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Asymptotické chování diferenciálních modelů

Katedra matematické analýzy a aplikací matematiky
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Asymptotic behaviour of differential models

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Abstrakt: V moderní vědě a technice vede matematický popis složitých fyzikálních procesů často k diferenciálních modelům. Proto je téma diferenciálních rovnic v zájmu vědců a odborníků již mnoho let. Tato práce je věnována existenci a asymptotickému chování řešení obyčejných diferenciálních rovnic se singularitou v nezávisle proměnné. Uvádíme dva různé singulární problémy, proto je práce rozdělena do dvou částí. První část se zaměřuje na studium nelineárních diferenciálních rovnic druhého řádu na neomezené oblasti $[0,\infty)$, které mohou mít časovou singularitu v počátku. Množina všech možných řešení je popsána v závislosti na jejich asymptotického chování v nekonečnu. Jsou odvozeny existenční výsledky a vlastnosti tlumených a únikových řešení. Prostřednictvím těchto výsledků je dokázána existence rostoucího řešení s limitou $u(\infty) = L$, homoklinického řešení, které hraje důležitou roli v aplikacích. Dále je zkoumána existence Kneserových řešení a jsou odvozeny asymptotické vlastnosti těchto řešení v rámci teorie regulárně měnících se funkcí. Analytické výsledky týkající se Kneserových řešení jsou znázorněny pomocí numerických simulací. Druhá část práce zkoumá analytické a numerické vlastnosti systémů lineárních obyčejných diferenciálních rovnic s neintegrovatelnou nehomogenní složkou a časovou singularitou prvního druhu na kompaktním intervalu. Asymptotické chování řešení je analyzováno v singulárním bodě $t = 0$. Důraz je kladen na stanovení struktury obecných lineárních dvoubodových okrajových podmínek zaručujících existenci a jednoznačnost řešení, která jsou spojitá na uzavřeném intervalu zahrnující singulární bod. Dále je studována kolokační metoda, která aproximuje analytické řešení spojitou po částech polynomiální funkcí, a je odvozen její řád konvergence. Teoretické výsledky jsou doloženy numerickými simulacemi.

Klíčová slova: obyčejné diferenciální rovnice, asymptotické vlastnosti, časová singularita, Kneserova řešení, neoscilatorická řešení, regulární variace, metoda kolokace, konvergence.

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Abstract: In modern science and technology, the mathematical description of complex physical processes often leads to differential models. Therefore, the topic of differential equations has been of interest for scientists and engineers for many years. This thesis is devoted to the existence and asymptotic behaviour of solutions to ordinary differential equations with a singularity in the independent variable. Two different singular problems are presented, and therefore, the thesis is divided into two parts. The first part focusses on the study of second order nonlinear differential equations on unbounded domain $[0,\infty)$ which may have a time singularity at the origin. A set of all solutions is described according to their asymptotic behaviour at infinity. Existence results and properties of damped and escape solutions are derived. By means of these results, the existence of an increasing solution with $u(\infty) = L$, a homoclinic solution, playing an important role in applications is proved. Furthermore, the existence of Kneser solutions is investigated and asymptotic properties of such solutions and their first derivatives are derived in the framework of regularly varying functions. The analytical findings concerning Kneser solutions are illustrated by numerical simulations.The second part of the thesis investigates analytical and numerical properties of systems of linear ordinary differential equations with an unsmooth nonintegrable inhomogeneity and a time singularity of the first kind on a compact interval. The asymptotic behaviour of solutions at the singular point $t = 0$ is analysed. The focus is on specifying the structure of general linear two-point boundary conditions guaranteeing the existence and uniqueness of solutions which are continuous on a closed interval including the singular point. Moreover, the collocation method which approximates the analytical solution by a continuous piecewise polynomial function is analysed, and its convergence order is derived.

Key words: ordinary differential equations, asymptotic properties, time singularity, Kneser solutions, nonoscillatory solutions, regular variations, collocation method, convergence.

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Prohlašuji, že jsem disertační práci zpracovala samostatně pod vedením paní prof. RNDr. Ireny Rachůnkové, DrSc. a všechny použité zdroje jsem uvedla v seznamu literatury.

V Olomouci dne ....

Contents

Notation

1 Preface **9**
- 1.1 Recent state summary . . . 9
- 1.2 Thesis objectives . . . 13
- 1.3 Theoretical framework . . . 14
- 1.4 Applied methods . . . 17
- 1.5 Original results . . . 17
- 1.6 Structure of the thesis . . . 24

I Singular nonlinear second order differential equation **27**

2 Introduction **29**
- 2.1 Motivation . . . 29
- 2.2 Statement of the problem . . . 31

3 Solvability of the problem **33**
- 3.1 Preliminary results . . . 33
- 3.2 Existence of a solution . . . 37
- 3.3 Uniqueness of a solution . . . 39

4 Bubble-type solution **43**
- 4.1 Damped solutions . . . 43
- 4.2 Escape solutions . . . 46
- 4.3 Homoclinic solution . . . 52

5 Kneser solutions **57**
- 5.1 Singular problem . . . 58
- 5.2 Regular problem . . . 67

6 Asymptotic properties of Kneser solutions **75**
- 6.1 Regularly varying functions . . . 75
- 6.2 Asymptotic behaviour . . . 78
- 6.3 Numerical simulations . . . 82

II Linear systems with a singularity of the first kind **89**

7 Introduction **91**
- 7.1 Motivation . . . 91
- 7.2 Statement of the problem . . . 92

8 The three case studies **95**
8.1 Preliminary results . . . 95
8.2 Eigenvalues of $M(0)$ with negative real parts . . . 97
8.3 Eigenvalues of $M(0)$ with positive real parts . . . 103
8.4 Zero eigenvalues of $M(0)$ . . . 111

9 General problems **119**
9.1 General IVPs, TVPs . . . 119
9.2 General BVPs . . . 121

10 Numerical analysis **127**
10.1 Collocation method . . . 127
10.2 Convergence results . . . 128
10.3 Numerical experiments . . . 140

Conclusion **149**

List of publications **151**

References **153**

Curriculum Vitae **161**

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Notation

Throughout the thesis, the following standard notation is used. Other nonstandard notation is introduced in the text as it is needed.

$\mathbb{N}$	set of all natural numbers;
$\mathbb{R}^n$	n-dimensional vector space of real-valued vectors;
$\mathbb{C}^n$	n-dimensional vector space of complex-valued vectors;
$\mathbb{R}^{m\times n}$	$m\times n$-dimensional space of real-valued matrices;
$\mathbb{C}^{m\times n}$	$m\times n$-dimensional space of complex-valued matrices;
$\lvert x\rvert$	$\lvert x\rvert := \max\{\lvert x_i\rvert : 1\le i\le n\}$; maximum norm for a vector $x\in\mathbb{C}^n$;
$\lvert A\rvert$	$\lvert A\rvert := \max_{1\le i\le m}\sum_{j=1}^{n}\lvert a_{ij}\rvert$; induced operator norm for a matrix $A\in\mathbb{C}^{m\times n}$;
$C[a,b]$	Banach space of real vector-valued functions continuous on $[a,b]$ equipped with the norm $\lVert y\rVert_{C[a,b]} := \max\{\lvert y(t)\rvert : t\in[a,b]\}$;
$C^p[a,b]$	Banach space of real vector-valued functions p-times continuously differentiable on $[a,b]$ equipped with the norm $\lVert y\rVert_{C^p[a,b]} := \sum_{k=0}^{p}\lVert y^{(k)}\rVert_{C[a,b]}$;
$C(I)$	set of real vector-valued functions continuous on the interval I;
$C_{m\times n}[a,b]$	Banach space of real-valued matrix functions continuous on $[a,b]$ equipped with the norm $\lVert y\rVert_{C_{m\times n}[a,b]} := \max\{\lvert y(t)\rvert : t\in[a,b]\}$;
$C^p_{m\times n}[a,b]$	Banach space of real-valued matrix functions p-times continuously differentiable on $[a,b]$ equipped with the norm $\lVert y\rVert_{C^p_{m\times n}[a,b]} := \sum_{k=1}^{p}\lVert M^{(k)}\rVert_{\delta}$;
$\mathrm{Lip}_{loc}(I)$	set of locally Lipschitz continuous functions on the interval I;
$O(g(x))$	$f(x)=O(g(x))$ for $x\to 0^+$ if $\exists\, c>0, \delta>0 : \lvert f(x)\rvert\le c\lvert g(x)\rvert$ for $0<x<\delta$;

If it cannot be confusing, we omit the subscripts m and n for simplicity of notation, and write $C[a,b]=C_{m\times n}[a,b]$, $C^p_{m\times n}[a,b]=C^p[a,b]$. We also use the shorthand notation:

$$\lVert y\rVert := \max\{\lvert y(t)\rvert : t\in[0,1]\};\ \text{for } y\in C[0,1];$$
$$\lVert y\rVert_{\delta} := \max\{\lvert y(t)\rvert : t\in[0,\delta]\};\ \text{norm restricted to the interval } [0,\delta],\ \delta>0;$$
$$\lVert M\rVert_{\delta} := \max\{\lvert M(t)\rvert : t\in[0,\delta]\};\ \text{norm for } M\in C_{m\times n}[0,\delta];$$

1 Preface

This thesis is devoted to the existence and asymptotic behaviour of solutions of ordinary differential equations (ODEs), in particular, of problems which may have a time singularity. The thesis consists of two parts.

Part I is concerned with nonlinear second order differential equation

$$\big(p(t)u'(t)\big)' + q(t)f(u(t)) = 0, \quad t \in [0,\infty), \tag{1.1}$$

subject to initial conditions

$$u(0) = u_0, \quad u'(0) = 0. \tag{1.2}$$

Here, $f \in C(\mathbb{R})$, $f(L_0) = f(0) = f(L) = 0$, $L_0 < 0 < L$ and $xf(x) > 0$ for $x \in (L_0,0) \cup (0,L)$. Further, $p,q \in C[0,\infty)$ are positive on $(0,\infty)$ and $p(0) = 0$. Part I deals with the asymptotic behaviour of solutions in a neighbourhood of infinity and collects results from papers [1, 2, 3]. Global existence results on $[0,\infty)$ are obtained and the structure of all possible solutions is described according to their asymptotic properties.

Part II presents results from [4, 5, 6, 7] devoted to boundary value problems (BVPs) of the form

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t}, \quad t \in (0,1], \quad y \in C[0,1], \quad B_0y(0) + B_1y(1) = \beta, \tag{1.3}$$

where y is n-dimensional real function, M is $n \times n$ continuous matrix function, f is a n-dimensional function which is at least continuous on $[0,1]$ and $B_0, B_1 \in \mathbb{R}_{m\times n}$, $\beta \in \mathbb{R}^n$. In Part II, the behaviour of solutions near the singular point is analysed. The stress is laid on the structure of boundary conditions which are necessary and sufficient for the existence of a unique continuous solution on the closed interval $[0,1]$ including the singular point $t = 0$.

1.1 Recent state summary

In numerous applications from physics, chemistry and mechanics, the mathematical description of complex processes leads to differential models. These models take often the form of systems of time-dependent partial differential equations (PDEs). For the investigation of stationary solutions many of these models can be reduced to singular ODEs of the second order or to singular systems of ODEs of the first order, especially when symmetries in geometry of the problem appear. Therefore, papers providing analytical results on their structural properties, stability and convergence of different numerical methods, and results of numerical simulations are available.

In this thesis we study asymptotic behaviour of solutions of singular second order nonlinear differential equations in Part I and asymptotic behaviour of singular linear systems of first order differential equations in Part II. A problem is denoted as *singular* if the right-hand side does not fulfil the Carathéodory conditions. Otherwise, a problem is called *regular*. In particular, the system of first order ODEs

$$x'(t) = f(t,x),\ t \in I \subset \mathbb{R}, \tag{1.4}$$

where $x \in \mathbb{R}^n, f : I \times \mathbb{R}^n \to \mathbb{R}^n$, is understood to be *singular* if function f does not fulfil the Carathéodory conditions. For differential equations of the second order we use the terminology corresponding to the equivalent first order system. The solvability of such problems is not covered by the Carathéodory theory and requires a different approach. Two types of singularities are distinguished, time and space singularities. System (1.4) has a time singularity if $f(\cdot,x) : I \to \mathbb{R}^n$ is not integrable for some $x \in \mathbb{R}^n$. Both problems studied in this thesis are singular with a time singularity at the origin, which means that

$$\int_0^{\varepsilon} |f(t,x)|\,\mathrm{d}t = \infty$$

for some $x \in \mathbb{R}^n$ and each sufficiently small $\varepsilon > 0$.

In **Part I**, we study the second order differential equation (1.1):

$$(p(t)u'(t)) + q(t)f(u(t)) = 0,\ t \geq 0.$$

This equation can be assumed as a special case of (1.4). For $v = pu'$, we obtain

$$u'(t) = \frac{1}{p(t)}v(t), \quad v'(t) = -q(t)f(u(t)), \quad t \geq 0.$$

In general, the case

$$\int_0^1 \frac{\mathrm{d}s}{p(s)} = \infty$$

is also considered. Therefore, the differential equation (1.1) may have a time singularity at $t = 0$.

Regular equation

There exists an extensive literature which is devoted to a qualitative analysis of solutions of the nonlinear equation (1.1), or to its quasilinear generalizations in the regular setting, where $p > 0$ on $[0,\infty)$. A general survey concerning asymptotic properties of solutions of nonautonomous ODEs is provided in the monograph [60]. In last decades, great effort has been devoted to the asymptotic analysis in the case that (1.1) is the Emden-Fowler equation

$$u''(t) + q(t)|u|^{\gamma}\operatorname{sgn} u = 0, \quad \gamma > 0, \gamma \neq 1.$$

We refer to [122] for a history survey. The subject of oscillation and nonoscillation of the Emden-Fowler equation has been extensively studied, for example see [74, 79, 80, 96]. The Emden-Fowler equations of arbitrary order are thoroughly discussed in [60]. Equation (1.1) with a nonconstant p and a more general f is investigated in [21, 73, 82, 123, 124], where the equation is regular and the nonlinearity f in all these papers has behaviour which is characterized by the assumption $xf(x) > 0$ for all $x \neq 0$ and is globally monotone. For further discussion we refer to [32, 33, 45].

A possible generalization of the Emden-Fowler equation has the form with p-Laplacian

$$\big(p(t)\Phi_\alpha(u')\big)' + q(t)\Phi_\gamma(u) = 0,\ \alpha > 0, \gamma > 0,$$

where $\Phi_\alpha(u) := |u|^\alpha \operatorname{sgn} u$. This generalized Emden-Fowler equation is called sub-half-linear, half-linear or super-half-linear if $\alpha > \gamma$, $\alpha = \gamma$ or $\alpha < \gamma$, respectively. Existence results for solutions with certain asymptotic behaviour for sub-half-linear case can be found in [53, 75, 77], and those for the half-linear case in [30, 54, 76], whereas the super-half-linear case is studied in [31, 89].

The theory of regular variations [23] provides a powerful tool for the asymptotic analysis. The systematic study of differential equations by means of regularly varying functions can be found in [88]. Asymptotic results for related equations or systems which are characterized by regularly varying functions are obtained in [39, 52, 78, 90, 91, 112, 113]. Oscillation and nonoscillation criteria for related two-dimensional systems of linear and nonlinear ODEs are established in [86] and in [38, 94], respectively. We also refer to [37, 62], where Kneser solutions of two-dimensional systems of ODEs are studied.

Singular equation

The singular equation

$$(p(t)u'(t))' + p(t)f(u(t)) = 0, \tag{1.5}$$

which is a special case of (1.1) with $p \equiv q$, is studied in [99, 100], [104]–[110]. The dynamical system approach and the lower and upper function method are used in [106, 107] where the existence of an escape solution and the existence of a homoclinic solutions of this problem is reached. All types of possible solutions of (1.5), (1.2) with conditions which guarantee their existence and specify their asymptotic behaviour are described in [104, 105, 108]. In particular, the existence of damped oscillatory solutions with decreasing amplitudes of problem (1.5), (1.2) is described in [110]. Sufficient conditions for convergence of such oscillatory solutions to zero are given in [100]. Problem (1.5), (1.2) can be transformed into a problem of the existence of a positive solutions on the half-line. For $p(t) = t^k, k \in \mathbb{N}$, and for $p(t) = t^k, k \in (1,\infty)$, such problem is solved by variational methods in [22] and [24], respectively.

Moreover, the existence and uniqueness of damped solution to (1.1), (1.2) is studied in [111] provided that the derivative of function p is continuous and positive and satisfies another more restrictive assumptions. In addition, conditions guaranteeing that all damped solutions are oscillatory with decreasing amplitudes are derived.

For other problems with singularities we refer to [11, 12, 95, 101, 102], where in [95] the existence theory for singular two-point BVPs on finite and semi-infinite intervals is introduced. Existence theory for a variety of singular problems based on regularization and sequential technique is presented in [101, 102]. For further development see [11, 12] and references therein.

Part II is devoted to singular systems of first order ordinary differential equations. A popular model class for theoretical investigations is the singular BVP

$$y'(t) = \frac{M(t)}{t^\alpha} y(t) + f(t, y(t)),\ t \in (0,1], \quad b(y(0), y(1)) = 0,$$

where $\alpha \geq 1$, $n \times n$ matrix function M and n-dimensional vector functions f and b are continuous. Two types of time singularities are distinguished depending on the value of α. For $\alpha = 1$ the singularity is denoted as a *singularity of the first kind* and for $\alpha > 1$ the problem is called singular with an *essential singularity* or a *singularity of the second kind*. Part II deals with the linear systems of the form

$$y'(t) = \frac{M(t)}{t} y(t) + \frac{f(t)}{t},\ t \in (0,1], \tag{1.6}$$

where in general $\frac{M(t)}{t}$ and $\frac{f(t)}{t}$ may not be integrable on $[0,1]$ which yields a time singularity of the first kind at $t = 0$.

Analytical properties of problems with a continuous inhomogeneity

$$y'(t) = \frac{M(t)}{t} y(t) + f(t),\ t \in (0,1], \quad B_0 y(0) + B_1 y(1) = \beta, \tag{1.7}$$

are discussed in [47, 121], where the attention is focused on the existence, uniqueness and smoothness of solutions. In particular, the structure of the boundary conditions which are necessary and sufficient for (1.7) to have a unique continuous solution on $[0,1]$ is of special interest. Problems of type (1.7), where f may additionally depend on the space variable y and have a space singularity are studied for example in [13, 87, 116, 117]. Moreover, we refer to papers [9, 17, 18, 20, 50, 61, 97], where the solvability of similar linear singular problems is discussed. Interesting results for linear BVPs with time singularities in weight-spaces can be found in [56, 57, 58]. The equation (1.6) with unsmooth inhomogeneity is investigated in [118] where the existence only of a unique continuous particular solution without any boundary conditions is examined. The nonlinear equation of the form (1.6) is studied in [119].

To compute the numerical solution of (1.7) the polynomial collocation [26] is proposed in [46]. See also [121] for second order systems. The choice of the collocation is motivated by its advantageous convergence properties for (1.7) while in the presence of a singularity other high order methods show order reductions and become inefficient [49]. Consequently, for singular BVPs two open domain MATLAB codes based on collocation have been implemented [19, 65]. The code `sbvp` can be applied to explicit first order ODEs [19], while `bvpsuite` can be used to solve arbitrary mixed order problems in implicit formulation. Its scope also includes the differential algebraic equations [65]. Both codes have been used to numerically simulate singular BVPs important for applications and they have proved to work dependably and efficiently [27, 66]. This is our motivation to propose the polynomial collocation for the approximation of (1.3).

Due to very advantageous properties of the collocation method, this approach has been used in a variety of other openly available programs. We enclose some examples for the existing software packages designed to deal with regular and singular ODEs: the standard MATLAB code `bvp4c` [114] and the related solver `bvp5c` [55], two FORTRAN codes, `BVP_SOLVER` specified in [115], and COLNEW described in [15] and based on one of the best established BVP solvers COLSYS [14]. For most of the basic solvers, error estimation routines and grid adaptation strategies implemented in these codes, analytical justification in context of singular systems is given. Typically, to enhance the efficiency of the code, the order of the basic solver varies depending on the tolerances specified by the user. Collocation has proved to be also a useful tool to treat other problem classes, dynamical system in ODEs [72] and algebraic-differential equations, see [81].

1.2 Thesis objectives

The thesis has several objectives, above all to obtain new contributions to the theory of singular differential equation and to extend results for a more general class of problems.

The first aim of Part I is to investigate the existence of all three types of solutions of problem (1.1), (1.2) from Definitions 2.4 and 2.5. This means to generalize the existence result about damped solutions from [111], and moreover, to prove the existence of escape and homoclinic solutions to problem (1.1), (1.2). The effort to show the existence of a homoclinic solution is emphasised by its important role in applications. The second goal is to describe in more details the asymptotic behaviour of damped nonoscillatory solutions of problem (1.1), (1.2). In particular, investigation of the existence of damped Kneser solutions of equation (1.1) is of special interest. The aim is also to derive asymptotic formulas for such solutions and their first derivatives in the framework of regularly varying functions which has shown to be a powerful tool for the study of asymptotic properties of solutions of differential equations. We confront our results with numerical simulations to provide a helping insight into the analysis.

The aim of Part II is to investigate analytical properties of linear BVPs (1.7) with a variable coefficient matrix and an unsmooth inhomogeneity. We are interested to recover solution y which is at least continuous, $y \in C[0,1]$. In particular, our attention is focused on specifying the structure of general linear two-point boundary condition guaranteeing the existence and uniqueness of solutions which are continuous on the closed interval including the singular point $t = 0$. We also specify conditions for f and M which are sufficient for $y \in C^r[0,1]$, $r \in \mathbb{N}$. The motivation for the above analysis of the variable coefficient case is twofold:

First of all, in order to investigate the nonlinear case one can choose to study the properties of its linearization, see [47]. In this context a related linear BVP with a variable coefficient matrix has to be studied. More precisely, the technique applied in [47] is based on the *assumption that a solution to the nonlinear problem exists*. Next, the nonlinear problem is linearized at the exact solution and the well-posedness of this linearization is studied. However, we are not going to follow this technique and plan in an upcoming paper to *show the existence of the solution of the nonlinear BVP*, instead of assuming its existence.

Secondly, for us, the investigation of the structural properties of (1.3) is necessary and interesting in its own right as a prerequisite for numerical analysis. A knowledge of qualitative properties such as the existence, uniqueness and smoothness of a solution in the linear case is required for the convergence theory of the collocation method. The aim is to prove that the convergence order of the collocation is at least equal to the stage order of the method as predicted by numerical simulations.

