-1}{k} < R_D\left(\frac{\lfloor l\cdot x\rfloor}{l}, \frac{\lceil m\cdot y\rceil}{m}\right) \leq \frac{n}{k}$. Therefore, $\frac{\lceil k\cdot R_D\left(\frac{\lfloor l\cdot x\rfloor}{l}, \frac{\lceil m\cdot y\rceil}{m}\right)\rceil}{k} = \frac{n}{k} = R^*_D(x,y)$.

- Let $n = 0$. This implies $D\left(0, \frac{\lfloor l\cdot x\rfloor}{l}\right) \geq \frac{\lceil m\cdot y\rceil}{m}$, which means that $R^*_D(x,y) = 0$. It also holds that $D(0,x) \geq y$, because $D(0,x) \geq D\left(0, \frac{\lfloor l\cdot x\rfloor}{l}\right) \geq \frac{\lceil m\cdot y\rceil}{m} \geq y$. It means that $R_D(x,y) = 0$, and therefore in case $n = 0$ we again get the equality $R^*_D(x,y) = \frac{\lceil k\cdot R_D\left(\frac{\lfloor l\cdot x\rfloor}{l}, \frac{\lceil m\cdot y\rceil}{m}\right)\rceil}{k}$.

Corollary 7.13 (Hliněná and Biba [27]) *Let $g_D : [0,1]^2 \to [0,1]$ and $g^*_D : [0,1]^2 \to [0,1]$ be the estimaties of modus ponens with commutative disjunctions D and $D^m_{k,k}$ respectively. Then the following equality holds:*

$$g^*_D(b,r) = \frac{\left\lceil k \cdot g_D\left(\frac{\lceil k\cdot b\rceil}{k}, \frac{\lceil m\cdot r\rceil}{m}\right)\right\rceil}{k}.$$

Since $f_\to(b,r) = C_{I_D}(b,r)$, it may seem that one can obtain discrete operator $f^*_\to$ simply from conjunction C_{I_D} using Definition 7.7 However, this is not a correct procedure - residual conjunction to I^*_D is different. The following fact is proved in a similar manner as Theorem 7.12

Theorem 7.14 (Hliněná and Biba [27]) *Let $D : [0,1]^2 \to [0,1]$ be a continuous disjunction. Let $I^*_D : [0,1]^2 \to [0,1]$ be a material implication given by discrete disjunction $D^m_{k,l}$. Then the discrete residual conjunction to I^*_D is given by*

$$C_{I^*_D}(b,r) = \frac{\left\lceil k \cdot C_{I_D}\left(\frac{\lceil l\cdot b\rceil}{l}, \frac{\lceil m\cdot r\rceil}{m}\right)\right\rceil}{k}.$$

The last example shows estimation of modus ponens with the disjunction $(S_P)^{25}_{5,5}$ derived from probabilistic sum.

Example 7.15 (Hliněná and Biba [27]) *Let $C_{I^*_D}$ be a residual conjunction obtained from the disjunction $(S_P)^{25}_{5,5}$. $C_{I^*_D}$ is given by Table 3.*

Observe that $C_{I^*_D}(b,1) = 0$ if $b = 0$ and $C_{I^*_D}(b,1) = 1$ otherwise. This fact holds for any conjunction $C_{I^*_D}$ obtained using disjunction D without non-trivial zero divisors. It is generalized in the following theorem:

Theorem 7.16 (Hliněná and Biba [27]) *Let $D^m_{k,l}$ be a discrete disjunction without non-trivial zero divisors, then $C_{I^*_D}(0,1) = 0$ and $C_{I^*_D}(b,1) = 1$ for all $b > 0$.*

Proof. Let $D^m_{k,l}$ be a disjunction without non-trivial zero divisors, i.e.

$$x < 1, y < 1 \Leftrightarrow D(x, y) < 1.$$

Since $I^*_D(x, y) = D^m_{k,l}(y, 1 - x)$, we have $I^*_D(x, y) = 1 \Leftrightarrow x = 0 \vee y = 1$. From definition of C_I we have

$$C_{I^*_D}(b, 1) = \inf\{h \in [0, 1]; I^*_D(b, h) = 1\}.$$

The set at the right side is either $[0, 1]$ (if $b = 0$), or $\{1\}$. Infima of these sets are 0 and 1 respectively, therefore the proof is complete.