1.3 Theoretical framework

In this section, we present the theoretical framework for the subsequent analysis. In Part I we continue the research initiated by Rachůnková, Tomeček et al. in [99, 100], [104]–[110] and [111]. In particular, we extend the results reached for equation (1.5) to the more general equation (1.1). All these papers are establish on the basic assumptions: Function f is Lipschitz continuous on the domain where the solution is searched for, f has prescribed sign conditions and two or three zeros. The coefficient function p is assumed to fulfil:

$$p \in C[0,\infty) \cap C^1(0,\infty),\ p(0) = 0,\ p' > 0 \text{ on } (0,\infty),\ \lim_{t\to\infty} \frac{p'(t)}{p(t)} = 0. \tag{1.8}$$

In [106, 107] the existence of three types of solutions (damped, homoclinic and escape) in the sense of Definitions 2.4, 2.5 is proved under additional conditions posed on p and f. We are mainly motivated by papers [104, 105, 108, 109] where another technique based on differential inequalities is applied. The existence and asymptotic behaviour of all types of possible solutions of problem (1.5), (1.2) are derived without additional conditions. Problem (1.5), (1.2) with f having just two zeros in $f(0) = f(L) = 0$ is studied in [104] and [109]. In particular, in [104] the problem (1.5), (1.2) is investigated provided either f has a sublinear behaviour near $-\infty$ or

$$\int_0^1 \frac{\mathrm{d}s}{p(s)} < \infty.$$

The case when f has a superlinear behaviour near $-\infty$ is studied [109]. In paper [105], the existence of an escape solution and also of a homoclinic solution of (1.5), (1.2) is proved under the assumptions that f has at least two zeros $f(0) = f(L) = 0$ and a sublinear or linear behaviour near $-\infty$. In [108], sufficient conditions for the existence of at least one escape solution of problem (1.5), (1.2) which is fundamental for the existence of a homoclinic solution, are presented. The existence and asymptotic properties of oscillatory solutions of problem (1.5), (1.2) are described in [99, 100, 110].

The investigation of the more general problem (1.1), (1.2) has started in [111], where the existence and uniqueness of damped solution to (1.1), (1.2) is studied under the basic assumptions on function f which are specified at the beginning of this chapter, under assumption (1.8) on function p and under assuming q to be continuous on $[0,\infty)$ and positive on $(0,\infty)$. Moreover, the existence and uniqueness of damped oscillatory solutions with decreasing amplitudes is proved under other additional conditions.

In Part II we continue the investigation initiated by de Hoog and Weiss [46]–[49] and later followed by Weinmüller et al. [65]–[69], [121] where the boundary value problem (1.7) is studied. Here we summarize results obtained in the framework given in [47, 69], which are extended in Part II to the more general problem (1.3) with unsmooth $f(t)/t$. We discuss the case of negative real parts of eigenvalues, the case of positive real parts of eigenvalues and zero eigenvalues of matrix $M(0)$ separately, since this is the key to understand the rest of the theory. Here, $M \in C^1[0,1]$ which yields $M(t) = M(0) + tD(t),\ D \in C[0,1]$.

- Let all eigenvalues of $M(0)$ have negative real parts. Then, $y \in C[0,1]$ if and only if $y(0) = 0$. Therefore, the initial value problem (IVP)

  $$y'(t) = \frac{M(t)}{t} y(t) + f(t), \quad y(0) = 0,$$

  has a unique solution. Moreover, $y \in C^{r+1}[0,1]$ if $f \in C^r[0,1], D \in C^r[0,1], r \geq 0$.

- In the case of only positive real parts of eigenvalues of $M(0)$, we have to specify the boundary conditions at $t = 1$ and solve a terminal value problem (TVP). In particular, the TVP

  $$y'(t) = \frac{M}{t} y(t) + f(t), \quad B_1 y(1) = \beta,$$

where $B_1 \in \mathbb{R}^{n\times n}$ is nonsingular and $\beta \in \mathbb{R}^n$, has a unique solution $y \in C[0,1]$. This solution satisfies $y(0) = 0$. If $f \in C^r[0,1]$, $D \in C^r[0,1]$ and $\sigma_+ > r+1$ then $y \in C^{r+1}[0,1]$.

- Let all eigenvalues of $M(0)$ be zero. Consider the IVP which takes the form

$$y'(t) = \frac{M(t)}{t} y(t) + f(t), \quad M(0)y(0) = 0, \quad B_0 y(0) = \beta,$$

where the $m \times m$ matrix $B_0\tilde{R}$ is nonsingular, $\beta \in \mathbb{R}^m$, and $m = \dim X_0^{(e)}$. The initial condition $M(0)y(0) = 0$ is necessary and sufficient for the solution to by continuous. The remaining m conditions necessary for its uniqueness are specified by $B_0 y(0) = \beta$. For $f \in C^r[0,1], D \in C^r[0,1], r \geq 0$, $y \in C^{r+1}[0,1]$.

In the numerical part, the polynomial collocation is used to approximate the analytical solution by a continuous piecewise polynomial function. For regular problems with appropriately smooth data a convergence order $O(h^k)$ can be guaranteed [16], where k denotes the number of collocation points. Moreover, for special choice of the collocation nodes, for example the Gaussian points, superconvergence order $O(h^{2k})$ holds at the mesh points.

The collocation for singular problems (1.7) with only nonpositive real parts of eigenvalues of $M(0)$ is discussed in [46]. For smooth solutions of such problems, the polynomial collocation converges with at least the stage order $O(h^k)$ uniformly in t when polynomials of degree k are used to define the basic numerical scheme. Similar results hold in context of TVP with nonnegative real parts of eigenvalues. The counterexamples given in [46] show that the superconvergence breaks down in general for singular problems. However, for problems with nonpositive real parts of eigenvalues of $M(0)$ the small superconvergence $O(h^{k+1})$ is proved to hold for certain choice of collocation points. In the case of a general spectrum of $M(0)$ the approach to convergence analysis in [46] is not feasible. In [68] a new representation of the global error of collocation method applied to (1.7) is derived and the global error is shown to be equal at least to the stage order $O(h^k)$ for sufficiently smooth data.

In [119] local existence and uniqueness analysis is provided for a certain class of nonlinear differential equation of the type $tu'(t) = g(t,u(t))$, see also [118]. The boundary conditions are disregarded and the problem is solved numerically by collocation applied to the integral equation resulting after the integration of the ODE system. It turns out that the global error of the collocation scheme is $O(h^k|\ln h|)$ provided that the problem data are appropriately smooth and h is sufficiently small.

1.4 Applied methods

For nonlinear differential equations and linear equations with variable coefficients only seldom the explicit solution formula is known. There are two approaches to solve such equations. The aim of the first one is to obtain qualitative properties of solutions, such as existence, uniqueness, asymptotic behaviour and other characterisations. The second approach, which heavily relies on the first one, is the use of numerical methods to solve the problem. In this thesis both, analytical and numerical methods are applied.

Many problems in differential equations can be reduced to operator equations in Banach spaces where methods of functional analysis are effective, especially those based on the fixed point theory. The Schauder Fixed Point Theorem plays an important role for existence results. We apply this theorem in Part I with the help of the Arzelà-Ascoli Theorem which yields the compactness of operators. By means of the methods of a priori estimates we can apply general existence principles to wide range of problems even when all assumptions are not fulfilled. The Banach Fixed Point Theorem is especially useful as it both guarantees the existence and the uniqueness of a solution. We apply the Banach Fixed Point Theorem for linear system in Part II since it turns out to be very helpful to deal with difficulties caused by the singularity at $t = 0$.

From the great variety of numerical methods we decided to use the polynomial collocation to compute a numerical solution. This method shows advantageous convergence properties compared to other direct high order methods which may be affected by order reductions and become inefficient in the presence of a singularity.

1.5 Original results

In this thesis new contributions to the theory of singular nonlinear ordinary differential equations of second order (1.1) and linear first order boundary value problems with a singularity of the first kind and unsmooth inhomogeneities (1.3) are presented. The thesis is based on the results published in [1]–[7].

Main original results of Part I are the existence of a homoclinic solution of problem (1.1), (1.2) proved in Chapter 4 and published in [2], the existence of a Kneser solution to problem (1.1), (1.2) proved in Chapter 5 and published in [3], and asymptotic formulas for Kneser solutions in Chapter 6, published in [1].

The existence results concerning escape, homoclinic and Kneser solutions to problem (1.1), (1.2) are proved under these basic assumptions on functions f, p and q:

- Function f is continuous on $\mathbb{R}$, has three zeros $f(L_0) = f(0) = f(L_0) = 0$, $L_0 < 0 < L$, satisfies the sign condition $xf(x) > 0$ on $(L_0, L) \setminus \{0\}$, and is Lipschitz continuous on $[L_0, L]$. Moreover, we assume that there exists $\bar{B} \in (L_0, 0)$ such that $F(\bar{B}) = F(L)$, where $F(x) = \int_0^x f(s)\, \mathrm{d}s$.

- The coefficient function p is assumed to be continuous on $[0,\infty)$, positive on $(0,\infty)$ and $p(0)=0$.
- The coefficient function q is considered to be continuous on $[0,\infty)$ and positive on $(0,\infty)$.
- The coefficient functions p and q are connected with the asymptotic relation

$$\lim_{t\to 0^+} \frac{1}{p(t)} \int_0^t q(s)\,\mathrm{d}s = 0.$$

We would like to stress that $p(0)=0$ and the case $\int_0^1 \frac{\mathrm{d}t}{p(t)} = \infty$ is also considered here. Hence, the differential operator in equation (1.1) may have a singularity at $t=0$. This is a fundamental difference from the papers investigating problem (1.1), (1.2) in regular settings. Moreover, we would like to point out that for our results, the nonlinearity f does not have globally monotone behaviour and that the condition $xf(x)>0$ does not need be fulfilled for all $x\neq 0$.

In Theorem 4.15, the existence result for homoclinic solutions are successfully generalized to problem (1.1), (1.2) where $p\neq q$. The existence of such solution has great importance in applications and is derived by means of properties of the sets of initial values of damped and escape solutions which are nonempty, see Theorem 4.1 for damped solutions and Theorem 4.14 for escape solutions, and open, see the proof of Theorem 4.13. The main result - the existence of a homoclinic solution of problem (1.1), (1.2) is proved under these additional conditions:

$$(pq)'>0 \text{ on } (0,\infty),\ \ \lim_{t\to\infty} \frac{(p(t)q(t))'}{q^2(t)} = 0,\ \liminf_{t\to\infty} q(t)>0,\ \lim_{t\to\infty} \frac{p}{q}>0.$$

Then if either

$$\int_1^\infty \frac{\mathrm{d}s}{p(s)}<\infty,\ \int_1^\infty \left(\int_s^\infty \frac{\mathrm{d}\tau}{p(\tau)}\right)^2 q(s)\,\mathrm{d}s=\infty,\quad \liminf_{t\to 0^+} \frac{f(x)}{x}>0,\ \liminf_{t\to 0^-} \frac{f(x)}{x}>0,$$

or

$$\int_1^\infty \frac{\mathrm{d}s}{p(s)}=\infty,\quad \liminf_{t\to\infty} p(t)>0,$$

the problem (1.1), (1.2) has at least one homoclinic solution.

The aim of our future investigation is to derived the existence of a homoclinic solution also in the case when damped nonoscillatory solutions of (1.1), (1.2) may appear. This case is excluded in Theorem 4.15. Moreover, we intend to generalize problem (1.1), (1.2) to the problem with ϕ-Laplacian

$$\big(p(t)\phi(u'(t))\big)' + q(t)f(u(t)) = 0,\quad u(0)=u_0,\ u'(0)=0,$$

where ϕ is an increasing homeomorphism with $\phi(\mathbb{R}) = \mathbb{R}$.

Furthermore, the existence of Kneser solutions is investigated and asymptotic properties of such solutions and their first derivatives are derived in the framework of regularly varying functions. In particular, the new existence results about Kneser solutions of singular problem (1.5), (1.2) for $p \equiv q$ proved in Theorems 5.5 and 5.6 are generalizations of those published in [3]. For the existence of Kneser solutions the additional conditions need to be satisfied:

$$p \equiv q,\ p \in C^1(0,\infty),\ p^2 \text{ is nondecreasing on } (0,\infty),\ \frac{p'(t)\int_0^t p(s)\,\mathrm{d}s}{p^2(t)} \geq c > \frac{1}{2} \text{ for } t > 0.$$

Then there exists a Kneser solution of problem (1.1), (1.2), if there exists either $A_0 \in (0,L)$ or $B_0 \in (L_0,0)$ such that

$$\frac{f(x)}{F(x)} \geq \frac{2}{2c-1}$$

holds for $x \in (0,A_0]$ and $u_0 \in (0,A_0]$ or $x \in [B_0,0)$ and $u_0 \in [B_0,0)$. By our knowledge, for $p \neq q$, the existence of Kneser solutions of singular problem (1.1), (1.2) remains to be an open question.

In general case $p \neq q$, a regular equation on $[a,\infty), a > 0$, is studied in Section 5.2. More precisely, two classes of initial value problems are discussed,

$$\begin{gathered}\big(p(t)u'(t)\big)' + q(t)f((u(t)) = 0,\ \ t \in [a,\infty),\\ u(a) = u_0 \in (0,L),\ 0 \leq u(t) \leq L \text{ for } t \in [a,\infty),\end{gathered} \tag{1.9}$$

and

$$\begin{gathered}\big(p(t)u'(t)\big)' + q(t)f((u(t)) = 0,\ \ t \in [a,\infty),\\ u(a) = u_0 \in (L_0,0),\ L_0 \leq u(t) \leq 0 \text{ for } t \in [a,\infty).\end{gathered} \tag{1.10}$$

In Theorem 5.16 the existence of a nonoscillatory solution of problem (1.9) is proved. This solution is either a Kneser solution or a monotonically increasing solution whose limit is L for t tending to infinity. The dual assertion stated in Theorem 5.17 yields the existence of a nonoscillatory solution of problem (1.10) which is either a Kneser solution or a monotonically decreasing solution whose limit is L_0 for t tending to infinity.

In both, the regular and the singular case, the new asymptotic formulas in the form of upper bounds are derived for damped Kneser solutions to the ODE (1.1) and their first derivatives in Theorem 6.9 and Theorem 6.10. These theorems claim that if functions p,q are continuous on $[a,\infty)$, $a \geq 0$ and regularly varying functions of index $\alpha \geq 1$, $\beta > 0$ respectively, such that $\beta - \alpha > -1$, and function f fulfils the conditions:

$$L_0 < 0 < L,\ f \in C[L_0,L],\ f(L_0) = f(0) = f(L) = 0,\ xf(x) > 0 \text{ for } x \in (L_0,L)\setminus\{0\},$$

$$\exists r > 1 : \liminf_{t\to 0}\frac{|f(x)|}{|x|^r} > 0,\ \limsup_{t\to 0}\frac{|f(x)|}{|x|^r} < \infty,$$

then each Kneser solution of (1.1) on $t \in [a,\infty)$ which satisfies either

$$u(a) = u_0 \in (0,L),\ 0 \le u(t) \le L \text{ for } t \in [a,\infty),$$

or

$$u(a) = u_0 \in (L_0,0),\ L_0 \le u(t) \le 0 \text{ for } t \in [a,\infty),$$

has the following asymptotic behaviour:

$$\lim_{t\to\infty} t^{\frac{\beta-\alpha+2}{r-1}-\varepsilon}|u(t)| = 0.$$

Moreover,
if $\beta > r\alpha - r - 1$ then

$$\lim_{t\to\infty} t^{\alpha-\varepsilon}|u'(t)| = 0,$$

if $\beta \le r\alpha - r - 1$ then

$$\lim_{t\to\infty} t^{\frac{\beta-\alpha+r+1}{r-1}-\varepsilon}|u'(t)| = 0,$$

for arbitrary small $\varepsilon > 0$.

According to the terminology concerning the Emden-Fowler equation

$$\big(p(t)|u'(t)|^{\alpha}\operatorname{sgn}(u'(t))\big)' + q(t)|u(t)|^{\gamma}\operatorname{sgn}(u(t)) = 0, \quad \alpha,\gamma \ge 0,$$

equation (1.1) can be treated as a super-linear equation since here $\alpha = 1$ and $\gamma = r > 1 = \alpha$, holds for $f(x)$ with x in a neighbourhood of the origin. The super-half-linear Emden-Fowler equation is studied in [31, 89] where a different sign condition is posed on the nonlinear term when compared to results presented in this thesis. Therefore, the results in [31, 89] cannot be used for our problem.

The goal of our future investigation is to prove the existence of Kneser solutions to equation (1.1) with a time singularity at $a = 0$ and $p \ne q$, and to more precisely describe their asymptotic behaviour in the form of asymptotic equivalence as t tends to infinity.

In Part II original contributions to BVPs (1.3) are described. The investigation concerns linear system of first order differential equations with a singularity of the first kind and with a generally nonintegrable inhomogeneity subject to general linear two-point boundary conditions. While for BVP (1.7) with smooth inhomogeneity and its applications, comprehensive literature is available, this is not the case for problem (1.3). The ODE system (1.6) was investigated in [118], where only particular solutions without boundary conditions are considered, whereas in Part II a general structure of linear two–point boundary conditions is of interest.

The main analytical result in Part II is the existence and uniqueness of a continuous or even smooth solution of problem (1.3) derived in Chapters 8 and 9 and published in [4] for the case with a constant coefficient matrix, and in [6] for the case with a variable coefficient matrix. The new numerical result presented in Chapter 10 is the convergence of a numerical solution obtained by a collocation method. The convergence order is proved to be $O(h^k)$, where k denotes the number of collocation points. The numerical analysis is published in [4] for IVPs with a constant coefficient matrix and it is completed in [7] for the general BVPs with a variable coefficient matrix.

In particular, the analytical results concerning the case of the constant coefficient matrix M_0 having eigenvalues only with negative real parts are presented in Theorem 8.5, the case of eigenvalues of M_0 with positive real parts is discussed in Theorem 8.11 under less restrictive assumptions than those used in the proof in [4]. The case of M_0 with zero eigenvalues is described in Theorem 8.15. Conditions that are necessary for a solution to be continuous on $[0,1]$ and conditions that are required for the uniqueness of a solution are given in each case. These results are necessary prerequisites for the investigation of problem (1.3) with a variable coefficient matrix. The three key cases are discussed separately in Theorem 8.6, Theorem 8.12 and Theorem 8.18 for eigenvalues of $M(0)$ with negative real parts, eigenvalues of $M(0)$ with positive real parts and zero eigenvalues of $M(0)$, respectively. In more details:

- If all eigenvalues of $M(0)$ have negative real parts and $f \in C[0,1]$, then there exists a unique continuous solution y of the BVP (1.3). However, the boundary conditions in (1.3) are here reduced to the initial conditions $M(0)y(0) = -f(0)$, which is necessary and sufficient for $y \in C[0,1]$. Moreover, if $f \in C^r[0,1]$, $M \in C^r[0,1]$ and $M^{(r)}(0) = 0$, then $y \in C^r[0,1]$.

- If all eigenvalues of $M(0)$ are assumed to have positive real parts, then there exists a unique continuous solution y of the BVP (1.3) provided that $f \in C[0,1]$ and $B_1 \in \mathbb{R}^{n\times n}$ is nonsingular. Here, again the general boundary conditions are reduced to a particular form, namely to the terminal conditions $B_1 y(1) = \beta$. The smoothness of y depends not only on the smoothness of the inhomogeneity f and the matrix M but also on the size of the smallest positive real part σ_+ of the eigenvalues of $M(0)$. In particular, if $f \in C^r[0,1]$, $M \in C^r[0,1]$, $M^{(r)}(0) = 0$, and $\sigma_+ > r$, then $y \in C^r[0,1]$.

- In the case that all eigenvalues of $M(0)$ are equal zero, we have to assume a special structure in f close to the singularity, namely, $f(t) = O(t^\alpha h(t))$ for $t \to 0$, where $h \in C[0,\delta]$, $\delta > 0$ and $\alpha > 0$. Moreover, if M had the form $M(t) = M(0) + t^\gamma D(t)$, $\gamma > 0$, $t \in [0,1]$, $D \in C[0,1]$, then there exists a unique continuous solution y of the BVP (1.3) reduced to an IVP, where $B_1 \equiv 0$ and $B_0\tilde{R} \in \mathbb{R}^{m\times m}$ is nonsingular. The matrix $\tilde{R}$ consists of the linearly independent columns of the projection R onto the m-dimensional space spanned by eigenvectors associated with zero eigenvalues of $M(0)$. The necessary and sufficient condition for y to be continuous is here $M(0)y(0) = 0$. Moreover, if $f \in C^r[0,1]$, $D \in C^r[0,1]$, $\alpha \geq r+1$, and $\gamma \geq r+1$, then $y \in C^{r+1}[0,1]$.

In the most general case, the spectrum of $M(0)$ is arbitrary, contains zero eigenvalues, and both eigenvalues with positive and negative real parts. In order to state the main analytic result for BVPs, the following projections are needed. Let S, R, H and N denote the projection onto the subspace spanned by the eigenvectors associated with eigenvalues of $M(0)$ with positive real parts, the subspace spanned by eigenvectors associated with zero eigenvalues of $M(0)$, the subspace spanned by principal eigenvectors associated with zero eigenvalues of $M(0)$, and the subspace spanned by eigenvectors associated with eigenvalues of $M(0)$ with negative real parts, respectively. Moreover, we define $Z := R+H$, $P := R+S$. We also use $\tilde{P}$, $\tilde{R}$ to denote the matrices consisting of the maximal set of linearly independent columns of the respective projections.

The existence of a unique continuous solution, $y \in C[0,1]$ of BVP (1.3) is proved in Theorem 9.9 provided that

$$f \in C[0,1], \;\; Zf(t) = O(t^{\alpha} h(t)),$$

where $\alpha > 0$ and h is continuous at zero,

$$M \in C[0,1], \; ZM(t) = ZM(0) + t^{\gamma} ZD(t),$$

here $\gamma > 0$ and D is a continuous matrix function, and

$$B_0, B_1 \in \mathbb{R}^{m \times n}, \; \beta \in \mathbb{R}^m, \;\; \text{where } m = \text{rank} P,$$

such that the $m \times m$ matrix

$$B_0 \tilde{R} + B_1 \tilde{P}$$

is nonsingular. This solution satisfies two of initial conditions

$$Hy(0) = 0, \quad M(0)Ny(0) = -Nf(0),$$

which are necessary and sufficient for $y \in C[0,1]$. The smoothness result $y \in C^r[0,1]$ follows by applying the smoothness results derived separately for components of the solutions associated with eigenvalues of $M(0)$ with negative real parts, positive real parts and zero eigenvalues.

In the numerical part, the collocation method applied to approximate the solution of the analytical problem (1.3) is analysed. We portion the interval $[0,1]$ by the equidistant mesh $\Delta = \{0 - t_0 < t_1 < \ldots < t_I = 1\}$ with a step h. The analytical solution is approximated by a continuous piecewise polynomial function p of degree less or equal k which fulfils the problem at k collocation points $t_{jl} = t_j + u_l h$, $l = 1, \ldots k$ where $0 < u_1 < \ldots < u_k \leq 1$, and which fulfils the boundary conditions. It turns out that the collocation retains its classical stage order k uniformly in t for a scheme with k collocation points, provided that the analytical solutions are appropriately smooth. The convergence behaviour is investigated separately for general IVPs, TVPs and BVPs.

In context of an IVPs with appropriately smooth solution, polynomial collocation method executed with k arbitrary collocation points retains its classical stage order $O(h^k)$ uniformly in t. This convergence result is derived in Theorem 10.2. For certain choice of collocation points satisfying

$$\int_0^1 w(s)\,\mathrm{d}s = 0, \quad w(t) = (t-u_1)(t-u_2)\cdots(t-u_k)$$

the so-called small superconvergence order $O(h^{k+1})$ is proved to hold uniformly in t, only in the case where $M(0)$ has multiple zero eigenvalues the order of convergence is affected by logarithm terms:

$$\|y-p\| \leq const.h^{k+1}|\ln(h)|_+^{d-1},$$

where d is the dimension of the largest Jordan box of $M(0)$ associated with the eigenvalue $\lambda = 0$ and

$$(x)_+ = \begin{cases} x, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

The small superconvergence is proved in Theorem 10.3. Similarly, for TVP the convergence of collocation with stage order $O(h^k)$ is proved in Theorem 10.7.

The main result for BVPs from the numerical point of view is the existence and convergence of collocation solution of the corresponding numerical scheme to (1.3). Provided that the unique analytical solution is sufficiently smooth, in particular $y \in C^{k+2}[0,1]$, and $f \in C^{k+1}[0,1]$, $M \in C^{k+2}[0,1]$, $\sigma_+ > k+2$ and h is sufficiently small, then there exists a unique collocation solution p and for the global error the following estimate holds

$$\|p-y\| \leq const.h^k.$$

Here, σ_+ is the smallest real part of positive eigenvalues. This assertion stated in Theorem 10.9 represents an extension of previous results obtained for (1.7) to the more general class of problems (1.3) with a singularity of the first kind and with a nonintegrable inhomogeneity.

The aim of future study is to investigate the nonlinear problem

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t,y(t))}{t}, \; t \in (0,1], \quad B_0y(0) + B_1y(1) = \beta.$$

and to prove the existence of a solution of the nonlinear BVP which we intend to publish in an upcoming paper.

The thesis presents original results achieved by the author during the PhD studies of the Mathematical Analysis at the Palacky University Olomouc in cooperation with:

- prof. RNDr. Irena Rachůnková, DrSc., Department of Mathematical Analysis and Mathematical Applications, Faculty of Science Palacký University Olomouc, Chapters 5–10, see [1], [4]–[7].
- Ao. Univ. Prof. Dr. Ewa B. Weinmüller, Department for Analysis and Scientific Computing, Vienna University of Technology, Chapters 5–10, see [1], [4]–[7].
- prof. RNDr. Svatoslav Staněk, CSc., Department of Mathematical Analysis and Mathematical Applications, Faculty of Science Palacký University Olomouc, Chapters 7–9, see [4], [5].
- Michael Hubner, Department for Analysis and Scientific Computing, Vienna University of Technology, Chapters 6, 10, see [1], [7].
- Mgr. Martin Rohleder, Department of Mathematical Analysis and Mathematical Applications, Faculty of Science Palacký University Olomouc, Chapters 2–4, see [2].
- Mgr. Jakub Stryja, Ph.D., Department of Mathematics and Descriptive Geometry, VŠB - Technical University Ostrava, Chapters 2–4, see [2].

The results were presented at several international conferences and published in multiple peer-reviewed journals. The list of author's publications is listed at the end of the thesis.

1.6 Structure of the thesis

The thesis is divided into two parts and organized in the following manner:

Part I consists of five chapters and it is devoted to singular nonlinear second order differential equation. In Chapter 2, basic properties of solutions of an auxiliary problem are derived. These preliminaries are fundamental for further investigation of existence and uniqueness properties stated in Chapter 3. Once the results for the auxiliary problem are obtained, we proceed to the existence of three types of solutions of the original problem in Chapter 4. In particular, we prove the existence and other properties of damped and escape solutions. By means of these results we derive the main objective of this chapter - the existence of a homoclinic solution in Theorem 4.15. Chapter 5 investigates the existence of Kneser solutions. Although the thesis is mainly focused on singular problems, here the regular problem is also discussed. In the last chapter of Part I, regularly varying functions at infinity are introduced and asymptotic formulas for Kneser solutions of both, the singular problem and the regular problem with regularly varying coefficients, are derived.

Part II deals in four chapters with linear systems of first order differential equations with a singularity of the first kind and unsmooth data. In Chapter 7 motivation and necessary notation is introduced. In Chapter 8, three case studies are carried out, the case of only negative real parts of eigenvalues of $M(0)$, positive real parts of eigenvalues of $M(0)$, and zero eigenvalues of matrix $M(0)$. The three case studies are used to formulate results for general IVPs, TVPs, and BVPs in Chapter 9. Chapter 10 is devoted to numerical analysis, in particular, to collocation methods and the convergence analysis of the collocations schemes applied to solve the problem numerically. Finally, numerical examples are provided to illustrate the theoretical findings.

The thesis is completed with the list of author's publications, the list of references and author's curriculum vitae.

Part I

Singular nonlinear second order differential equation

2 Introduction

Part I covers the singular initial value problem

$$\big(p(t)u'(t)\big)' + q(t)f(u(t)) = 0, \quad u(0) = u_0 \in [L_0, L], \quad u'(0) = 0$$

on the unbounded domain $[0,\infty)$. Here, $f \in C(\mathbb{R})$, $f(L_0) = f(0) = f(L) = 0$, $L_0 < 0 < L$ and $xf(x) > 0$ for $x \in (L_0, 0) \cup (0, L)$. Further, $p, q \in C[0,\infty)$ are positive on $(0,\infty)$ and $p(0) = 0$. The integral $\int_0^1 \frac{\mathrm{d}s}{p(s)}$ may be divergent which yields the time singularity at $t = 0$. The set of all solutions of the problem is described. Existence results and properties of damped and escape solutions are derived. By means of these results, the existence of an increasing solution with $u(\infty) = L$ (a homoclinic solution) playing an important role in applications is proved. Furthermore, the existence of Kneser solutions is investigated and asymptotic properties of such solutions and their first derivatives are derived in the framework of regularly varying functions. The analytical findings concerning Kneser solutions are illustrated by numerical simulations obtained by the collocation method. The content of Part I is mainly based on results published in [1]–[3]. In particular, results devoted to the existence and properties of damped, escape and homoclinic solutions are published in [2], results concerning existence of Kneser solutions are generalizations of those published in [3], and asymptotic properties of Kneser solutions of problem with regularly varying coefficients are published in [1].

2.1 Motivation

In this part, we investigate the second order nonlinear ordinary differential equations arising in modelling of multi-phase fluids in hydrodynamics. Our motivation has its origin in the Cahn-Hilliard theory which is devoted to study behaviour of nonhomogeneous fluids and leads to the system of partial differential equations (see e.g. [35, 71, 83] and references therein):

$$\rho_t + div(\rho\vec{v}) = 0,$$
$$\frac{d\vec{v}}{dt} + \nabla(\mu(\rho) - \gamma\Delta\rho) = 0.$$

Here ρ denotes density of the fluid, $\vec{v}$ velocity of the fluid, $\mu(\rho)$ chemical potential and γ is a constant parameter. By considering the case when the motion of the fluid is zero, the system of PDEs is reduced to a single equation of the form

$$\gamma\Delta\rho = \mu(\rho) - \mu_0,$$

which describes the state of the fluid in $\mathbb{R}^n$. Here μ_0 is a suitable constant. We search for a solution with the spherical symmetry which depends only on the radial variable r.

By introducing the polar system of coordinates in $\mathbb{R}^n$ and a convenient system of units, the PDE is reduced to the ODE posed on unbounded domain

$$\gamma\left(\rho'' + \frac{n-1}{r}\rho'\right) = \mu(\rho) - \mu_0,\ r \in (0,\infty).$$

This equations is known as *the density profile equation* and together with the boundary conditions

$$\rho'(0) = 0,\ \lim_{r\to\infty} \rho(r) = \rho_l,$$

describes the formation of microscopical bubbles in a fluid, in particular, vapour inside a liquid. The first boundary condition follows from the spherical symmetry and is also necessary for the smoothness of a solution of the equation at $r = 0$, see Remark 2.2. The second condition indicates density of the external liquid surrounding the bubble. In the simplest model in $\mathbb{R}^3$, the chemical potential is considered as a three-degree polynomial with three distinct real roots and the problem is reduced to the form

$$(r^2u')' = \lambda^2 r^2 (u+1)u(u-\xi), \tag{2.1}$$

$$u'(0) = 0,\ \lim_{r\to\infty} u(r) = \xi, \tag{2.2}$$

where $\lambda \in (0,\infty)$ and $\xi \in [0,1]$ are parameters which reflect different physical situations, see [64, 83]. The constant solution $u \equiv \xi$ corresponds to the case of homogeneous fluid without bubbles. A great physical significance has a strictly increasing solution, the so called bubble-type solution, having just one zero. This solutions indicates the gas density inside the bubble, the bubble radius and the surface tension. For further development we also refer to [43, 120]. Numerical investigations of the problem can be found in [35, 64, 83].

The equation (2.1) arises in many other areas: in population genetics where it serves as a model for spatial distribution of the genetic composition of a population [40, 41]; in the homogeneous nucleation theory [8]; in relativistic cosmology for description of particles which can be treated as domains in the universe [84]; in the nonlinear field theory, in particular, when describing bubbles generated by scalar fields of the Higgs type in the Minkowski spaces [36]. Other problems close to (2.1), (2.2) can be found in [10, 22, 25, 59, 60, 63, 85].

Motivated by the above problems we consider the generalization of the density profile equation

$$(p(t)u'(t))' + q(t)f(u(t)) = 0$$

on the unbounded domain $[0,\infty)$. In particular, we study the equation subject to the initial conditions

$$u(0) = u_0 \in [L_0, L],\ u'(0) = 0.$$

By means of the existence and properties of different types of solutions we derive the existence of a bubble-type solution satisfying

$$\lim_{t\to\infty} u(t) = L.$$

2.2 Statement of the problem

We investigate the equation

$$\big(p(t)u'(t)\big)' + q(t)f(u(t)) = 0 \tag{2.3}$$

with the initial conditions

$$u(0) = u_0, \quad u'(0) = 0, \quad u_0 \in [L_0, L], \tag{2.4}$$

where

$$L_0 < 0 < L, \quad f(L_0) = f(0) = f(L) = 0, \tag{2.5}$$

$$f \in C(\mathbb{R}), \quad xf(x) > 0 \text{ for } x \in (L_0, L) \setminus \{0\}, \tag{2.6}$$

$$p \in C[0,\infty), \quad p(0) = 0, \quad p(t) > 0 \text{ for } t \in (0,\infty), \tag{2.7}$$

$$q \in C[0,\infty), \quad q(t) > 0 \text{ for } t \in (0,\infty). \tag{2.8}$$

At the beginning, we specify smoothness of solutions that we are interested in. Further, we define different types of solutions according to their asymptotic behaviour.

Definition 2.1 Let $c \in (0,\infty)$. A function $u \in C^1[0,c]$ with $pu' \in C^1[0,c]$ which satisfies equation (2.3) for every $t \in [0,c]$ and which satisfies the initial conditions (2.4) is called a *solution* of problem (2.3), (2.4) on $[0,c]$. If u is a solution of problem (2.3), (2.4) on $[0,c]$ for every $c > 0$, then u is called a *solution* of problem (2.3), (2.4).

Remark 2.2 The condition $u'(0) = 0$ is necessary for the smoothness of the solution in the case where $p \equiv q$ is an increasing function. To see this, let us consider a solution u of $(pu')' + pf(u) = 0$. Since $u \in C^1[0,\infty)$, the assumption $p(0) = 0$ yields $p(0)u'(0) = 0$. Since f is continuous on $[L_0, L]$ and $u(0) \in (L_0, L)$, there exist $M > 0$ and $\delta > 0$ such that $|f(u(t))| \le M$ for $t \in (0,\delta)$. We now integrate (2.3) and use the monotonicity of p to obtain

$$|u'(t)| = \left| \frac{1}{p(t)} \int_0^t p(s) f(u(s))\,\mathrm{d}s \right| \le \frac{M}{p(t)} \int_0^t p(s)\,\mathrm{d}s \le Mt, \quad t \in (0,\delta).$$

Consequently, $u'(0) = 0$ holds.

Definition 2.3 A solution u of problem (2.3), (2.4) is said to be *oscillatory* if $u \not\equiv 0$ in any neighbourhood of ∞ and if u has a sequence of zeros tending to ∞. Otherwise, u is called *nonoscillatory*.

Definition 2.4 Consider a solution of problem (2.3), (2.4) with $u_0 \in (L_0, L)$ and denote

$$u_{sup} = \sup\{u(t) \colon t \in [0,\infty)\}.$$

If $u_{sup} = L$, then u is called a *homoclinic solution* of problem (2.3), (2.4).
If $u_{sup} < L$, then u is called a *damped solution* of problem (2.3), (2.4).

Note that if we extend functions p and q in equation (2.3) from the half-line onto $\mathbb{R}$ as even functions, then a homoclinic solution of (2.3), (2.4) has the same limit L as $t \to -\infty$ and $t \to \infty$. This is a motivation of Definition 2.4.

Definition 2.5 Let u be a solution of problem (2.3), (2.4) on $[0,c]$, where $c \in (0,\infty)$. If u satisfies

$$u(c) = L, \quad u'(c) > 0,$$

then u is called an *escape solution* of problem (2.3), (2.4) on $[0,c]$.

Definition 2.6 A solution u of equation (2.3) on $[a,\infty), a \geq 0$, is called a *Kneser solution* if there exists $t_0 > a$ such that

$$u(t)u'(t) < 0 \text{ for } t \in [t_0,\infty).$$

Different types of solutions with respect to their asymptotic behaviour are illustrated in Figure 2.1.

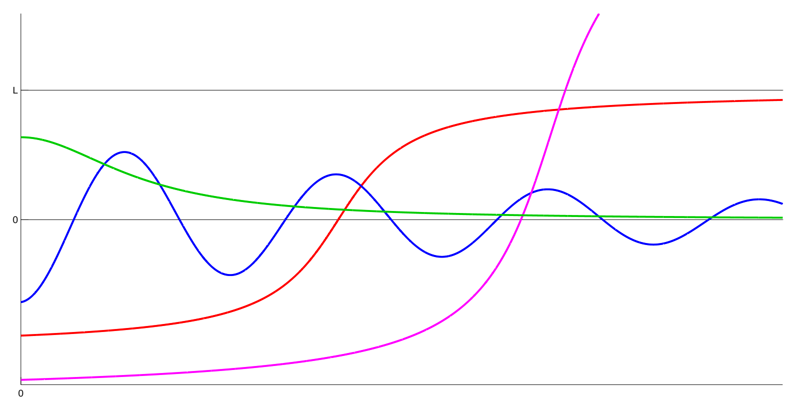

Figure 2.1: Different types of solutions

Our first goal is to study properties of solutions determined by Definitions 2.4, 2.5 and 2.6 and to find conditions guaranteeing their existence.

3 Solvability of the problem

In order to derive the existence of all three types of solutions of problem (2.3), (2.4), we introduce the auxiliary equation

$$\big(p(t)u'(t)\big)' + q(t)\tilde{f}(u(t)) = 0, \tag{3.1}$$

where

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in [L_0, L], \\ 0 & \text{for } x < L_0, \quad x > L. \end{cases} \tag{3.2}$$

By means of results about existence and properties of all three types of solutions of problem (3.1), (2.4), we proceed to the existence of escape and homoclinic solution of problem (2.3), (2.4), which is proved in the next chapter in Theorem 4.14 and Theorem 4.15.

3.1 Preliminary results

In this section, we provide auxiliary lemmas concerning properties of solutions of problem (3.1), (2.4). The assertions stated below are needed in the sequel for the existence and uniqueness results.

Lemma 3.1 *Let* (2.5)–(2.8) *hold and let u be a solution of problem* (3.1), (2.4).

a) *Assume that there exists $t_1 \geq 0$ such that $u(t_1) \in (0, L)$ and $u'(t_1) = 0$. Then*

$$u(t) \geq 0 \Rightarrow u'(t) < 0 \textit{ for } t \in (t_1, \theta_1],$$

where θ_1 is the first zero of u on (t_1, ∞). If such θ_1 does not exist, then $u'(t) < 0$ for $t \in (t_1, \infty)$.

b) *Assume that there exists $t_2 \geq 0$ such that $u(t_2) \in (L_0, 0)$ and $u'(t_2) = 0$. Then*

$$u(t) \leq 0 \Rightarrow u'(t) > 0 \textit{ for } t \in (t_2, \theta_2], \tag{3.3}$$

where θ_2 is the first zero of u on (t_2, ∞). If such θ_2 does not exist, then $u'(t) > 0$ for $t \in (t_2, \infty)$.

Proof:

a) Let $t_1 \geq 0$ be such that $u(t_1) \in (0, L)$ and $u'(t_1) = 0$. First, we assume that there exists $\theta_1 > t_1$ satisfying $u(t) > 0$ on (t_1, θ_1) and $u(\theta_1) = 0$. Then, by (2.6) and (2.8), $q(t)\tilde{f}(u(t)) > 0$, and hence

$$(pu')'(t) < 0, \quad t \in (t_1, \theta_1).$$

Since $(pu')(t_1) = 0$ and since pu' is decreasing on (t_1, θ_1), we obtain $pu' < 0$ on (t_1, θ_1), and, due to (2.7), $u' < 0$ on (t_1, θ_1). Furthermore, after integrating (3.1) over (t_1, θ_1), we get

$$pu'(\theta_1) = -\int_{t_1}^{\theta_1} q(s)\tilde{f}(u(s))\,\mathrm{d}s < 0.$$

Thus $pu' < 0$ on $(t_1, \theta_1]$. If u is positive on $[t_1, \infty)$, we obtain as before $pu' < 0$ on (t_1, ∞). The inequality $u'(t) < 0$ for $t \in (t_1, \infty)$ follows from (2.7).

b) We argue similarly as in a). □

Further properties can be described by means of the function

$$\tilde{F}(x) = \int_0^x \tilde{f}(z)\,\mathrm{d}z, \quad x \in \mathbb{R}.$$

Condition (3.4) stated below is illustrated in Figure 3.1.

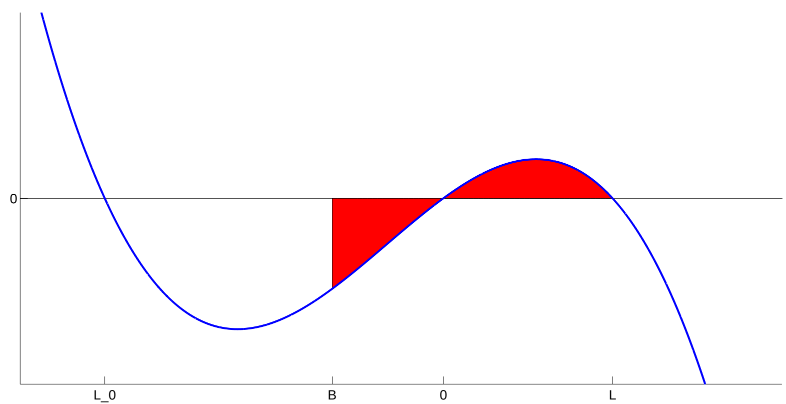

Figure 3.1: Function f satisfying condition (3.4)

Lemma 3.2 *Assume that* (2.5)–(2.8) *hold and that*

$$\text{there exists } \bar{B} \in (L_0, 0) : \tilde{F}(\bar{B}) = \tilde{F}(L), \tag{3.4}$$

$$pq \text{ is nondecreasing on } [0, \infty). \tag{3.5}$$

Let u be a solution of problem (3.1), (2.4) *such that there exist $b \geq 0$, $\theta > b$ satisfying*

$$u(b) \in (\bar{B}, 0), \quad u'(b) = 0, \quad u(\theta) = 0, \quad u(t) < 0 \text{ for } t \in [b, \theta).$$

Then u fulfils either

$$u'(t) > 0 \text{ for } t \in (b, \infty), \quad \lim_{t \to \infty} u(t) \in (0, L), \tag{3.6}$$

or

$$\exists c \in (\theta, \infty), \quad u(c) \in (0, L), \quad u'(c) = 0, \quad u'(t) > 0 \text{ for } t \in (b, c). \tag{3.7}$$

Furthermore, if

$$pq \text{ is increasing on } [0,\infty), \tag{3.8}$$

then the assertion holds also for $u(b) = \bar{B}$, $u'(b) = 0$.

Proof: According to Lemma 3.1, $u'(t) > 0$ for $t \in (b,\theta]$. Assume that there exists $c > \theta$ such that $u'(c) = 0$ and $u'(t) > 0$ for $t \in (b,c)$. Let $u(c) \geq L$. Then there exists $b_1 \in (\theta, c]$ such that $u(b_1) = L$, $u' > 0$ on (b,b_1). We multiply equation (3.1) by pu', integrate over (b,b_1) and obtain

$$\int_b^{b_1} \left(p(t)u'(t)\right)' p(t)u'(t)\,\mathrm{d}t = -\int_b^\theta (pq)(t)\tilde{f}(u(t))u'(t)\,\mathrm{d}t - \int_\theta^{b_1} (pq)(t)\tilde{f}(u(t))u'(t)\,\mathrm{d}t.$$

By (3.5), we get

$$\begin{aligned} 0 &\leq \frac{\left(p(b_1)u'(b_1)\right)^2}{2} \leq -(pq)(\theta)\int_b^\theta \tilde{f}(u(t))u'(t)\,\mathrm{d}t - (pq)(\theta)\int_\theta^{b_1} \tilde{f}(u(t))u'(t)\,\mathrm{d}t \\ &\leq (pq)(\theta)\left(\tilde{F}(u(b)) - \tilde{F}(u(b_1))\right) = (pq)(\theta)\left(\tilde{F}(u(b)) - \tilde{F}(L)\right). \end{aligned}$$

Hence,

$$\tilde{F}(u(b)) \geq \tilde{F}(L). \tag{3.9}$$

On the other hand, since $\bar{B} < u(b) < 0$, we get by (3.4)

$$\tilde{F}(L) = \tilde{F}(\bar{B}) > \tilde{F}(u(b)). \tag{3.10}$$

This contradicts (3.9). Consequently, $u(c) \in (0,L)$ and (3.7) holds.