$r \setminus b$	0	$]0, \frac{1}{5}]$	$]\frac{1}{5}, \frac{2}{5}]$	$]\frac{2}{5}, \frac{3}{5}]$	$]\frac{3}{5}, \frac{4}{5}]$	$]\frac{4}{5}, 1]$
0	0	0	0	0	0	0
$]0, \frac{1}{25}]$	0	0	0	0	0	$\frac{1}{5}$
$]\frac{1}{25}, \frac{2}{25}]$	0	0	0	0	0	$\frac{1}{5}$
$]\frac{2}{25}, \frac{3}{25}]$	0	0	0	0	0	$\frac{1}{5}$
$]\frac{3}{25}, \frac{4}{25}]$	0	0	0	0	0	$\frac{1}{5}$
$]\frac{4}{25}, \frac{5}{25}]$	0	0	0	0	0	$\frac{1}{5}$
$]\frac{5}{25}, \frac{6}{25}]$	0	0	0	0	$\frac{1}{5}$	$\frac{2}{5}$
$]\frac{6}{25}, \frac{7}{25}]$	0	0	0	0	$\frac{1}{5}$	$\frac{2}{5}$
$]\frac{7}{25}, \frac{8}{25}]$	0	0	0	0	$\frac{1}{5}$	$\frac{2}{5}$
$]\frac{8}{25}, \frac{9}{25}]$	0	0	0	0	$\frac{1}{5}$	$\frac{2}{5}$
$]\frac{9}{25}, \frac{10}{25}]$	0	0	0	0	$\frac{2}{5}$	$\frac{2}{5}$
$]\frac{10}{25}, \frac{11}{25}]$	0	0	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$
$]\frac{11}{25}, \frac{12}{25}]$	0	0	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$
$]\frac{12}{25}, \frac{13}{25}]$	0	0	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$
$]\frac{13}{25}, \frac{14}{25}]$	0	0	0	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{3}{5}$
$]\frac{14}{25}, \frac{15}{25}]$	0	0	0	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{3}{5}$
$]\frac{15}{25}, \frac{16}{25}]$	0	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$
$]\frac{16}{25}, \frac{17}{25}]$	0	0	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{4}{5}$
$]\frac{17}{25}, \frac{18}{25}]$	0	0	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{4}{5}$
$]\frac{18}{25}, \frac{19}{25}]$	0	0	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{4}{5}$
$]\frac{19}{25}, \frac{20}{25}]$	0	0	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{4}{5}$
$]\frac{20}{25}, \frac{21}{25}]$	0	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	1
$]\frac{21}{25}, \frac{22}{25}]$	0	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	1	1
$]\frac{22}{25}, \frac{23}{25}]$	0	$\frac{3}{5}$	$\frac{4}{5}$	1	1	1
$]\frac{23}{25}, \frac{24}{25}]$	0	$\frac{4}{5}$	1	1	1	1
$]\frac{24}{25}, 1]$	0	1	1	1	1	1

Table 3: Estimation of modus ponens with material implication I_D^*

8 Conclusions

The thesis shows new results concerning various classes of fuzzy implications given by one-variable functions. We described the properties of these classes of generated implications and also the intersections with already known classes of $(S, N)-$ and $R-$ implications. We also described the possibility of defining fuzzy preference structures using fuzzy implications. Some possibilities of generating a fuzzy implication are given without closer study, properties of these classes is not fully known to-day.

The thesis contains also some new results about many-valued modus ponens rule. The most interesting of results are those about discrete case of many-valued modus ponens. Many-valued modus ponens is used in fuzzy inference, fuzzy regulation etc. The discrete modus ponens can be used especially in cases when the input is not physical parameter (which can be measured with good precision), but instead, input is a qualitative characteristic (see "Likert scale"). The possible application of this discrete many-valued modus ponens is in the decision-making process. In my future work I would like to continue research in discrete modus ponens and, particularly, its possible applications in multicriteria decision making.

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