Let $u'(t) > 0$ on (b,∞). Then u is increasing and it has a limit for $t \to \infty$. Let $\lim_{t\to\infty} u(t) > L$. Then there exists $b_1 > \theta$ such that $u(b_1) = L$, $u'(b_1) > 0$, which yields a contradiction as before. Let $\lim_{t\to\infty} u(t) = L$. Then

$$\lim_{t\to\infty} \tilde{F}(u(t)) = \tilde{F}(L),$$

and, by (3.10), there exists $T > b$ such that $\tilde{F}(u(T)) > \tilde{F}(u(b))$. Hence, after multiplying (3.1) by pu' and integrating over (b,T), we get

$$0 < \frac{\left(p(T)u'(T)\right)^2}{2} \leq (pq)(\theta)\left(\tilde{F}(u(b)) - \tilde{F}(u(T))\right) < 0.$$

This contradiction yields $\lim_{t\to\infty} u(t) \in (0,L)$ and (3.6) holds.

Let us assume that (3.8) is fulfilled and $u(b) = \bar{B}$, $u'(b) = 0$. We follow the steps in the first part of this proof. If there exists b_1 such that $u(b_1) = L, u' > 0$ on (b,b_1), then by multiplying equation (3.1) by pu' and integrating over (b,b_1), we obtain the contradiction

$$0 \leq \frac{\left(p(b_1)u'(b_1)\right)^2}{2} < (pq)(\theta)\left(\tilde{F}(\bar{B}) - \tilde{F}(L)\right) = 0.$$

Consequently, if there exists $c \in (0,\infty)$ such that $u'(c) = 0$, $u'(t) > 0$ for $t \in (b,c)$, then $u(c) \in (0,L)$.

Let $u'(t) > 0$ for $t \in (b,\infty)$. Due to the above arguments, $\lim_{t\to\infty} u(t) \le L$. Assume that $\lim_{t\to\infty} u(t) = L$. Then

$$\tilde{F}(u(b)) = \tilde{F}(\bar{B}) = \tilde{F}(L) = \lim_{t\to\infty} \tilde{F}(u(t)).$$

After multiplying equation (3.1) by pu', integrating over (b,θ) and over (θ,t) for $t > \theta$, we get due to (3.3)

$$0 < \frac{(p(\theta)u'(\theta))^2}{2} < (pq)(\theta)\tilde{F}(\bar{B}),$$
$$-\frac{(p(\theta)u'(\theta))^2}{2} < \frac{(p(t)u'(t))^2}{2} - \frac{(p(\theta)u'(\theta))^2}{2} < (pq)(\theta)\left(-\tilde{F}(u(t))\right).$$

Therefore,

$$(pq)(\theta)\tilde{F}(\bar{B}) > \frac{(p(\theta)u'(\theta))^2}{2} > (pq)(\theta)\tilde{F}(u(t)).$$

We let $t \to \infty$ and get

$$(pq)(\theta)\tilde{F}(\bar{B}) > \frac{(p(\theta)u'(\theta))^2}{2} \ge (pq)(\theta)\tilde{F}(\bar{B}).$$

This contradiction completes the proof. □

The next lemma can be proved analogously.

Lemma 3.3 *Assume* (2.5)–(2.8), (3.4), (3.5). *Let u be a solution of problem* (3.1), (2.4) *such that there exist $a \ge 0$, $\theta > a$ satisfying*

$$u(a) \in (0,L), \quad u'(a) = 0, \quad u(\theta) = 0, \quad u(t) > 0 \text{ for } t \in [a,\theta). \tag{3.11}$$

Then u fulfils either

$$u'(t) < 0 \text{ for } t \in (a,\infty), \quad \lim_{t\to\infty} u(t) \in (\bar{B},0), \tag{3.12}$$

or

$$\exists b \in (\theta,\infty)\colon u(b) \in (\bar{B},0), \quad u'(b) = 0, \quad u'(t) < 0 \text{ for } t \in (a,b). \tag{3.13}$$

Remark 3.4 Let assumptions (2.5)–(2.8) hold. Both equations (2.3) and (3.1) have a constant solutions $u(t) \equiv L$, $u(t) \equiv 0$ and $u(t) \equiv L_0$. By Lemma 3.1, the solution $u(t) \equiv 0$ is the only solution satisfying conditions $u(t_0) = 0$, $u'(t_0) = 0$, for some $t_0 > 0$. Assume moreover

$$f \in \mathrm{Lip}_{loc}([L_0,L] \setminus \{0\}). \tag{3.14}$$

Then, by (3.2), (3.14), the solution $u(t) \equiv L$ is the only solution satisfying conditions $u(t_0) = L$, $u'(t_0) = 0$, for some $t_0 > 0$. Similarly, $u(t) \equiv L_0$ s the only solution satisfying conditions $u(t_0) = L_0$, $u'(t_0) = 0$, for some $t_0 > 0$.

Lemma 3.5 *Assume* (2.5)–(2.8) *and* (3.14). *Let u be a solution of problem* (3.1), (2.4) *with $u_0 \in (L_0, \bar{B}]$. Assume that there exist $\theta > 0$, $a > \theta$ such that*

$$\begin{aligned} u(\theta) &= 0, \quad u(t) < 0 \, \textit{for } t \in [0,\theta), \\ u'(a) &= 0, \quad u'(t) > 0 \, \textit{for } t \in (\theta, a). \end{aligned}$$

Then

$$u(a) \in (0,L), \quad u'(t) > 0 \, \textit{for } t \in (0,a). \tag{3.15}$$

Proof: Directly from Lemma 3.1, we have $u' > 0$ on $(0,a)$. Therefore,

$$pu'(t) > 0, \quad t \in (0,a). \tag{3.16}$$

On contrary to (3.15), assume that $u(a) \geq L$. Then, by (3.14) and Remark 3.4, we have $u(a) > L$. Therefore, there exists $a_0 \in (\theta, a)$ such that $u(t) > L$ on $(a_0, a]$. After integrating equation (3.1) over (a_0, a), we get

$$pu'(a) - pu'(a_0) = \int_{a_0}^{a} q(s)\tilde{f}(u(s))\,\mathrm{d}s = 0.$$

By virtue of (3.2), $pu'(a_0) = 0$, contrary to (3.16). □

3.2 Existence of a solution

This section is devoted to the existence result of a solution to the auxiliary problem (3.1), (2.4). The Schauder Fixed Point Theorem is used in order to obtain the existence of a solution on each compact interval $[0,b]$. Furthermore, the boundedness of function $\tilde{f}$ allows to extend the solution on $[0,\infty)$.

Theorem 3.6 (Existence of a solution of problem (3.1), (2.4)) *Let assumptions* (2.5)–(2.8) *and*

$$\lim_{t\to 0^+} \frac{1}{p(t)} \int_0^t q(s)\,\mathrm{d}s = 0 \tag{3.17}$$

hold. Then for each $u_0 \in [L_0, L]$ problem (3.1), (2.4) *has a solution u. If in addition conditions* (3.4), (3.5) *and* (3.14) *hold, then the solution u satisfies:*

$$\textit{if } u_0 \in [\bar{B}, L], \quad \textit{then } u(t) > \bar{B}, \quad t \in (0,\infty), \tag{3.18}$$

$$\textit{if } u_0 \in (L_0, \bar{B}), \quad \textit{then } u(t) > u_0, \quad t \in (0,\infty). \tag{3.19}$$

Proof: The existence of a solution of problem (3.1), (2.4) with $u_0 = L_0$, $u_0 = 0$ and $u_0 = L$ follows from Remark 3.4. Assume that $u_0 \in (L_0, 0) \cup (0, L)$. By integrating equation (3.1), we get an equivalent form

$$u(t) = u_0 - \int_0^t \frac{1}{p(s)} \int_0^s q(\tau)\tilde{f}(u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s, \quad t \in [0,\infty).$$

By (2.6), (3.2), there exists $M > 0$ such that

$$\left|\tilde{f}(x)\right| \le M,\ x \in \mathbb{R}.$$

Further, we put

$$\varphi(t) = \frac{1}{p(t)} \int_0^t q(s)\,\mathrm{d}s,\ t > 0, \quad \varphi(0) = 0. \tag{3.20}$$

According to (3.17), function $\varphi \in C[0,b]$. Choose an arbitrary $b > 0$. By (3.17), there exists $\varphi_b > 0$ such that $|\varphi(t)| \le \varphi_b$ for each $t \in [0,b]$. Consider the Banach space $C[0,b]$ with the maximum norm and define an operator $\mathscr{F} : C[0,b] \to C[0,b]$,

$$(\mathscr{F}u)(t) = u_0 - \int_0^t \frac{1}{p(s)} \int_0^s q(\tau)\tilde{f}(u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s.$$

Put $\Lambda = \max\{|L_0|, L\}$ and consider the ball $\mathscr{B}(0,R) = \left\{u \in C[0,b] : \|u\|_{C[0,b]} \le R\right\}$, where $R = \Lambda + Mb\varphi_b$. The norm of operator $\mathscr{F}$ is estimated as follows

$$\|\mathscr{F}u\|_{C[0,b]} = \max_{t\in[0,b]} \left| u_0 - \int_0^t \frac{1}{p(s)} \int_0^s q(\tau)\tilde{f}(u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s \right| \le \Lambda + Mb\varphi_b = R,$$

which yields that $\mathscr{F}$ maps $\mathscr{B}(0,R)$ on itself. Choose a sequence $\{u_n\} \subset C[0,b]$ such that $\lim_{n\to\infty} \|u_n - u\|_{C[0,b]} = 0$. Since the function $\tilde{f}$ is continuous, we get

$$\lim_{n\to\infty} \|\mathscr{F}u_n - \mathscr{F}u\|_{C[0,b]} \le \lim_{n\to\infty} \left\|\tilde{f}(u_n) - \tilde{f}(u)\right\|_{C[0,b]} \left(\int_0^b \frac{1}{p(s)} \int_0^s q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \right) = 0,$$

that is the operator $\mathscr{F}$ is continuous. Choose an arbitrary $\varepsilon > 0$ and put $\delta = \frac{\varepsilon}{M\varphi_b}$. Then, for $t_1, t_2 \in [0,b]$ such that $|t_1 - t_2| < \delta$ and for $u \in \mathscr{B}(0,R)$, we have

$$|(\mathscr{F}u)(t_1) - (\mathscr{F}u)(t_2)| = \left| \int_{t_1}^{t_2} \frac{1}{p(s)} \int_0^s q(\tau)\tilde{f}(u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s \right| \le M\varphi_b|t_2 - t_1| < \varepsilon.$$

Hence, functions in $\mathscr{F}(\mathscr{B}(0,R))$ are equicontinuous, and, by the Arzelà–Ascoli theorem, the set $\mathscr{F}(\mathscr{B}(0,R))$ is relatively compact. Consequently, the operator $\mathscr{F}$ is compact on $\mathscr{B}(0,R)$.

The Schauder Fixed Point Theorem yields a fixed point $u^\star$ of $\mathscr{F}$ in $\mathscr{B}(0,R)$. Therefore,

$$u^\star(t) = u_0 - \int_0^t \frac{1}{p(s)} \int_0^s q(\tau)\tilde{f}(u^\star(\tau))\,\mathrm{d}\tau\,\mathrm{d}s.$$

Hence, $u^\star(0) = u_0$,

$$\left(p(t)(u^\star)'(t)\right)' = -q(t)\tilde{f}\left(u^\star(t)\right), \quad t \in [0,b].$$

Since $|(u^\star)'(t)| \le M\varphi(t)$ and, by (3.17), $\lim_{t\to 0^+}(u^\star)'(t) = 0 = (u^\star)'(0)$. According to (3.2), $\tilde{f}(u^\star(t))$ is bounded on $[0,\infty)$ and hence, by Theorem 11.5 in [60], $u^\star$ can be extended to interval $[0,\infty)$ as a solution of equation (3.1).

In order to derive estimates (3.18) and (3.19), we use (3.4), (3.5) and (3.14) and apply Lemmas 3.2, 3.3 and 3.5. For more details, see the proof of Theorem 2.5 in [111].

□

Remark 3.7 Under assumptions (2.5)–(2.8) and (3.17), each solution of problem (3.1), (2.4) is defined on the half-line $[0,\infty)$. In addition, the set of these solutions with $u_0 \in (L_0,0)\cup(0,L)$ is composed of three disjoint classes $\mathscr{S}_d$ (damped solutions), $\mathscr{S}_h$ (homoclinic solutions), and $\mathscr{S}_e$ (escape solutions). Here

1. $u \in \mathscr{S}_d$ if and only if $u_{sup} < L$,
2. $u \in \mathscr{S}_h$ if and only if $u_{sup} = L$,
3. $u \in \mathscr{S}_e$ if and only if $u_{sup} > L$.

3.3 Uniqueness of a solution

Once the existence of a solution to (3.1), (2.4) is presented, we prove the uniqueness together with the continuous dependence on initial values under some additional assumptions.

Theorem 3.8 (Uniqueness and continuous dependence on initial values) *Let assumptions* (2.5)–(2.8)*,* (3.17) *hold and let*

$$f \in \mathrm{Lip}\,[L_0,L]\,. \tag{3.21}$$

Then for each $u_0 \in [L_0,L]$, problem (3.1)*,* (2.4) *has a unique solution. Further, for each $b > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for any $B_1,B_2 \in [L_0,L]$*

$$|B_1 - B_2| < \delta \Rightarrow \|u_1 - u_2\|_{C^1[0,b]} < \varepsilon.$$

Here, u_i is a solution of problem (3.1)*,* (2.4) *with $u_0 = B_i$, $i = 1,2$.*

Proof: For $i \in \{1,2\}$ choose $B_i \in [L_0,L]$. By Theorem 3.6, there exists a solution u_i of problem (3.1), (2.4) with $u_0 = B_i$. We integrate (3.1) where $u = u_i$, and get by (2.4)

$$u_i(\xi) = B_i - \int_0^{\xi} \frac{1}{p(s)} \int_0^s q(\tau)\tilde{f}(u_i(\tau))\,\mathrm{d}\tau\,\mathrm{d}s, \quad \xi \in [0,\infty)\,. \tag{3.22}$$

Denote

$$\rho(t) = \max\{|u_1(\xi) - u_2(\xi)|\colon \xi \in [0,t]\}, \quad t \in [0,\infty)\,.$$

Then (3.22) yields

$$\rho(t) \le |B_1 - B_2| + \int_0^t \frac{1}{p(s)} \int_0^s q(\tau)\left|\tilde{f}(u_1(\tau)) - \tilde{f}(u_2(\tau))\right| \mathrm{d}\tau\,\mathrm{d}s, \quad t \in [0,\infty)\,.$$

By (3.21), there exists a Lipschitz constant $K \in (0,\infty)$ for f on $[L_0, L]$. Then K is the Lipschitz constant for $\tilde{f}$ on $\mathbb{R}$ and

$$\rho(t) \le |B_1 - B_2| + K \int_0^t \frac{1}{p(s)} \int_0^s q(\tau)\rho(\tau)\,\mathrm{d}\tau\,\mathrm{d}s, \quad t \in [0,\infty). \tag{3.23}$$

Denote (*cf.* (3.17) and (3.20))

$$\varphi(s) = \frac{1}{p(s)} \int_0^s q(\tau)\,\mathrm{d}\tau, \quad s \in (0,\infty), \quad \varphi(0) = 0.$$

Choose $b > 0$. Then, due to (3.17), there exists $\varphi_b \in (0,\infty)$ such that

$$|\varphi(s)| \le \varphi_b, \quad s \in [0,b]. \tag{3.24}$$

Since ρ is nondecreasing on $[0,b]$, we get by (3.23)

$$\rho(t) \le |B_1 - B_2| + K \int_0^t \rho(s)\varphi(s)\,\mathrm{d}s, \quad t \in [0,b],$$

and, using the Gronwall lemma, we arrive at

$$\rho(t) \le |B_1 - B_2| e^{Kb\varphi_b}, \quad t \in [0,b]. \tag{3.25}$$

Similarly, by (3.22), we get for $i \in \{1,2\}$

$$u_i'(t) = -\frac{1}{p(t)} \int_0^t q(s)\tilde{f}(u_i(s))\,\mathrm{d}s, \quad t \in (0,\infty), \quad u_i'(0) = 0.$$

Therefore,

$$\left|u_1'(t) - u_2'(t)\right| \le \frac{K}{p(t)} \int_0^t q(s)|u_1(s) - u_2(s)|\,\mathrm{d}s, \quad t \in [0,\infty).$$

After applying (3.24) and (3.25), we get

$$\max\{\left|u_1'(t) - u_2'(t)\right| : t \in [0,b]\} \le |B_1 - B_2| K\varphi_b e^{Kb\varphi_b}.$$

Consequently,

$$\|u_1 - u_2\|_{C^1[0,b]} \le |B_1 - B_2|(1 + K\varphi_b)e^{Kb\varphi_b} < \varepsilon,$$

provided $|B_1 - B_2| < \delta$, where

$$\delta = \frac{\varepsilon}{(1 + K\varphi_b)e^{Kb\varphi_b}}.$$

If $B_1 = B_2$, then $u_1(t) = u_2(t)$ on each $[0,b] \subset \mathbb{R}$ which yields the uniqueness of a solution of problem (3.1), (2.4). $\square$

Example 3.9 In order to illustrate a problem with a unique solution, we put

$$p(t) = t^{\alpha}, \quad q(t) = t^{\beta}, \quad t \in [0, \infty).$$

Power functions p, q with $\alpha > 0$, $\beta \geq 0$ satisfy conditions (2.7), (2.8), (3.8) (consequently (3.5)). Moreover, if $\beta > \alpha - 1$ then (3.17) holds. Further, we choose $f \in C(\mathbb{R})$ defined for $x \in [L_0, L]$ as a polynomial function:

$$f(x) = k|x|^{\gamma} \operatorname{sgn} x (x - L_0)(L - x), \quad x \in [L_0, L], \tag{3.26}$$

where $\gamma > 0$, $k > 0$. Clearly, assumptions (2.5), (2.6) and (3.14) are satisfied. Moreover, assumption (3.4) holds if $0 < L < -L_0$, and due to Theorem 3.6, we obtain the existence of a solution. In addition, if $\gamma \geq 1$, then f is Lipchitz continuous on $[L_0, L]$ and the uniqueness follows.

4 Bubble-type solution

The goal of this chapter is to prove the existence of a bubble-type solution of problem (2.3), (2.4). Such solution is strictly increasing and satisfies

$$\lim_{t\to\infty} u(t) = L.$$

We point out that a homoclinic solution is actually a bubble-type solution.This relation is described in Lemma 4.9. The proof of the existence of a homoclinic solution is based on a characterisation of the set of escape and damped solutions. Therefore we first investigate damped and escape solutions in more details.

4.1 Damped solutions

In this section, we specify an interval for starting values u_0 where the existence of damped solutions is guaranteed. Moreover, we provide conditions under which each damped solution is oscillatory. Note that by virtue of Definition 2.4 and (3.2), all results of this section are proved for the original problem (2.3), (2.4), since function f coincides with function $\tilde{f}$ on $[L_0, L]$ and $L_0 \le u(t) \le L$ holds for $t \in [0\infty)$ if u is a damped solution. In addition, all lemmas of Section 3 are valid for damped solutions of problem (2.3), (2.4).

Theorem 4.1 (Existence of damped solutions of problem (2.3), (2.4)) *Assume that the assumptions* (2.5)–(2.8)*,* (3.4)*,* (3.5)*,* (3.14) *and* (3.17) *are fulfilled. Then for each* $u_0 \in (\bar{B}, L)$*, problem* (2.3)*,* (2.4) *has a solution u. The solution u is damped and satisfies* (3.18)*. Moreover, if* (3.21) *holds, then the solution u is unique.*

Proof: Choose $u_0 \in (\bar{B}, L)$. By Theorem 3.6, there exists a solution u of the auxiliary problem (3.1), (2.4) satisfying $u(t) > \bar{B}$ for $t \in (0, \infty)$. The idea to prove that u is damped is based on Lemmas 3.1– 3.3 and it is presented in the proof of the Theorem 2.6. in [111]. Since the estimate $\bar{B} < u < L$ holds, then $\tilde{f}(u(t)) = f(u(t))$ for $t \ge 0$. Consequently u is a solution of problem (2.3), (2.4). We refer to [111] for more details.

□

Remark 4.2 If moreover (3.8) is fulfilled, then, by Lemma 3.2, the assertion of Theorem 4.1 holds for $u_0 = \bar{B}$, too. Functions satisfying (3.8) are presented in Example 3.9.

In the study of oscillatory and nonoscillatory properties of solutions, two cases are distinguished according to the convergence or divergence of the integral $\int_1^\infty \frac{1}{p(s)}\,\mathrm{d}s$:

I. We assume that the function p fulfils

$$\int_1^\infty \frac{1}{p(s)}\,\mathrm{d}s < \infty. \tag{4.1}$$

II. We assume that the function p fulfils

$$\int_1^\infty \frac{1}{p(s)}\,\mathrm{d}s = \infty. \tag{4.2}$$

Definition 4.3 A function u is called *eventually positive*, if there exists $t_0 > 0$ such that $u(t) > 0$ for $t \in (t_0,\infty)$. A function u is called *eventually negative*, if there exists $t_0 > 0$ such that $u(t) < 0$ for $t \in (t_0,\infty)$.

The asymptotic behaviour of damped nonoscillatory solutions is specified in the next lemma.

Lemma 4.4 *Assume* (2.5)–(2.8)*,* (3.4)*,* (3.5)*,* (4.1) *and*

$$\lim_{t\to\infty} \int_1^t \frac{1}{p(s)} \int_1^s q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s = \infty. \tag{4.3}$$

Let u be a damped solution of problem (2.3)*,* (2.4) *with $u_0 \in (L_0,0)\cup(0,L)$ which is nonoscillatory. Then*

$$\lim_{t\to\infty} u(t) = 0. \tag{4.4}$$

If moreover

$$\liminf_{t\to\infty} p(t) \int_t^\infty \frac{1}{p(s)}\,\mathrm{d}s > 0 \tag{4.5}$$

then

$$\lim_{t\to\infty} u'(t) = 0. \tag{4.6}$$

Proof: Assume that u is a damped nonoscillatory solution of problem (2.3), (2.4) with $u_0 \in (L_0,0)\cup(0,L)$. Then u is either eventually positive or eventually negative. Firstly, we prove that $\lim_{t\to\infty} u(t) = 0$. Since u is nonoscillatory, Lemma 3.1 guarantees the existence of $t_0 > 1$ such that u is either increasing or decreasing on $[t_0,\infty)$. Therefore, there exists $\lim_{t\to\infty} u(t) = c$. Since $u_{sup} < L$, we have $c < L$. After integrating equation (2.3) from t_0 to t and dividing this by $p(t)$, we get

$$\begin{aligned} u'(t) &= \frac{p(t_0)u'(t_0)}{p(t)} - \frac{1}{p(t)} \int_{t_0}^t q(s) f(u(s))\,\mathrm{d}s, \\ u(t) &= u(t_0) + \int_{t_0}^t \frac{p(t_0)u'(t_0)}{p(s)}\,\mathrm{d}s - \int_{t_0}^t \frac{1}{p(s)} \int_{t_0}^s q(\tau) f(u(\tau))\,\mathrm{d}\tau\,\mathrm{d}s. \end{aligned} \tag{4.7}$$

Let u be eventually positive. Then $c \in [0,L)$. Assume $c \in (0,L)$. Then there exists $M > 0$ such that $f(u(t)) \geq M$ for $t \geq t_0$. From (4.7), we obtain

$$u(t) \leq u(t_0) + p(t_0)u'(t_0) \int_{t_0}^{t} \frac{1}{p(s)}\,\mathrm{d}s - M \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s,$$

$$\lim_{t\to\infty} u(t) \leq u(t_0) + p(t_0)u'(t_0) \int_{t_0}^{\infty} \frac{1}{p(s)}\,\mathrm{d}s - M \lim_{t\to\infty} \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s = -\infty,$$

which contradicts $c \in (0,L)$. Hence $c = 0$.

Let u be eventually negative. If u is negative on $[0,\infty)$, then, by Lemma 3.1 b), we get $u'(t) > 0$ for $t \in (0,\infty)$ and thus $c \in (L_0, 0]$. Now, assume that there exist $a \geq 0$ and $\theta > a$ satisfying (3.11) and $u(t) < 0$ for $t > \theta$. By Lemma 3.3, it occurs either (3.12) or (3.13). If (3.12) holds, then $c \in (\bar{B}, 0)$. If (3.13) holds, then, by Lemma 3.1 b), $c \in (\bar{B}, 0]$. Assume that $c \in (L_0, 0)$. Then there exists $M > 0$ such that $-f(u(t)) \geq M$ for $t \geq t_0$ and, similarly as in the eventually positive case, we derive a contradiction. Therefore, $c = 0$ and (4.4) is proved.

Secondly, we prove that $\lim_{t\to\infty} u'(t) = 0$. Assume in addition that (4.5) is valid. Let u be eventually negative. Then, by (2.6)–(2.8) there exists $t_1 > 0$ such that $u'(t) > 0$ for $t \geq t_1$. By (4.5) there exist $c > 0$ and $t_2 \geq t_1$ such that

$$p(t) \int_{t}^{\infty} \frac{1}{p(s)}\,\mathrm{d}s \geq c > 0\,, \quad t \in [t_2, \infty).$$

From (3.1), (2.5), (2.6) and (2.8) it follows

$$(p(t)u'(t))' = -q(t)\tilde{f}(u(t)) > 0 \text{ for } t \in [t_2, \infty).$$

Function pu' is increasing on interval $[t_2, \infty)$ and we have

$$p(\tau)u'(\tau) \leq p(s)u'(s) \leq p(t)u'(t) \text{ for } t_2 \leq \tau \leq s \leq t.$$

Therefore

$$u(t) - u(\tau) = \int_{\tau}^{t} u'(s)\,\mathrm{d}s = \int_{\tau}^{t} \frac{p(s)u'(s)}{p(s)}\,\mathrm{d}s \geq p(\tau)u'(\tau) \int_{\tau}^{t} \frac{1}{p(s)}\,\mathrm{d}s,$$

$$\lim_{t\to\infty} (u(t) - u(\tau)) \geq p(\tau)u'(\tau) \int_{\tau}^{\infty} \frac{1}{p(s)}\,\mathrm{d}s,$$

$$-u(\tau) \geq u'(\tau)p(\tau) \int_{\tau}^{\infty} \frac{1}{p(s)}\,\mathrm{d}s \geq u'(\tau)c > 0,$$

$$0 = \limsup_{\tau\to\infty} (-u(\tau)) \geq c \limsup_{\tau\to\infty} u'(\tau) \geq 0.$$

By (4.4) we get (4.6). For eventually positive solutions we proceed analogously. □

The oscillation criteria for damped solutions of problem (2.3), (2.4) are studied in [2, 111]. The following result is a corollary to Theorem 3.5 and Theorem 3.6 in [2].

Theorem 4.5 *Let assumptions* (2.5)–(2.8)*,* (3.4)*,* (3.5) *hold and either* (4.1) *and*

$$\liminf_{x\to 0^+}\frac{f(x)}{x}>0,\quad \liminf_{x\to 0^-}\frac{f(x)}{x}>0,$$
$$\int_1^\infty \ell^2(s)q(s)\,\mathrm{d}s=\infty,\quad \textit{where } \ell(t)=\int_t^\infty \frac{1}{p(s)}\,\mathrm{d}s, \tag{4.8}$$

or (4.2) *and*

$$\int_1^\infty q(s)\,\mathrm{d}s=\infty. \tag{4.9}$$

Then each damped solution of problem (2.3)*,* (2.4) *with* $u_0\in(L_0,0)\cup(0,L)$ *to be oscillatory.*

Example 4.6 A power function $p(t)=t^\alpha$ fulfils (4.1) if $\alpha>1$. In addition, let $q(t)=t^\beta$, $\beta\ge 0$, $\beta>\alpha-1$ hold. Condition (4.8) yields

$$\ell(t)=\int_t^\infty\frac{\mathrm{d}s}{s^\alpha}=\frac{t^{1-\alpha}}{\alpha-1},\quad \int_1^\infty \ell^2(s)q(s)\,\mathrm{d}s=\frac{1}{(\alpha-1)^2}\int_1^\infty s^{2-2\alpha+\beta}\,\mathrm{d}s=\infty,$$

provided $\beta\ge 2\alpha-3$. Since the implications

$$\alpha\in(1,2]\Rightarrow 2\alpha-3\le\alpha-1<\beta,$$

$$\alpha>2\Rightarrow\beta\ge 2\alpha-3>\alpha-1,$$

are valid, we deduce that p and q satisfy (2.7), (2.8), (3.8), (3.17) and (4.8) if $\alpha\in(1,2]$, $\alpha-1<\beta$ or $2<\alpha$, $2\alpha-3\le\beta$. Further, let $f\in C(\mathbb{R})$ be sublinear for $x\in[-2,1]$ such that

$$f(x)=\begin{cases}-|x|^a(x+2), & x\in[-2,0],\\ x^b(1-x), & x\in[0,1],\end{cases} \tag{4.10}$$

where $0<a\le b\le 1$. Then $L_0=-2$, $L=1$ and f fulfils (2.5), (2.6), (3.4), (3.14) and (4.8). Consequently, each damped solution is oscillatory by Theorem 4.5.

Example 4.7 In the second case, the integral in (4.2) is divergent for a power function $p(t)=t^\alpha$, $\alpha\le 1$. The condition guaranteeing oscillation of all damped solutions (4.9) holds for $q(t)=t^\beta$, $\beta\ge 0$. Moreover, let $\alpha\in(0,1]$, $\beta>\alpha-1$ and let $f(x)=k|x|^\gamma\operatorname{sgn}x(x-L_0)(L-x)$ for $x\in[L_0,L]$ where $0<L<-L_0$, $\gamma>0$, $k>0$. Then (2.7), (2.8), (3.8) and (3.17) hold. Therefore, each damped solution is oscillatory by Theorem 4.5.

4.2 Escape solutions

In this section, we prove some important properties of escape and homoclinic solutions. In order to obtain existence results, the monotonicity of escape and homoclinic solutions is needed, see Lemma 4.8 and Lemma 4.9. Moreover, we specify asymptotic behaviour of homoclinic solutions in Lemma 4.10. Note that, by Theorem 4.1, a solution of problem (2.3), (2.4) is damped if $u_0 \in (\bar{B}, L)$, $\bar{B} < 0$. Therefore, we can restrict our consideration about escape and homoclinic solutions on $u_0 \in (L_0, 0)$.

Lemma 4.8 (Lemma 4.1 in [2], Escape solution is increasing) *Let assumptions* (2.5)–(2.8) *hold. If a solution u of problem* (3.1)*,* (2.4) *with* $u_0 \in (L_0, 0)$ *is an escape solution, then*

$$\exists c \in (0,\infty): \; u(c) = L, \quad u'(t) > 0 \textit{ for } t \in (0,\infty). \tag{4.11}$$

Lemma 4.9 (Lemma 4.2 in [2], Homoclinic solution is increasing) *Let assumptions* (2.5)–(2.8)*,* (3.4)*,* (3.5) *and* (3.14) *hold. If a solution u of problem* (3.1)*,* (2.4) *with* $u_0 \in (L_0, 0)$ *is homoclinic, then*

$$\lim_{t\to\infty} u(t) = L, \quad u'(t) > 0 \textit{ for } t \in (0,\infty). \tag{4.12}$$

In order to prove further asymptotic properties of homoclinic solutions, we need the condition

$$\liminf_{t\to\infty} p(t) > 0. \tag{4.13}$$

Lemma 4.10 *Assume that* (2.5)–(2.8)*,* (3.4)*,* (3.5) *and* (3.14) *hold. Further, assume that either condition* (4.1) *is valid or conditions* (4.2) *and* (4.13) *are fulfilled. If a solution u of problem* (3.1)*,* (2.4) *with* $u_0 \in (L_0, 0)$ *is homoclinic, then u fulfils*

$$\lim_{t\to\infty} u'(t) = 0. \tag{4.14}$$

Proof: According to Lemma 4.9, u fulfils (4.12). Hence, there exists $t_0 > 0$ such that $u(t_0) = 0$, $u > 0$ and $\tilde{f}(u) > 0$ on (t_0, ∞). We have $(pu')' < 0$ and function pu' is decreasing on (t_0, ∞). Since $p > 0$, $u' \geq 0$ on $[0, \infty)$, there exists

$$\lim_{t\to\infty} p(t)u'(t) \geq 0. \tag{4.15}$$

Assume (4.1), then we have $\lim_{t\to\infty} \frac{1}{p(t)} = 0$ and $\lim_{t\to\infty} p(t) = \infty$. Since pu' is decreasing, we obtain from (4.15)

$$0 \leq \lim_{t\to\infty} p(t)u'(t) < p(t_0)u'(t_0) < \infty,$$

and (4.14) follows.
Assume (4.2) and (4.13). By (4.15), we have

$$\lim_{t\to\infty} p(t)u'(t) = K \geq 0.$$

Let $K > 0$. Then $p(t)u'(t) \geq K$ for $t \geq t_0$ and

$$u'(t) \geq \frac{K}{p(t)}, \quad t \geq t_0,$$

$$u(t) - u(t_0) \geq K \int_{t_0}^{t} \frac{\mathrm{d}s}{p(s)}, \quad t \geq t_0.$$

We let $t \to \infty$ and get by (4.2) and (4.12) that $L \geq K \cdot \infty$, a contradiction. Therefore, $K = 0$ and, due to (4.13), we have (4.14). □

The following lemma is essential for the existence of escape solutions. For illustration of sequences $\{b_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ used below, see Figure 4.1.

Lemma 4.11 *Let* (2.5)–(2.8), (3.4), (3.14) *and either* (4.1), (4.8), *or* (4.2), (4.13) *be satisfied. Further we assume*

$$(pq)' > 0 \text{ on } (0,\infty), \tag{4.16}$$

$$\lim_{t\to\infty} \frac{(p(t)q(t))'}{q^2(t)} = 0, \tag{4.17}$$

$$\liminf_{t\to\infty} \frac{p(t)}{q(t)} > 0, \tag{4.18}$$

$$\liminf_{t\to\infty} q(t) > 0. \tag{4.19}$$

Choose $C \in (L_0, \bar{B})$ *and* $\{B_n\}_{n=1}^{\infty} \subset (L_0, C)$. *Let for each* $n \in \mathbb{N}$, u_n *be a solution of problem* (3.1), (2.4) *with* $u_0 = B_n$ *and let* $(0, b_n)$ *be the maximal interval such that*

$$u_n(t) < L, \quad u_n'(t) > 0, \quad t \in (0, b_n). \tag{4.20}$$

Finally, assume that for $n \in \mathbb{N}$ *there exist* $\gamma_n \in (0, b_n)$ *such that*

$$u_n(\gamma_n) = C \text{ and } \{\gamma_n\}_{n=1}^{\infty} \text{ is unbounded.} \tag{4.21}$$

Then the sequence $\{u_n\}_{n=1}^{\infty}$ *contains an escape solution of problem* (3.1), (2.4).

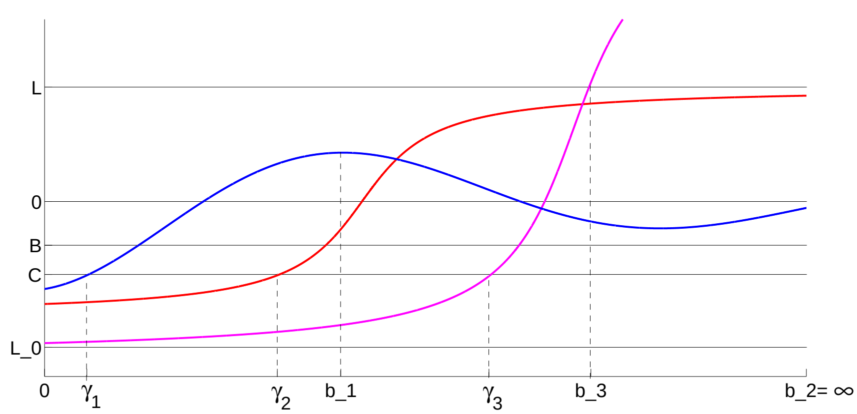

Figure 4.1: Illustration to Lemma 4.11

Proof: Since the sequence $\{\gamma_n\}_{n=1}^\infty$ is unbounded, there exists a subsequence going to infinity as $n \to \infty$. For simplicity, let us denote it by $\{\gamma_n\}_{n=1}^\infty$. Then we have

$$\lim_{n\to\infty} \gamma_n = \infty, \quad \gamma_n < b_n, \quad n \in \mathbb{N}. \tag{4.22}$$

Assume on the contrary that for any $n \in \mathbb{N}$, u_n is not an escape solution of problem (3.1), (2.4).

Step 1. Choose $n \in \mathbb{N}$. Then we have two possibilities:

1. u_n is a damped solution. Then, if (4.1) and (4.8) hold, we get, by Theorem 4.5, that u_n is oscillatory. If (4.2) and (4.13) hold, we can use Theorem 4.5, because (4.19) yields (4.9), and we get that u_n is oscillatory, again.

2. u_n is a homoclinic solution, which yields $b_n = \infty$ (*cf.* Lemma 4.9) and we write $u_n(b_n) = \lim_{t\to\infty} u_n(t) = L$. By Lemma 4.10, u_n fulfils (4.14) and hence $u_n'(b_n) = 0$.

Therefore, we have

$$u_n(b_n) \in (0, L], \quad u_n'(b_n) = 0, \tag{4.23}$$

for both $b_n < \infty$ and $b_n = \infty$. In addition,

$$\exists \bar{\gamma}_n \in [\gamma_n, b_n) : u_n'(\bar{\gamma}_n) = \max\{u_n'(t) : t \in [\gamma_n, b_n)\}. \tag{4.24}$$

Due to (3.1), u_n fulfils

$$\tilde{f}(u_n(t))u_n'(t) = -\frac{p(t)u_n'(t)(p(t)u_n'(t))'}{p(t)q(t)}, \quad t \in (0, b_n). \tag{4.25}$$

Further, we put

$$E_n(t) = \frac{(p(t)u_n'(t))^2}{2}\frac{1}{p(t)q(t)} + \tilde{F}(u_n(t)), \quad t \in (0, b_n). \tag{4.26}$$

Then, by (4.25),

$$\begin{aligned}\frac{\mathrm{d}E_n(t)}{\mathrm{d}t} &= \frac{(p(t)u_n'(t))\,(p(t)u_n'(t))'}{p(t)q(t)} + \frac{(p(t)u_n'(t))^2}{2}\left(\frac{1}{p(t)q(t)}\right)' + \tilde{f}(u_n(t))u_n'(t) \\ &= \frac{(p(t)u_n'(t))^2}{2}\left(\frac{1}{p(t)q(t)}\right)' = -\frac{(p(t)u_n'(t))^2}{2}\frac{(p(t)q(t))'}{(p(t)q(t))^2}, \quad t \in (0, b_n).\end{aligned}$$

By virtue of (2.8), (4.16) and (4.20), we have

$$\frac{\mathrm{d}E_n(t)}{\mathrm{d}t} = -\frac{u_n'^2(t)}{2q^2(t)}\,(p(t)q(t))' < 0, \quad t \in (0, b_n). \tag{4.27}$$

After integrating (4.27) over (γ_n, b_n) and using (4.20), (4.24), we obtain

$$E_n(\gamma_n) - E_n(b_n) = \int_{\gamma_n}^{b_n} \frac{u_n'^2(t)\,(p(t)q(t))'}{2q^2(t)}\,\mathrm{d}t \le u'(\overline{\gamma}_n) \int_{\gamma_n}^{b_n} \frac{u_n'(t)\,(p(t)q(t))'}{2q^2(t)}\,\mathrm{d}t$$
$$\le u'(\overline{\gamma}_n) K_n \int_{\gamma_n}^{b_n} u_n'(t)\,\mathrm{d}t,$$

where

$$K_n = \sup\left\{ \frac{(p(t)q(t))'}{2q^2(t)} : t \in (\gamma_n, b_n) \right\} \in (0,\infty).$$

Hence, we have

$$E_n(\gamma_n) \le E_n(b_n) + u_n'(\overline{\gamma}_n) K_n (L - C). \tag{4.28}$$

By virtue of (2.7), (2.8), (4.20) and (4.21), we get from (4.26)

$$E_n(\gamma_n) > \tilde{F}(u_n(\gamma_n)) = \tilde{F}(C). \tag{4.29}$$

Since $\tilde{F}$ is increasing on $[0,L]$, (4.23) and (4.26) give for $b_n < \infty$

$$E_n(b_n) \le \tilde{F}(u_n(b_n)) \le \tilde{F}(L). \tag{4.30}$$

Let $b_n = \infty$, which means that u_n is homoclinic and $\lim_{t\to\infty} u_n(t) = L$. Then there exists $t_0 > 0$ such that $u_n(t) > 0$ and $\tilde{f}(u_n(t)) > 0$ for $t \in [t_0, \infty)$. Therefore, pu_n' is decreasing on $[t_0, \infty)$. Due to (2.7) and (4.20), $p > 0, u_n' > 0$ on $(0,\infty)$, and hence

$$0 \le \lim_{t\to\infty} p(t) u_n'(t) < p(t_0) u_n'(t_0) < \infty.$$

Therefore, by using (4.19), (4.23), we get

$$0 \le \limsup_{t\to\infty} \frac{p(t)}{q(t)} u_n'(t) < \infty$$

and

$$\lim_{t\to\infty} \frac{p(t)}{q(t)} u_n'^2(t) = 0.$$

Consequently, (4.30) is valid also for $b_n = \infty$. Further, using (4.28)–(4.30), we derive

$$\tilde{F}(C) < E_n(\gamma_n) \le \tilde{F}(L) + u_n'(\bar{\gamma}_n) K_n (L - C), \tag{4.31}$$

and hence,

$$\frac{\tilde{F}(C) - \tilde{F}(L)}{L - C} \frac{1}{K_n} < u_n'(\bar{\gamma}_n). \tag{4.32}$$

Step 2. At this point, we consider the sequence $\{u_n\}_{n=1}^{\infty}$. Assumptions (4.17) and (4.22) imply

$$\lim_{n\to\infty} K_n = 0, \tag{4.33}$$

which by (4.32) yields

$$\lim_{n\to\infty} u_n'(\bar{\gamma}_n) = \infty. \tag{4.34}$$

Since $\tilde{F} \geq 0$ on $[l_0, L]$, we get from (4.26)

$$E_n(\bar{\gamma}_n) \geq \frac{p(\bar{\gamma}_n) u_n'^2(\bar{\gamma}_n)}{2q(\bar{\gamma}_n)}, \quad n \in \mathbb{N}.$$

Further, since E_n is decreasing on $(0, b_n)$ according to (4.27), we derive from (4.31)

$$\frac{p(\overline{\gamma}_n) u_n'^2(\overline{\gamma}_n)}{2q(\overline{\gamma}_n)} \leq E_n(\overline{\gamma}_n) \leq E_n(\gamma_n) \leq \tilde{F}(L) + u_n'(\overline{\gamma}_n) K_n(L - C), \quad n \in \mathbb{N}.$$

Consequently,

$$u_n'(\overline{\gamma}_n) \left(\frac{p(\overline{\gamma}_n)}{2q(\overline{\gamma}_n)} u_n'(\overline{\gamma}_n) - K_n(L - C) \right) \leq \tilde{F}(L) < \infty, \quad n \in \mathbb{N}. \tag{4.35}$$

Due to (4.18), (4.33) and (4.34),

$$\lim_{n\to\infty} \left(\frac{p(\overline{\gamma}_n)}{2q(\overline{\gamma}_n)} u_n'(\overline{\gamma}_n) - K_n(L - C) \right) = \infty. \tag{4.36}$$

Conditions (4.34)–(4.36) yield a contradiction. Therefore, the sequence $\{u_n\}_{n=1}^{\infty}$ contains an escape solution of problem (3.1),(2.4). □

All next existence theorems for escape and homoclinic solutions have the following common assumptions:

$$\text{(2.5)–(2.8), (3.4), (3.17), (3.21) and (4.16)–(4.19).} \tag{4.37}$$

Provided that each damped solution is oscillatory, we establish the existence results for two cases which are characterized by conditions (4.1) and (4.2). Therefore, we use in addition either assumptions

$$\text{(4.1) and (4.8)} \tag{4.38}$$

or assumptions

$$\text{(4.2) and (4.13).} \tag{4.39}$$

Under these assumptions, we prove that problem (3.1), (2.4) has at least one escape solution.

Theorem 4.12 (Existence of an escape solution of problem (3.1), (2.4)) *Assume that* (4.37) *and either* (4.38) *or* (4.39) *hold. Then there exists at least one escape solution of problem* (3.1)*,* (2.4).

Proof: Choose $n \in \mathbb{N}$, $C \in (L_0, \bar{B})$ and $B_n \in (L_0, C)$. By Theorem 3.8, there exists a unique solution u_n of problem (3.1), (2.4) with $u_0 = B_n$. By Lemma 3.1 b), there exists a maximal $a_n > 0$ such that $u_n' > 0$ on $(0, a_n)$. Since $u_n(0) < 0$, there exists a maximal $\tilde{a}_n > 0$ such that $u_n < L$ on $[0, \tilde{a}_n)$. If we put $b_n = \min\{a_n, \tilde{a}_n\}$, then (4.20) holds. If u_n is damped, then, by Theorem 4.5, u_n is oscillatory (*cf.* Step 1. in the proof of Lemma 4.11), and hence there exists $\gamma_n \in (0, b_n)$ such that $u_n(\gamma_n) = C$. If u_n is not damped, then it is either a homoclinic or an escape solution (*cf.* Remark 3.7), and clearly, there exists $\gamma_n \in (0, b_n)$ satisfying $u_n(\gamma_n) = C$. Consider a sequence $\{B_n\}_{n=1}^{\infty} \subset (L_0, C)$. Then we get the sequence $\{u_n\}_{n=1}^{\infty}$ of solutions of problem (3.1), (2.4) with $u_0 = B_n$, and the corresponding sequence of $\{\gamma_n\}_{n=1}^{\infty}$. Assume that $\lim_{n\to\infty} B_n = L_0$. Then, by Theorem 3.8, the sequence $\{u_n\}_{n=1}^{\infty}$ converges locally uniformly on $[0, \infty)$ to the constant function $u \equiv L_0$. Therefore, $\lim_{n\to\infty} \gamma_n = \infty$ and (4.21) is valid. Consequently, according to Lemma 4.11, the sequence $\{u_n\}_{n=1}^{\infty}$ contains an escape solution of problem (3.1), (2.4). □

4.3 Homoclinic solution

The goal of this section is to give sufficient conditions for the existence of escape and homoclinic solutions of problem (2.3), (2.4). First, we analyse problem (3.1), (2.4) and we proceed by generalizing these results to problem (2.3), (2.4).

The next theorem provides the existence of a homoclinic solution of problem (3.1), (2.4). The proof is based on a description of sets of initial values of damped and escape solutions.

Theorem 4.13 (Existence of a homoclinic solution of problem (3.1), (2.4)) *Assume that* (4.37) *and either* (4.38) *or* (4.39) *hold. Then there exists a homoclinic solution of problem* (3.1)*,* (2.4)*.*

Proof: Step 1. Let $\mathscr{M}_d \subset (L_0, 0)$ be the set of all $u_0 \in (L_0, 0)$ such that the corresponding solutions of problem (3.1), (2.4) are damped. By Theorem 4.1 and its proof, $\mathscr{M}_d$ is nonempty. Let us choose $u_0 \in \mathscr{M}_d$ and let u be the corresponding solution of problem (3.1), (2.4). Then, according to Theorem 4.5, u is oscillatory. Therefore, there exist $0 < a_1 < b_1$ such that

$$u(a_1) = A_1 > 0, \quad u(b_1) = B_1 < 0.$$

Choose $\varepsilon > 0$ satisfying

$$\varepsilon < \frac{1}{2}\min\{A_1, |B_1|\}.$$

Let v be the solution of equation (3.1) satisfying $v(0) = v_0 \in (L_0, 0)$. By Theorem 3.8, there exists $\delta > 0$ such that

$$|v_0 - u_0| < \delta \Rightarrow \|u - v\|_{C^1[0,b_1]} < \varepsilon.$$

Consequently,

$$u(t)-\varepsilon < v(t) < u(t)+\varepsilon \text{ for } t \in [0,b_1]\,,$$

and

$$v(a_1) > \frac{A_1}{2} > 0, \quad v(b_1) < \frac{B_1}{2} < 0.$$

Therefore, if $|v_0-u_0| < \delta$, then v is not an increasing function, and so v is damped (*cf.* Lemmas 4.8, 4.9 and Remark 3.7) . Altogether, if $u_0 \in \mathscr{M}_d$, then $(u_0-\delta, u_0+\delta) \subset \mathscr{M}_d$, and finally $\mathscr{M}_d$ is open in $(L_0,0)$.

Step 2. Let $\mathscr{M}_e \subset (L_0,0)$ be the set of all $u_0 \in (L_0,0)$ such that the corresponding solutions of problem (3.1), (2.4) are escape solutions. By Theorem 4.12, $\mathscr{M}_e$ is nonempty. Let us choose $u_0 \in \mathscr{M}_e$ and let u be the corresponding escape solution of problem (3.1), (2.4). Then u fulfils (4.11). Hence, there exists $c_1 > c$ such that

$$u(c_1) = L_1 > L.$$

Choose $\varepsilon > 0$ satisfying

$$\varepsilon < \frac{1}{2}(L_1 - L)\,.$$

Let v be the solution of equation (3.1) satisfying $v(0) = v_0 \in (L_0,0)$. By Theorem 3.8, there exists $\delta > 0$ such that

$$|v_0-u_0| < \delta \Rightarrow \|u-v\|_{C^1[0,c_1]} < \varepsilon.$$

Consequently,

$$u(t)-\varepsilon < v(t) < u(t)+\varepsilon \quad \text{for } t \in [0,c_1]$$

and

$$v(c_1) > \frac{1}{2}(L+L_1) > L.$$

Therefore, due to Remark 3.7, if $|v_0-u_0| < \delta$, then v is an escape solution. Finally, if $u_0 \in \mathscr{M}_e$, then $(u_0-\delta, u_0+\delta) \subset \mathscr{M}_e$, and $\mathscr{M}_e$ is open in $(L_0,0)$.

Step 3. Let $\mathscr{M}_h \subset (L_0,0)$ be defined by

$$\mathscr{M}_h = (L_0,0) \setminus (\mathscr{M}_d \cup \mathscr{M}_e)\,.$$

Since $\mathscr{M}_d \cup \mathscr{M}_e$ is nonempty and open set in $(L_0,0)$, $\mathscr{M}_h$ has to be nonempty and closed in $(L_0,0)$. In addition, if we choose $u_0 \in \mathscr{M}_h$, then the corresponding solution of problem (3.1), (2.4) fulfils $u_{sup} = L$, and, due to Remark 3.7, u is a homoclinic solution of problem (3.1), (2.4). $\square$

Finally, we extend the existence results from Theorem 4.12 and Theorem 4.13 to problem (2.3), (2.4) and reach the main purpose of this paper.

Theorem 4.14 (Existence of an escape solution of problem (2.3), (2.4)) *Assume that* (4.37) *and either* (4.38) *or* (4.39) *hold. Then there exist constant* $c \in (0,\infty)$ *and function u such that u is an escape solution of problem* (2.3), (2.4) *on* $[0,c]$.

Proof: By Theorem 4.12, there exists an escape solution u of problem (3.1), (2.4). By Lemma 4.8, u fulfils (4.11). Due to (3.2) u, is an escape solution of problem (2.3), (2.4) on $[0,c]$. □

Theorem 4.15 (Existence of a homoclinic solution of problem (2.3), (2.4)) *Assume that* (4.37) *and either* (4.38) *or* (4.39) *hold. Then there exists a homoclinic solution of problem* (2.3), (2.4).

Proof: By Theorem 4.13, there exists a homoclinic solution u of problem (3.1), (2.4). Due to (3.2), u is a homoclinic solution of problem (2.3), (2.4), as well. □

We conclude with examples where functions p,q and f were chosen in such a way that problem (2.3), (2.4) has at least one homoclinic solution.

Example 4.16 Consider $p(t) = t^\alpha$ and $q(t) = t^\beta$, where $\alpha \in (1,2]$, $\alpha - 1 < \beta \leq \alpha$ or $\alpha \in (2,3]$, $2\alpha - 3 \leq \beta \leq \alpha$. Note that (4.18) follows from $\beta \leq \alpha$ and that the inequality $2\alpha - 3 \leq \alpha$ yields $\alpha \leq 3$. Let $f \in C(\mathbb{R})$ be given by (4.10) where $a = b = 1$. Then $L_0 = -2$, $L = 1$. Then assumptions (4.37) and (4.38) are satisfied. Due to Theorem 4.15 there exists a homoclinic solution of problem (2.3), (2.4).

Example 4.17 Consider $p(t) = t^\alpha$ and $q(t) = t^\beta$, where $\alpha \in (0,1]$, $\beta \geq 0$, $\alpha - 1 < \beta \leq \alpha$. Let $f \in C(\mathbb{R})$ be given by (3.26) with $0 < L < -L_0$, $\gamma \geq 1$, $k > 0$. Then assumptions (4.37) and (4.39) of Theorem 4.15 are satisfied and hence, problem (2.3), (2.4) has a homoclinic solution.

Example 4.18 Consider the initial value problem

$$\begin{aligned} \left(t^2 u'(t)\right)' + \sqrt{t^3}\, u(t)\,(1-u(t))\,(u(t)+4) = 0, \quad t > 0, \\ u(0) = u_0 \in [-4,1], \quad u'(0) = 0. \end{aligned} \tag{4.40}$$

Here $L_0 = -4$, $L = 1$, $\bar{B} = \sqrt{6} - 3$, $f(x) = x(1-x)(x+4)$, $p(t) = t^2$, $q(t) = \sqrt{t^3}$. According to Example 3.9, we see that (3.8) and all assumptions of Theorems 3.6, 3.8 and 4.1 are satisfied. By Remark 4.2, for each $u_0 \in [\bar{B}, L)$ there exists a unique damped solution u of problem (4.40). Since all assumptions of Theorem 4.5 are valid, we get that if $u_0 \neq 0$, then the corresponding solutions u is oscillatory. In addition, assumptions (4.37) and (4.38) of Theorems 4.14 and 4.15 are fulfilled. Therefore, there exists at least one homoclinic solution of problem (4.40). Further, having in mind that $\mathscr{M}_e \subset (L_0, 0)$ is open (*cf.* Step 2 in the proof of Theorem 4.13, we get infinitely many escape solutions u of problem (4.40) on $[0,c]$, where c can be different for different solutions. By Theorem 4.1, we have $\mathscr{M}_e \subset (L_0, \bar{B})$.

Example 4.19 Consider problem (2.3), (2.4) with $p(t) = \sqrt[4]{t^7}$, $q(t) = \sqrt[4]{t^5} + \arctan t$, $t \in [0, \infty)$, and

$$f(x) = \begin{cases} x(1-x)(x+3) & \text{for } x > 0, \\ \frac{7}{13}x(1-x)(x+2) & \text{for } x \le 0. \end{cases}$$

Here $L_0 = -2$, $L = 1$, $\bar{B} = -1$ and we can verify that (3.8) and all assumptions of Theorems 3.6, 3.8, 4.1, 4.5 as well as assumptions (4.37) and (4.38) of Theorems 4.14 and 4.15 are satisfied. Therefore, the there exists at least one homoclinic solutions of problem (2.3), (2.4).

Example 4.20 Consider problem (2.3), (2.4) with

$$p(t) = t, \quad q(t) = \sqrt{t}, \quad t \in [0, \infty).$$

Then p and q satisfy (2.7), (2.8), (3.17), (4.2), (4.13) and (4.16)–(4.19). Further, let $f \in C(\mathbb{R})$ be such that

$$f(x) = x^2 \operatorname{sgn}(x)(1-x)(x+3).$$

Then $L_0 = -3$, $L = 1$ and f fulfils (2.5), (2.6), (3.4) and (3.21). Therefore, assumptions (4.37), (4.39) of Theorems 4.14 and 4.15 are fulfilled and problem (2.3), (2.4) has a homoclinic solution.

5 Kneser solutions

This chapter is devoted to the investigation of the existence of damped nonoscillatory solutions to problem (2.3), (2.4). Since under the assumptions of Theorem 5.8 each such solution is a Kneser solution, we establish existence criteria for Kneser solutions of (2.3), (2.4) and study the asymptotics of these solutions. In Section 5.1 the existence of Kneser solutions to singular problem (2.3), (2.4) is investigated. We restrict our attention to the case where $q \equiv p$ and study the equation:

$$(p(t)u'(t))' + p(t)f(u(t)) = 0, \tag{5.1}$$

and provide existence criteria for positive and negative Kneser solutions of such problem. The existence of Kneser solutions to general problem (2.3), (2.4)

$$(p(t)u'(t))' + q(t)f(u(t)) = 0,$$

is more involved. Therefore, in Section 5.2, we restrict our consideration on the interval $[a,\infty), a > 0$, where equation (2.3) is regular. For the reader's convenience, we recapitulate the definition of Kneser solution and we introduce damped Kneser solutions.

Definition 5.1 A solution u of equation (2.3) on $[a,\infty), a \ge 0$, is called a *Kneser solution* if there exists $t_0 > a$ such that

$$u(t)u'(t) < 0 \text{ for } t \in [t_0,\infty). \tag{5.2}$$

We say that u is a damped Kneser solutions provided in addition

$$\sup\{u(t) : t \in [a,\infty)\} < L.$$

Remark 5.2 Equation (2.3) have no damped Kneser solutions in the case that

$$\int_1^\infty \frac{\mathrm{d}s}{p(s)} = \infty. \tag{5.3}$$

This follows from (2.6), (2.7), (2.8) and the following arguments: Let u be a damped eventually positive solution of (2.3) on $[a,\infty)$, that is

$$u(t) > 0,\ t \ge a_1 \tag{5.4}$$

for some $a_1 \ge a \ge 0$. Then, pu' is decreasing for $t \ge a_0$. Assume that $pu' < 0$ for $t \ge t_1 \ge a_1$. By integrating the inequality $p(t)u'(t) < p(t_1)u'(t_1) = K < 0$, we obtain

$$u(t) < u(t_1) + K\int_{t_1}^t \frac{\mathrm{d}s}{p(s)}.$$

Therefore, as t tends to infinity, $\lim_{t\to\infty} u(t) = -\infty$ contradicts (5.4). This means that $u' > 0$ on $[t_0,\infty)$. Hence, any damped eventually positive solution of (2.3) is increasing and there exists no Kneser solution to (2.3). Similar arguments can be given for damped eventually negative solutions of (2.3).

In Theorem 4.1, it was proved that for each $u_0 \in [\bar{B}, L)$ there exists a solution of problem (2.3), (2.4) which is damped. On the other hand, if $u_0 \in (L_0, \bar{B})$, then the corresponding solution of problem (2.3), (2.4) need not be damped. We demonstrate the existence of damped nonoscillatory solutions in the next example.

Example 5.3 Consider the equation

$$\left(t^3 u'(t)\right)' + t^3 f(u(t)) = 0, \tag{5.5}$$

where

$$f(x) = \begin{cases} -12 - 2x & \text{for } x < -2, \\ x^3 & \text{for } x \in [-2,1], \\ 2 - x & \text{for } x > 1. \end{cases}$$

Here $L_0 = -6,\ L = 2,\ \bar{B} = -\sqrt[4]{3},\ p(t) = t^3$. By Theorem 4.1 and Remark 4.2 problem (5.5), (2.4) with $u_0 \in [-\sqrt[4]{3}, 2)$ has a unique solution u which is damped. What is more, we can check by a direct computation that for $u_0 \in [-2,1]$, problem (5.5), (2.4) has a solution

$$u(t) = \frac{8u_0}{8 + u_0^2 t^2},\ t \in [0,\infty).$$

If $u_0 \in (0,1]$, the solution u is positive and decreasing in $[0,\infty)$, and hence it is damped nonoscillatory. Similarly, we see that if $u_0 \in [-2,0)$, then u is negative and increasing on $[0,\infty)$, and so it is also damped and nonoscillatory. On the other hand, numerical simulations give damped oscillatory solutions provided $u_0 \in (1,2)$, see Figure 5.1. Moreover, the set of all initial values $u_0 < 0$ such that the corresponding solutions of problem (5.5), (2.4) are damped is open in $(-\infty, 0)$. See Theorem 17 in [105]. This theorem is also valid in the case where nonoscillatory solutions are not excluded, but only for $p \equiv q$ and under other additional assumptions. On this example we also demonstrate that there are problems for which more solutions are damped, not only those starting at $u_0 \in [\bar{B}, L)$.

5.1 Singular problem

The main purpose of this section is to prove the existence of Kneser solutions to singular problem (5.1), (2.4)

$$(p(t)u'(t))' + p(t)f(u(t)) = 0,\ t \in [0,\infty),\ u(0) = u_0 \in (L_0, L),\ u'(0) = 0.$$

In the whole section, Kneser solutions are considered on the half-line $[0,\infty)$.

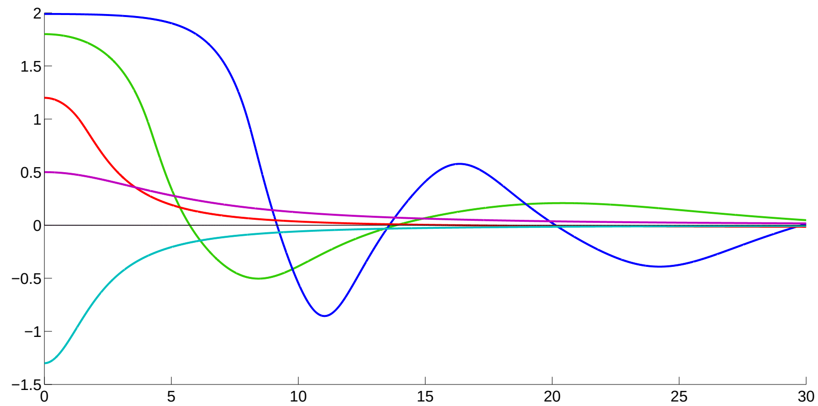

Figure 5.1: Simulations for Example 5.3

In order to prove *sufficient* conditions for the existence of Kneser solutions of equation (5.1), we need identities (5.7), (5.8) of the Pohozhaev type. In general, identities of the Pohozhaev type have shown to be useful to investigate the existence and nonexistence of positive radial solutions of quasilinear elliptic equations which are related to second order ordinary differential equations of the Emden–Fowler type, see for example [42]. Here we use the notation

$$P(t) = \int_0^t p(s)\,\mathrm{d}s, \quad F(x) = \int_0^x f(s)\,\mathrm{d}s.$$

In order to derive the existence of Kneser solutions, we need the additional assumption

$$p \in C^1(0,\infty). \tag{5.6}$$

Lemma 5.4 *Let u be a solution of equation* (5.1). *Let $f \in C(\mathbb{R})$ and let* (5.6) *hold. Then u fulfils*

$$p(t)u(t)u'(t) = \int_0^t p(s)u'^2(s)\,\mathrm{d}s - \int_0^t p(s)f(u(s))\,u(s)\,\mathrm{d}s,\ t > 0, \tag{5.7}$$

and

$$\begin{aligned} P(t)\left(\frac{u'^2(t)}{2} + F(u(t))\right) &= \int_0^t p(s)F(u(s))\,\mathrm{d}s \\ &\quad - \int_0^t \left(\frac{p'(s)P(s)}{p^2(s)} - \frac{1}{2}\right) p(s)u'^2(s)\,\mathrm{d}s,\ t > 0. \end{aligned} \tag{5.8}$$

Proof: Consider a solution u of equation (5.1). To derive identity (5.7), we multiply (5.1) by the solution u and obtain

$$\big(p(t)u'(t)\big)'\,u(t) + p(t)f(u(t))\,u(t) = 0,\ t > 0. \tag{5.9}$$

After using (5.9) together with the equality

$$\big(p(t)u'(t)u(t)\big)' = \big(p(t)u'(t)\big)' u(t) + p(t)u'^2(t),\ t > 0,$$

we get

$$\big(p(t)u'(t)u(t)\big)' = p(t)u'^2(t) - p(t)f\left(u(t)\right)u(t),\ t > 0. \tag{5.10}$$

After integrating (5.10) over $(0,t)$ identity (5.7) follows.

Further we multiply equation (5.1) by $\frac{P(t)u'(t)}{p(t)}$ and get

$$\frac{P(t)u'(t)}{p(t)} \big(p(t)u'(t)\big)' + P(t)f\left(u(t)\right)u'(t) = 0,\ t > 0. \tag{5.11}$$

By virtue of the equality

$$\left(P(t)F\left(u(t)\right)\right)' = p(t)F\left(u(t)\right) + P(t)f\left(u(t)\right)u'(t),\ t > 0,$$

and after integrating (5.11) over $(0,t)$ we obtain

$$\int_0^t \frac{P(s)u'(s)}{p(s)} \big(p(s)u'(s)\big)'\,\mathrm{d}s + P(t)F\left(u(t)\right) - \int_0^t p(s)F(u(s))\,\mathrm{d}s = 0,\ t > 0.$$

Furthermore,

$$\int_0^t \frac{P(s)u'(s)}{p(s)} \big(p(s)u'(s)\big)'\,\mathrm{d}s = \frac{P(t)u'^2(t)}{2} - \int_0^t \frac{p(s)u'^2(s)}{2}\,\mathrm{d}s + \int_0^t \frac{P(s)p'(s)}{p(s)}u'^2(s),$$

holds for $t > 0$ where the integration by parts was used. Consequently, we deduce

$$\begin{aligned} P(t)\left(\frac{u'^2(t)}{2} + F\left(u(t)\right)\right) &= \int_0^t p(s)F\left(u(s)\right)\,\mathrm{d}s \\ &\quad - \int_0^t \left(\frac{p'(s)P(s)}{p^2(s)} - \frac{1}{2}\right) p(s)u'^2(s)\,\mathrm{d}s,\ t > 0. \end{aligned}$$

□

The existence of a Kneser solution is guaranteed by two following theorems.

Theorem 5.5 *(On the existence of Kneser solutions I.)*
Assume (2.5)–(2.7)*,* (3.4)*,* (3.5)*,* (3.21)*,* (5.6)*,*

$$\lim_{t\to 0^+} \frac{P(t)}{p(t)} = 0, \tag{5.12}$$

and that there exist $c > \frac{1}{2}$ *and* $A_0 \in (0,L)$ *such that the inequalities*

$$\frac{p'(t)P(t)}{p^2(t)} \geq c,\ t \in (0,\infty), \tag{5.13}$$

$$\frac{xf(x)}{F(x)} \geq \frac{2}{2c-1},\ x \in (0,A_0] \tag{5.14}$$

hold. Then for each $u_0 \in (0, A_0]$ there exists a unique solution u of problem (5.1), (2.4). *The solution u is damped and satisfies*

$$u(t) > 0,\ u'(t) < 0,\ t \in (0,\infty). \tag{5.15}$$

Proof: By Theorems 3.8 and 4.1, there exists a unique solution u of problem (5.1), (2.4) with $u_0 \in (0, A_0]$. Suppose, on the contrary, that there exists $t_0 > 0$ such that $u(t) > 0$ on $[0, t_0)$ and $u(t_0) = 0$. Due to Lemma 3.1

$$0 < u(t) < A_0,\ u'(t) < 0,\ t \in (0, t_0]. \tag{5.16}$$

We use equality (5.7) for $t = t_0$

$$p(t_0)u(t_0)u'(t_0) = \int_0^{t_0} p(s)u'^2(s)\,\mathrm{d}s - \int_0^{t_0} p(s)f(u(s))\,u(s)\,\mathrm{d}s.$$

Since $u(t_0) = 0$, we get

$$\int_0^{t_0} p(s)u'^2(s)\,\mathrm{d}s = \int_0^{t_0} p(s)f(u(s))\,u(s)\,\mathrm{d}s. \tag{5.17}$$

Due to (5.8), where $t = t_0$

$$\begin{aligned} 0 &< P(t_0)\left(\frac{u'^2(t_0)}{2} + F(u(t_0))\right) \\ &= \int_0^{t_0} p(s)F(u(s))\,\mathrm{d}s - \int_0^{t_0}\left(\frac{p'(s)P(s)}{p^2(s)} - \frac{1}{2}\right)p(s)u'^2(s)\,\mathrm{d}s. \end{aligned}$$

According to (5.17) and (5.13) we get

$$\begin{aligned} \int_0^{t_0} p(s)F(u(s))\,\mathrm{d}s &> \int_0^{t_0}\left(\frac{p'(s)P(s)}{p^2(s)} - \frac{1}{2}\right)p(s)u'^2(s)\,\mathrm{d}s \\ &\geq \left(c - \frac{1}{2}\right)\int_0^{t_0} p(s)u'^2(s)\,\mathrm{d}s = \frac{2c-1}{2}\int_0^{t_0} p(s)f(u(s))\,u(s)\,\mathrm{d}s. \end{aligned}$$

This yields

$$\int_0^{t_0} p(s)F(u(s))\left(\frac{2}{2c-1} - \frac{f(u(s))\,u(s)}{F(u(s))}\right)\mathrm{d}s > 0. \tag{5.18}$$

Furthermore,

$$\frac{f(u(s))\,u(s)}{F(u(s))} \geq \frac{2}{2c-1},\ s \in (0, t_0),$$

is satisfied according to (5.14) and (5.16). Therefore

$$\left(\frac{2}{2c-1} - \frac{f(u(s))\,u(s)}{F(u(s))}\right) \leq 0,\ s \in (0, t_0),$$

contrary to (5.18). The obtained contradiction implies $u(t) > 0$ on $[0, \infty)$. Due to Lemma 3.1, we get $u'(t) < 0$ for $t \in (0, \infty)$. Hence, (5.15) is valid. Since $A_0 \in (0, L)$, solution u is damped. $\square$

A dual theorem for an initial value u_0 from a left neighbourhood of zero is proved with similar arguments.

Theorem 5.6 *(On the existence of Kneser solutions II.)*
Assume (2.5)–(2.7), (3.4), (3.5), (3.21), (5.6) *and* (5.12) *hold. Let condition* (5.13) *hold with a constant* $c > \frac{1}{2}$ *and assume that there exists* $B_0 \in (L_0, 0)$ *such that the inequality*

$$\frac{xf(x)}{F(x)} \geq \frac{2}{2c-1},\ x \in [B_0, 0) \tag{5.19}$$

is satisfied. Then for each $u_0 \in [B_0, 0)$ *there exists a unique solution u of problem* (5.1), (2.4). *The solution u is damped and satisfies*

$$u(t) < 0,\ u'(t) > 0,\ t \in (0, \infty). \tag{5.20}$$

Proof: By Theorems 3.8 and 4.1, there exists a unique solution u of auxiliary problem (3.1), (2.4) with $u_0 \in [B_0, 0)$. In the contradiction with (5.20), we suppose the existence of $t_0 > 0$ such that $u(t) < 0$ on $[0, t_0)$ and $u(t_0) = 0$. By virtue of Lemma 3.1, the inequality

$$B_0 < u(t) < 0,\ u'(t) > 0,\ t \in (0, t_0] \tag{5.21}$$

holds.

By using the identities of Pohozhaev type (5.7), (5.8) and assumptions (5.13), (5.19) and (5.21) as in the proof of Theorem 5.5, we arrive at a contradiction which implies that t_0 cannot exist. Therefore (5.20) holds. Moreover, $\tilde{f}(u(t)) = f(u(t))$ for $t \geq 0$, and therefore u is a Kneser solution to (5.1), (2.4). Clearly, u is damped.

□

Corollary 5.7 Let u be a solution of equation (5.1) and let assumption (5.14) or (5.19) be fulfilled. If the function $p \in C^1(0, \infty)$ satisfies condition (5.13) only on $(0, t_0]$ then, according to the proof of Theorem 5.5 or Theorem 5.6, each solution u of problem (5.1), (2.4) with $u_0 \in (0, A_0]$ or $u_0 \in [B_0, 0)$ has no zero on $(0, t_0]$.

The asymptotic behaviour of damped nonoscillatory solutions is specified in Lemma 4.4. Below, we show a connection between damped nonoscillatory solutions vanishing at infinity (see Definition 2.3 and 2.4) and damped Kneser solutions.

Theorem 5.8 *Let* (2.5)–(2.7), (3.4), (3.5), (4.1) *and* (4.3) *hold. Then a damped solution of* (5.1) (2.4) *is a Kneser solution if and only if it is a nonoscillatory solution vanishing at infinity.*

Proof: Let u be a damped Kneser solution of problem (5.1), (2.4). Then by (5.2), u is nonoscillatory and due to Lemma 4.4 vanishes at infinity. Let u be a damped nonoscillatory solution which is eventually positive. Then there exists $a > 0$ such that

$$0 < u(t) < L,\ t \geq a. \tag{5.22}$$

By Lemma 4.4, u fulfils (4.4) and hence there exists $t_0 \geq a$ such that

$$u'(t_0) \leq 0. \tag{5.23}$$

After applying (2.6), (2.7)) and (5.22) to equation (5.1), we get $(p(t)u'(t))' < 0$ for $t \geq a$, which together with (5.23) yields

$$u'(t) < 0,\ t \in (t_0, \infty). \tag{5.24}$$

Consequently, by virtue of Definition 2.6 and inequalities (5.22), (5.24), u is a Kneser solution of equation (5.1). If u is eventually negative, we argue similarly.

□

Example 5.9 Let $L_0 < B_0 < 0 < A_0 < L,\ \alpha > 1,\ r > 1$. Consider problem (5.1), (2.4), where

$$p(t) = t^{\alpha},\ t \in [0, \infty), \tag{5.25}$$

$$f(x) = \begin{cases} \dfrac{|B_0|^r (L_0 - x)}{B_0 - L_0} & \text{for } x \in [L_0, B_0), \\ |x|^r \operatorname{sgn} x & \text{for } x \in [B_0, A_0], \\ \dfrac{A_0^r (x - L)}{A_0 - L} & \text{for } x \in (A_0, L], \end{cases} \tag{5.26}$$

such that $F(L_0) > F(L)$. To ensure the existence of Kneser solutions, we apply Theorems 5.5 and 5.6. We see that p satisfies condition (5.13), because

$$\frac{p'(t)P(t)}{p^2(t)} = \frac{\alpha}{\alpha + 1} = c > \frac{1}{2},\ \ t \in (0, \infty).$$

Since

$$\frac{xf(x)}{F(x)} = r + 1 \text{ for } x \in [B_0, 0) \cup (0, A_0],$$

conditions (5.14) and (5.19) are reduced to a simple inequality

$$r + 1 \geq \frac{2}{2c - 1}. \tag{5.27}$$

Lower bounds $\frac{2}{2c-1}$ corresponding to different values of α are given in the table below.

α	2	3	4	5	6	7	8	9	10	11
$\frac{2}{2c-1}$	6.00	4.00	3.33	3.00	2.80	2.67	2.57	2.50	2.44	2.40

For instance, put $\alpha = 3$. Then $c = \frac{3}{4}$ and $\frac{2}{2c-1} = 4$. Therefore, due to (5.27), if $r \geq 3$ we can apply Theorems 5.5 and 5.6. For each $u_0 \in [B_0, 0) \cup (0, A_0]$, problem (5.1), (2.4), where p is given by (5.25) and f by (5.26) has a damped Kneser solution u.

In addition, the function f (5.26) with $r \geq 3$ has a superlinear behaviour at a neighbourhood of origin and the function $p(t) = t^3$ belongs to a set of regularly varying functions defined in Section 6.1. According to Section 6.2, Theorem 6.9 and Theorem 6.10, the asymptotic formulas

$$\lim_{t\to\infty} t^{\frac{2}{r-1}-\varepsilon}|u(t)| = 0,$$
$$\lim_{t\to\infty} t^{\frac{r+1}{r-1}-\varepsilon}|u'(t)| = 0,$$

hold for the Kneser solution u and for arbitrary small $\varepsilon > 0$.

Example 5.10 Consider problem (5.1), (2.4) where

$$p(t) = t^4 + t^3, \ t \in [0, \infty), \tag{5.28}$$

$$f(x) = \begin{cases} -\frac{x+3}{2} & \text{for } \ x < -1 \\ x^3 & \text{for } \ x \in [-1, 1] \\ 2 - x & \text{for } \ x > 1. \end{cases} \tag{5.29}$$

We verify that assumptions (5.13), (5.14) and (5.19) are satisfied. For $t > 0$

$$\frac{p'(t)P(t)}{p^2(t)} = \frac{(4t^3 + 3t^2)(\frac{t^5}{5} + \frac{t^4}{4})}{(t^4 + t^3)^2} = \frac{16t^8 + 32t^7 + 15t^6}{20(t^8 + 2t^7 + t^6)} \geq \frac{3}{4} = c > \frac{1}{2},$$

since

$$16t^8 + 32t^7 + 15t^6 \geq 15t^8 + 30t^7 + 15t^6.$$

Further

$$\frac{xf(x)}{F(x)} = 4 = \frac{2}{2c-1}, \ x \in [-1, 0) \cup (0, 1].$$

Hence, for each $u_0 \in [-1, 0) \cup (0, 1]$ and p, f given by (5.28), (5.29), problem (5.1), (2.4) has a damped Kneser solution u.

Moreover, function f is superlinear at zero and p is a regularly varying function of index 4. Therefore, we can apply asymptotic formulas for u and u':

$$\lim_{t\to\infty} t^{1-\varepsilon}|u(t)| = 0, \ \lim_{t\to\infty} t^{2-\varepsilon}|u'(t)| = 0,$$

for arbitrary small $\varepsilon > 0$ according to Theorems 6.9 and 6.10.

In the following examples we illustrate other types of function p which appear to satisfy condition (5.13)

$$\frac{p'(t)P(t)}{p^2(t)} = c > \frac{1}{2},\ t \in (0,\infty).$$

Example 5.11 Let us consider problem (5.1), (2.4), where

$$p(t) = t^3 + t\cos(t),\ t \in [0,\infty),$$

$$f(x) = \begin{cases} -\frac{x+3}{2} & \text{for } x < -1 \\ x^4 \operatorname{sgn} x & \text{for } x \in [-1,1] \\ 2-x & \text{for } x > 1. \end{cases}$$

The graph (Figure 5.2) of the function

$$\frac{p'(t)P(t)}{p^2(t)} = \frac{3t^2 + \cos(t) - t\sin(t)}{(t^3 + t\cos(t))^2}\left(\frac{t^4}{4} + \cos(t) + t\sin(t)\right)$$

shows that condition (5.13) is satisfied with $c = 0.7$

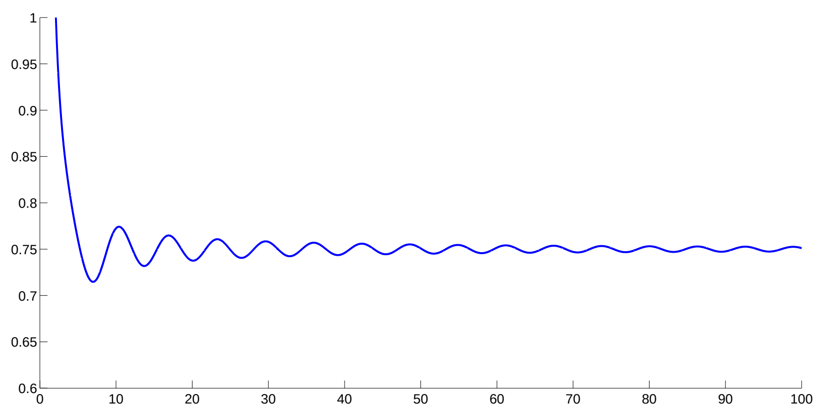

Figure 5.2: Condition (5.13) for $p(t) = t^3 + t\cos(t)$

$$\frac{p'(t)P(t)}{p^2(t)} \geq 0.7,\ t \in (0,\infty). \tag{5.30}$$

Moreover,

$$\lim_{t\to\infty} \frac{p'(t)P(t)}{p^2(t)} = \frac{3}{4}.$$

Since

$$\frac{xf(x)}{F(x)} = 5 = \frac{2}{2c-1},\ x \in [-1,0)\cup(0,1],$$

problem (5.1), (2.4) has for each $u_0 \in [-1,0) \cup (0,1]$ a damped Kneser solution u, according to Theorems 5.5, 5.6. Function p is positive and differentiable and satisfies condition

$$\lim_{t\to\infty} \frac{tp'(t)}{p(t)} = \alpha \in (0,\infty),$$

with $\alpha = 3$. Such functions belong to the set of regularly varying functions of index 3, see Remark 6.11 Therefore, we can apply Theorems 6.9 and 6.10 with $r = 4$. Asymptotic formulas for u and u' are given by

$$\lim_{t\to\infty} t^{2/3-\varepsilon}|u(t)| = 0,\ \lim_{t\to\infty} t^{5/3-\varepsilon}|u'(t)| = 0.$$

Example 5.12 Let us consider problem (5.1), (2.4), where

$$p(t) = \frac{t^3}{1+t},\ t \in [0,\infty),$$

$$f(x) = \begin{cases} -\frac{x+3}{2} & \text{for } x < -1 \\ x^5 & \text{for } x \in [-1,1] \\ 2-x & \text{for } x > 1. \end{cases}$$

The graph (Figure 5.3) of the function

$$\begin{aligned} \frac{p'(t)P(t)}{p^2(t)} &= \frac{t^2(2t+3)}{(t+1)^2}\left(\frac{t^3}{3} - \frac{t^2}{2} + t - \log(t+1)\right)\frac{(1+t)^2}{t^6} \\ &= \frac{1}{6t^4}\big(4t^4 + 3t^2 + 18t - 12t\log(t+1) - 18\log(t+1)\big) \end{aligned} \tag{5.31}$$

shows that $p'(t)P(t)/p^2(t)$ is monotone. In addition

$$\lim_{t\to\infty} \frac{p'(t)P(t)}{p^2(t)} = \frac{2}{3}$$

holds. We put $c = \frac{2}{3}$, then $\frac{2}{2c-1} = 6$ and condition (5.27) with $r = 5$ is satisfied. Therefore, problem (5.1), (2.4) has for each $u_0 \in [-1,0) \cup (0,1]$ a damped Kneser solution, due to Theorem 5.5 and 5.6. In addition, function p is a regularly varying function of index 2 and function f is superlinear at zero with $r = 5$. Therefore, the asymptotic formulas

$$\lim_{t\to\infty} t^{1/2-\varepsilon}|u(t)| = 0,\ \lim_{t\to\infty} t^{3/2-\varepsilon}|u'(t)| = 0$$

hold, according to Theorems 6.9, 6.10, see Section 6.2.

Remark 5.13 We are aware that analytical proofs of estimate (5.30) and monotonicity (5.31) cannot be replaced by any graph. However, we are able to illustrate the fulfilment of assumption (5.13) graphically only.

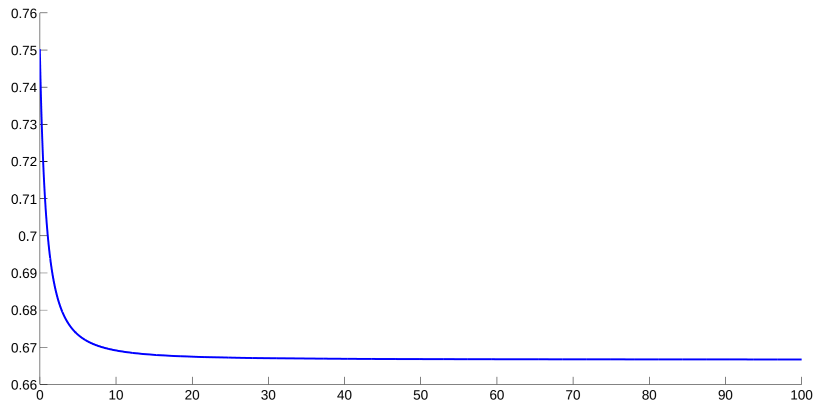

Figure 5.3: Condition (5.13) for $p(t) = \frac{t^3}{1+t}$

5.2 Regular problem

While investigating the existence of damped nonoscillatory solutions of general singular problem (2.3), (2.4) where $p \neq q$, we have been confronted with numerous difficulties. Therefore we begin to study the solvability of equation (2.3)

$$(p(t)u'(t))' + q(t)f(u(t)) = 0$$

on the interval $[a,\infty), a > 0$. Since $a > 0$, equation (2.3) is regular. We investigate (2.3) together with one of the following conditions:

$$u(a) = u_0 \in (0,L),\ 0 \le u(t) \le L \text{ for } t \in [a,\infty), \tag{5.32}$$

or

$$u(a) = u_0 \in (L_0,0),\ L_0 \le u(t) \le 0 \text{ for } t \in [a,\infty). \tag{5.33}$$

Problems (2.3), (5.32) and (2.3), (5.33) are studied under the assumptions:

$$L_0 < 0 < L,\ f \in C[L_0,L],\ f(L_0) = f(0) = f(L) = 0, \tag{5.34}$$

$$p \in C[a,\infty), p > 0 \text{ on } [a,\infty), \tag{5.35}$$

$$q \in C[a,\infty),\ q > 0 \text{ on } (a,\infty). \tag{5.36}$$

The existence result is shown using the Diagonalization Lemma.

Lemma 5.14 (Diagonalization Lemma)
Let $u_n \in C^1[a,n]$, $n \in \mathbb{N}$, $n > a$ be such that for each $b > a$ there exists $\rho_b > 0$ satisfying

$$|u_n^{(j)}(t)| \le \rho_b \ \textit{for } t \in [a,b], n \ge b,\ j = 0,1,$$

and

$$\{u_n'\}_{n \ge b} \textit{ is equicontinuous on } [a,b].$$

Then, there exists a subsequence $\{u_{k_n}\} \subset \{u_n\}$ and $u \in C^1[a,\infty)$ such that

$$\lim_{n\to\infty} u_{k_n}^{(j)}(t) = u^{(j)}(t) \text{ locally uniformly on } [a,\infty), j = 0,1.$$

Proof: Let $\{b_n\} \in \mathbb{N}$ be increasing, $b_n > a$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} b_n = \infty$. Then

(i) for $b_1 \in \mathbb{N}$ we have $|u_n^{(j)}(t)| < \rho_{b_1}$ for $t \in [a,b_1], n \geq b_1, j = 0,1,$ moreover, $\{u_n'\}_{n\geq b_1}$ is equicontinuous on $[a,b_1]$. Hence, by the Arzelà–Ascoli theorem, there is a subsequence $\{u_{k_{1,n}}\} \subset \{u_n\}_{n\geq b_1}$, for which $\{u_{k_{1,n}}^{(j)}(t)\}$ is uniformly convergent on $[a,b_1]$ for $j = 0,1$.

(ii) Next, there exists a subsequence $\{u_{k_{2,n}}\} \subset \{u_{k_{1,n}}\}$ such that $\{u_{k_{2,n}}^{(j)}\}$ is uniformly convergent on $[a,b_2]$ for $j = 0,1$.

(n) We can proceed inductively to obtain a subsequence $\{u_{k_{i,n}}\} \subset \{u_{k_{i-1,n}}\}$ such that $\{u_{k_{i,n}}^{(j)}\}$ is uniformly convergent on $[a,b_i]$ for $j = 0,1$.

Let $k_n := k_{n,n}$ for $n \in \mathbb{N}$ and consider the diagonal sequence $\{u_{k_n}\}$. Choose $\beta > a$. Then $[a,\beta] \subset [a,b_m]$ for some $m \in \mathbb{N}$. Since $\{u_{k_n}\}_{n\geq m}$ is taken from $\{u_{k_{m,n}}\}$ and we know that $\{u_{k_{m,n}}^{(j)}\}$ is uniformly convergent on $[a,b_m]$ for $j = 0,1$, we can see that $\{u_{k_n}^{(j)}\}$ is uniformly convergent on $[a,\beta]$ for $j = 0,1$. Consequently, $\{u_{k_n}^{(j)}\}_{n\geq m}$ is locally uniformly convergent on $[a,\infty)$. Let $\lim_{n\to\infty} u_{k_n}(t) = u(t)$ and $\lim_{n\to\infty} u_{k_n}'(t) = v(t)$ for $t \in [a,\infty)$. Then $u, v \in C[a,\infty)$ and letting $n \to \infty$ in

$$u_{k_n}(t) = u_{k_n}(a) + \int_a^t u_{k_n}'(s)ds,\ t \in [a,n], n \in \mathbb{N},$$

yields

$$u(t) = u(a) + \int_a^t v(s)ds,\ t \in [a,\infty).$$

Hence $u \in C^1[a,\infty)$ and $v = u'$ on $[a,\infty)$ and the result follows. □

The fact that $a > 0$ is crucial for the existence result stated in Theorem 5.15 since in this case equation (2.3) is regular on $[a,\infty)$ and the solvability of (2.3), (5.32) and (2.3), (5.33) can be shown using the following standard arguments.

Before proceeding, we define the function f^* by

$$f^*(x) = \begin{cases} \frac{L-x}{x-L+1}, & x > L, \\ f(x), & x \in [0,L], \\ \frac{x}{x-1}, & x < 0, \end{cases} \tag{5.37}$$

and consider the auxiliary equation

$$\big(p(t)u'(t)\big)' + q(t)f^*(u(t)) = 0. \tag{5.38}$$

Theorem 5.15 *Let assumptions* (5.34)–(5.36) *be satisfied and* $a > 0$. *Then problem* (2.3), (5.32) *(and problem* (2.3), (5.33)*) has at least one solution.*

Proof: Step 1. *Showing solvability of an auxiliary Dirichlet problem*:
Let $n > a$, $u_0 \in (0, L)$ and let us assume the Dirichlet boundary conditions

$$u(a) = u_0,\ u(n) = 0 \tag{5.39}$$

hold. We first prove the existence of a solution to problem (5.38), (5.39). The linear homogeneous problem

$$\big(p(t)u'(t)\big)' = 0,\ u(a) = 0, u(n) = 0, \tag{5.40}$$

has only the trivial solution $u \equiv 0$. Indeed, the assumption of the existence of a nontrivial solution results in the following contradiction. Let u be a nontrivial solution of (5.40). Then there exists $\theta \in (a, n)$ such that $u(\theta) \neq 0, u'(\theta) = 0$. After integrating (5.40) from θ to $t \in [a, n]$, we obtain

$$p(t)u'(t) = p(\theta)u'(\theta) = 0,\ t \in [a, n].$$

Since p is positive on $[a, \infty)$, $u' = 0$ on $[a, n]$ follows and hence, u has to be a constant function on $[a, n]$. Therefore, $u \equiv 0$ is the only solution to (5.40). Consequently, there exists the unique Green's function $G(t, s)$ to problem (5.40)

$$G(t,s) = \begin{cases} \left(1 - \dfrac{P(s)}{P(n)}\right) P(t) \ \text{ for } \ a \le t \le s \le n, \\ \left(1 - \dfrac{P(t)}{P(n)}\right) P(s) \ \text{ for } \ a \le s \le t \le n, \end{cases} \tag{5.41}$$

where $P(t) = \int_a^t \frac{\mathrm{d}\tau}{p(\tau)}, t \in [a, n]$. The Green's function (5.41) is bounded by

$$|G(t, s)| \le P(n) \text{ for } t, s \in [a, n].$$

Moreover, the partial derivative of $G(t, s)$ has the form

$$\frac{\partial G(t,s)}{\partial t} = \begin{cases} \left(1 - \dfrac{P(s)}{P(n)}\right) \dfrac{1}{p(t)} \ \text{ for } \ a \le t < s \le n, \\ -\dfrac{P(s)}{P(n)} \dfrac{1}{p(t)} \qquad \text{ for } \ a \le s < t \le n, \end{cases}$$

and it is also bounded,

$$\left| \frac{\partial G(t,s)}{\partial t} \right| \le \frac{1}{p_{min}},\ t \in [a, s) \cup (s, n], s \in [a, n],$$

where $p_{min} = \min\{p(t) : t \in [a, n]\} > 0$.

Then, the nonhomogeneous linear problem

$$\big(p(t)u'(t)\big)' = y(t),\ u(a) = u_0, u(n) = 0,$$

has the unique solution

$$u(t) = \frac{u_0}{P(n)} \int_t^n \frac{\mathrm{d}\tau}{p(\tau)} - \int_a^n G(t,s)y(s)\,\mathrm{d}s,\ t \in [a,n].$$

Let us define the operator $T : C[a,n] \to C[a,n]$,

$$(Tu)(t) = \frac{u_0}{P(n)} \int_t^n \frac{\mathrm{d}\tau}{p(\tau)} + \int_a^n G(t,s)q(s)f^*(u(s))\,\mathrm{d}s,\ t \in [a,n].$$

Let u be a fixed point of T. Then

$$\begin{aligned}
u(t) &= \frac{u_0}{P(n)} \int_t^n \frac{\mathrm{d}\tau}{p(\tau)} + \int_a^t \left(1 - \frac{P(t)}{P(n)}\right) P(s)q(s)f^*(u(s))\,\mathrm{d}s \\
&\quad + \int_t^n \left(1 - \frac{P(s)}{P(n)}\right) P(t)q(s)f^*(u(s))\,\mathrm{d}s, \\
u'(t) &= -\frac{u_0}{P(n)p(t)} - \frac{1}{p(t)} \int_a^t \frac{1}{P(n)} P(s)q(s)f^*(u(s))\,\mathrm{d}s \\
&\quad + \frac{1}{p(t)} \int_t^n \left(1 - \frac{P(s)}{P(n)}\right) q(s)f^*(u(s))\,\mathrm{d}s, \\
\big(p(t)u'(t)\big)' &= -\frac{1}{P(n)} P(t)q(t)f^*(u(t)) - \left(1 - \frac{P(t)}{P(n)}\right) q(t)f^*(u(t)) \\
&= -q(t)f^*(u(t)),\ t \in [a,n].
\end{aligned}$$

Therefore, $u \in C^1[a,n]$, $pu' \in C^1[a,n]$ and u is a solution of equation (5.38). Moreover, since $P(a) = 0$, we conclude

$$\begin{aligned}
u(a) &= \frac{u_0}{P(n)} P(n) + \int_a^n \left(1 - \frac{P(s)}{P(n)}\right) P(a)q(s)f^*(u(s))\,\mathrm{d}s = u_0, \\
u(n) &= \int_a^n \left(1 - \frac{P(n)}{P(n)}\right) P(s)q(s)f^*(u(s))\,\mathrm{d}s = 0,
\end{aligned}$$

and so, conditions (5.39) are satisfied.

In order to show the existence of a fixed point of the operator T, we use the Schauder Fixed Point Theorem. Let $\Omega \subset C[a,b]$,

$$\Omega = \left\{x \in C[a,n] : \|x\|_{C[a,n]} \le \rho\right\}, \tag{5.42}$$

where

$$\begin{aligned}
\rho &= |u_0| + P(n)MQ, \\
M &= \sup\{|f^*(x)| : x \in \mathbb{R}\}, \\
Q &= \int_a^n q(s)\,\mathrm{d}s.
\end{aligned}$$

Then,

$$\begin{aligned}\|Tu\|_{C[a,n]} &= \max_{t\in[a,n]}\left\{\frac{u_0}{P(n)}\int_t^n \frac{\mathrm{d}\tau}{p(\tau)} + \int_a^n G(t,s)q(s)f^*(u(s))\,\mathrm{d}s\right\}\\ &\le \frac{|u_0|}{P(n)}P(n) + P(n)M\int_a^n q(s)\,\mathrm{d}s = \rho.\end{aligned}$$

Consequently, $T(\Omega)$ is bounded in $C[a,n]$. Due to (5.42), $T(\Omega)\subset\Omega$. Since f^* is a continuous function, it follows from

$$\|Tu_m - Tu\|_{C[a,n]} \le \max_{t\in[a,b]}\{\int_a^n |G(t,s)|q(s)|f^*(u_m(s)) - f^*(u(s))|\,\mathrm{d}s\}$$

that T is continuous on Ω. Moreover, for $u\in\Omega$, $t_1,t_2\in[a,n]$, there exists $\xi\in(t_1,t_2)$ such that

$$\begin{aligned}|\,(Tu)\,(t_1) - (Tu)\,(t_2)| &\le \left|\frac{u_0}{P(n)}\int_{t_1}^{t_2}\frac{\mathrm{d}\tau}{p(\tau)}\right| + \int_a^n |G(t_1,s) - G(t_2,s)|q(s)|f^*(u(s))|\,\mathrm{d}s\\ &\le \frac{|u_0|}{P(n)}\frac{|t_1-t_2|}{p_{min}} + \int_a^n \frac{|\partial G(\xi,s)|}{\partial t}|t_1-t_2|q(s)|f^*(u(s))|\,\mathrm{d}s\\ &\le |t_1-t_2|\left(\frac{|u_0|}{P(n)}\frac{1}{p_{min}} + M\int_a^n \frac{q(s)}{p_{min}}\,\mathrm{d}s\right)\\ &\le |t_1-t_2|\left(\frac{|u_0|}{P(n)}\frac{1}{p_{min}} + \frac{MQ}{p_{min}}\right).\end{aligned}$$

This implies the compactness of T on Ω, due to the Arzelà–Ascoli Theorem. Since the operator T is continuous and compact on Ω and $T(\Omega)\subset\Omega$, there exists a fixed point $u = Tu$ due to the Schauder Fixed Point Theorem.

Step 2. *Showing solvability of problem* (2.3)*,* (5.39):
Let u be a solution of problem (5.38), (5.39). We prove that

$$0 \le u(t) \le L \text{ for } t\in[a,n]. \tag{5.43}$$

Let us assume that

$$u(t_0) = \max\{u(t) : t\in[a,n]\} > L. \tag{5.44}$$

Since $u(a) = u_0 \in (0,L)$ and $u(n) = 0$, it follows that $t_0\in(a,n)$ and $u'(t_0) = 0$. Therefore, we can find $\delta > 0$ such that $u(t) > L$ on $(t_0, t_0+\delta)\subset(a,n)$ and, by (5.37),

$$\big(p(t)u'(t)\big)' = -q(t)\frac{L-u(t)}{u(t)-L+1} > 0,\; t\in(t_0,t_0+\delta) \tag{5.45}$$

follows. After integrating (5.45) over $(t_0,t), t\in(t_0,t_0+\delta)$, we obtain

$$0 < -\int_{t_0}^t q(s)\frac{L-u(s)}{u(s)-L+1}\,\mathrm{d}s = p(t)u'(t).$$

Thus, $u' > 0$ on $(t_0, t_0+\delta)$, which contradicts (5.44). Analogously, the contradiction follows when assuming

$$\min\{u(t) : t \in [a,n]\} < 0.$$

Finally, it follows from (5.37) and (5.43) that u is a solution of equation (2.3) on $[a,n]$.

Step 3. *Showing solvability of problem* (2.3)*,* (5.32):
It follows from Step 2 that for each $n \in \mathbb{N}$, $n \geq a$ we have a solution u_n of equation (2.3) on $[a,n]$. This solution satisfies

$$u_n(a) = u_0,\ 0 \leq u_n(t) \leq L \text{ for } t \in [a,n]. \tag{5.46}$$

We prove that there exists a subsequence $\{u_\nu\} \subset \{u_n\}$ which locally uniformly converges on $[a,\infty)$ to a solution u of problem (2.3), (5.32). To this aim we consider an arbitrary compact interval $[a,b] \subset [a,\infty)$. Then

$$0 \leq u_n(t) \leq L,\ t \in [a,b],\ n > b,$$

holds. Consequently, there exists $\tau_n \in [a,b]$ such that $|u_n'(\tau_n)| \leq \frac{L}{b-a}$. Let us estimate the first derivative of the solution u_n on $[a,b]$. After integrating equation (2.3) from $t \in [a,b]$ to τ_n we obtain

$$\begin{aligned}
p(t)u_n'(t) &= p(\tau_n)u_n'(\tau_n) + \int_t^{\tau_n} q(s)f(u_n(s))\,\mathrm{d}s,\\
u_n'(t) &= \frac{p(\tau_n)}{p(t)}u_n'(\tau_n) + \frac{1}{p(t)}\int_t^{\tau_n} q(s)f(u_n(s))\,\mathrm{d}s,\\
|u_n'(t)| &\leq \frac{p_{max}}{p_{min}}\frac{L}{b-a} + \frac{1}{p_{min}}q_{max}f_{max}(b-a) =: \rho_b,\ t \in [a,b],
\end{aligned} \tag{5.47}$$

where

$$\begin{aligned}
p_{max} &= \max\{p(t) : t \in [a,b]\},\\
p_{min} &= \min\{p(t) : t \in [a,b]\},\\
q_{max} &= \max\{q(t) : t \in [a,b]\},\\
f_{max} &= \max\{|f(x)| : 0 \leq x \leq L\}.
\end{aligned}$$

According to (5.47), the sequence $\{u_n\}$ is equicontinuous on $[a,b]$. Equation (2.3) yields

$$|(p(t)u_n'(t))'| \leq |q(t)f(u_n(t))| \leq q_{max}f_{max},\ t \in [a,b].$$

Since the sequence $\{(pu_n')'\}$ is bounded on $[a,b]$, the sequence $\{pu_n'\}$ is equicontinuous on $[a,b]$. Since $p_{min} > 0$, we have by (5.47) for $t_1, t_2 \in [a,b]$

$$|u_n'(t_1) - u_n'(t_2)| \leq \frac{1}{p_{min}}\left(|p(t_1)u_n'(t_1) - p(t_2)u_n'(t_2)| + \rho_b|p(t_2) - p(t_1)|\right).$$

This implies that the sequence $\{u_n'\}$ is also equicontinuous on $[a,b]$. From the Arzelà-Ascoli Theorem and the Diagonalization Lemma 5.14 it follows, that there exists a subsequence $u_m \rightrightarrows^{loc} u$, $u_m' \rightrightarrows^{loc} u'$ on $[a,\infty)$ and u is a solution of equation (2.3) on $[a,\infty)$. By (5.46), u satisfies (5.32).

For problem (2.3), (5.33) we consider $u_0 \in (L_0, 0)$ and use the dual arguments. $\square$

Imposing some additional assumptions on f, p, and q, enables to derive two different limits of solutions to problem (2.3), (5.32) and (2.3), (5.33).

Theorem 5.16 *Let us assume that $a > 0$, conditions* (5.34)–(5.36) *hold, and*

$$f \in \mathrm{Lip}_{loc}(0,L], \quad f(x) > 0 \, \textit{for} \, x \in (0,L). \tag{5.48}$$

Then problem (2.3)*,* (5.32) *has a solution u, such that*

$$0 < u(t) < L \quad \textit{for} \, t \in [a,\infty). \tag{5.49}$$

If in addition

$$\lim_{t\to\infty} \frac{1}{p(t)} \int_a^t q(s)\, \mathrm{d}s = \infty, \tag{5.50}$$

with

$$\liminf_{t\to\infty} p(t) > 0, \tag{5.51}$$

then either

$$u'(t) > 0 \, \textit{for} \, t \geq a \, \textit{and} \, \lim_{t\to\infty} u(t) = L, \tag{5.52}$$

or

$$u \textit{ is a Kneser solution.} \tag{5.53}$$

Proof: According to Theorem 5.15, problem (2.3), (5.32) has a solution u. Let us assume that $u(b) = L$ for some $b > a$. Due to (5.32), $u'(b) = 0$. By virtue of the first condition in (5.48), $u \equiv L$ is the only solution satisfying $u(b) = L, u'(b) = 0$. Therefore, $u(t) < L$ for $t \in [a,\infty)$. Assume that $u(c) = 0$ for some $c > a$. Due to (5.32), $u'(c) = 0$. After integrating equation (2.3) over $(c,t), t \in [a,\infty)$, and using the second condition in (5.48), we conclude $u'(t) \leq 0$ for $t > c$ and $u'(t) \geq 0$ for $t < c$. This yields $u \equiv 0$ which contradicts $u(a) = u_0 > 0$. Therefore (5.49) holds. By (2.3), (5.32), (5.36), and (5.48)

$$(pu')'(t) = -q(t)f(u(t)) < 0, \; t \geq a,$$

and thus, pu' is decreasing on $[a,\infty)$.

(i) Assume that $u' > 0$ on $[a,\infty)$. Then there exists $\lim_{t\to\infty} u(t) =: l_0 \in (u_0, L]$. Let $l_0 \in (u_0, L)$ and let us denote $m_0 := \min\{f(x) : x \in [u_0, l_0]\} > 0$. Integration of (2.3) over $[a,t]$ yields

$$p(t)u'(t) - p(a)u'(a) \le -m_0 \int_a^t q(s)\,\mathrm{d}s,\; t \in [a,\infty),$$

$$0 < u'(t) \le \frac{1}{p(t)}\big(p(a)u'(a)\big) - m_0 \frac{1}{p(t)} \int_a^t q(s)\,\mathrm{d}s,\; t \in [a,\infty).$$

After letting $t \to \infty$ and using (5.50), (5.51), we arrive at $0 \le \liminf_{t\to\infty} u'(t) \le -\infty$, which is a contradiction. Therefore $l_0 = L$ and (5.52) holds.

(ii) Assume now that there exists $b \ge a$ such that $u'(b) \le 0$. Then $(pu')(b) \le 0$ and since pu' is decreasing, we can find $t_0 > b$ such that $pu' < 0$ on $[t_0,\infty)$. By (5.35), $u' < 0$ on $[t_0,\infty)$ and (5.53) follows.

□

The dual theorem formulated below can be proved using similar arguments.

Theorem 5.17 *Let* (5.34)–(5.36) *hold and* $a > 0$. *Moreover, we assume that*

$$f \in \mathrm{Lip}_{loc}[L_0, 0), \quad f(x) < 0 \textit{ for } x \in (L_0, 0). \tag{5.54}$$

Then problem (2.3), (5.33) *has a solution u, such that*

$$L_0 < u(t) < 0 \quad \textit{for } t \in [a,\infty).$$

If in addition (5.50) *and* (5.51) *hold, then either*

$$u'(t) < 0 \textit{ for } t \ge a \textit{ and } \lim_{t\to\infty} u(t) = L_0,$$

or

u is a Kneser solution.

It is important to notice that for $a = 0$, equation (2.3) becomes singular, because $p(0) = 0$, and the results obtained in the previous section cannot be immediately extended to cover the singular case. To see this, note that for the case $p(t) = t^\alpha$, $\alpha \in (0,1]$, $\int_1^\infty \frac{\mathrm{d}s}{p(s)} = \infty$ follows, and thus, problems (2.3), (5.32) and (2.3), (5.33) have no Kneser solutions, see Remark 5.2. Therefore, α has to be greater than 1. However, in such a case $\int_0^1 \frac{\mathrm{d}s}{p(s)} = \infty$ and the functions P and G in (5.41) are not defined at $t = a = 0$. Consequently, the singular case when $a = 0$ requires a quite new approach or at least a nontrivial modification compared with the case when $a > 0$.

To our knowledge, the existence of Kneser solutions for the case $a = 0$ and $p \neq q$ under assumptions (5.34), (2.7), and (2.8) remains an open problem.

6 Asymptotic properties of Kneser solutions

Asymptotic properties of Kneser solutions vanishing at infinity are described in this chapter. For the purpose of investigation asymptotic behaviour of such solutions, we introduce regularly varying functions in Section 6.1. The theory of regular variation has proved to be a useful tool for the study of asymptotic properties of nonoscillatory solutions of linear and nonlinear differential equations. Asymptotic formulas for damped Kneser solutions are derived in Section 6.2 provided that coefficient functions p and q are regularly varying at infinity. We conclude Part I with numerical simulations of Kneser solutions.

6.1 Regularly varying functions

In this section, functions regularly varying at infinity are introduced and some of their basic properties necessary for the analysis are shown. For a complete treatment of the theory of regularly varying functions, see for instance [23], and [88] for the applications to asymptotics of ordinary differential equation.

Definition 6.1 A function g, which is positive and measurable on $[\tau_0,\infty)$, $\tau_0>0$, is called *regularly varying of index* $\alpha\in\mathbb{R}$ if for each $\lambda>0$

$$\lim_{t\to\infty}\frac{g(\lambda t)}{g(t)}=\lambda^{\alpha}.$$

The set of all regularly varying functions of index α is denoted by $RV(\alpha)$.

Remark 6.2 A regularly varying function of index $\alpha=0$ is called a *slowly varying* function and the set of those functions is denoted by SV. A slowly varying function can be bounded or unbounded, but as $t\to\infty$ it can neither grow too fast to infinity, nor decay too fast to zero. This means that

$$\lim_{t\to\infty}t^{\varepsilon}L(t)=\infty,\ \lim_{t\to\infty}t^{-\varepsilon}L(t)=0$$

for any $\varepsilon>0$ holds.

Remark 6.3 Definition 6.1 implies that a regularly varying function g of α can be represented by

$$g(t)=t^{\alpha}L(t),\ t\in[\tau_0,\infty),$$

where L is a slowly varying function.

Remark 6.4 Let $g \in RV(\alpha)$, $h \in RV(\beta)$ and $k \in \mathbb{R}$, then

$$g+h \in RV(\max(\alpha,\beta)), \quad gh \in RV(\alpha+\beta), \quad g^k \in RV(k\alpha).$$

See for example Appendix in [88].

In what follows, the symbol $\sim$ is used to denote the asymptotic equivalence,

$$f(t) \sim g(t) \text{ as } t \to \infty \Leftrightarrow \lim_{t\to\infty} \frac{f(t)}{g(t)} = 1.$$

Theorem 6.5 (Karamata Integration Theorem)
Let $L(t) \in SV$, $c > 0$.

(i) If $\alpha > -1$, then

$$\int_c^t s^\alpha L(s) ds \sim \frac{1}{\alpha+1} t^{\alpha+1} L(t) \text{ as } t \to \infty.$$

(ii) If $\alpha < -1$, then

$$\int_t^\infty s^\alpha L(s) ds \sim -\frac{1}{\alpha+1} t^{\alpha+1} L(t) \text{ as } t \to \infty.$$

(iii) If $\alpha = -1$, then

$$l(t) = \int_c^t \frac{L(s)}{s} ds \in SV \text{ and } \lim_{t\to\infty} \frac{L(t)}{l(t)} = 0.$$

In order to investigate the asymptotic behaviour of the nonoscillatory solutions of problem (2.3), (5.32) and problem (2.3), (5.33), we first need to provide auxiliary lemmas for regularly varying functions.

Lemma 6.6 *Let $\alpha > 0$ and $g \in RV(\alpha)$. Then,*

$$\lim_{t\to\infty} g(t) \int_t^\infty \frac{\mathrm{d}s}{g(s)} = \infty. \tag{6.1}$$

Proof: According to Remark 6.3, the function g can be represented as $g(t) = t^\alpha L(t)$, $t \in [\tau_0, \infty)$, where $L \in SV$. For $\alpha > 1$, property (6.1) is a simple consequence of Theorem 6.5 (ii), since $-\alpha < -1$. Indeed, noticing that $L^{-1} \in SV$, for $t \to \infty$, we have

$$\int_t^\infty \frac{\mathrm{d}s}{g(s)} = \int_t^\infty s^{-\alpha} L^{-1}(s)\, \mathrm{d}s \sim \frac{1}{\alpha-1} t^{-\alpha+1} L^{-1}(t)$$

and therefore, the function

$$g(t) \int_t^\infty \frac{\mathrm{d}s}{g(s)} = t^\alpha L(t) \int_t^\infty s^{-\alpha} L^{-1}(s)\, \mathrm{d}s$$

is asymptotically equivalent to

$$\frac{1}{\alpha-1} t L(t) L^{-1}(t) = \frac{t}{\alpha-1}.$$

Thus, for $\alpha > 1$, property (6.1) follows.

Let us now consider $\alpha \in (0,1]$. Then,

$$\int_t^\infty \frac{\mathrm{d}s}{g(s)} = \int_t^\infty s^{-\alpha} L^{-1}(s)\,\mathrm{d}s = \int_t^\infty s^{-\alpha-1} s L^{-1}(s)\,\mathrm{d}s \geq t \int_t^\infty s^{-\alpha-1} L^{-1}(s)\,\mathrm{d}s,\; t \in [\tau_0, \infty).$$

According to Theorem 6.5 (ii) for $-\alpha-1 < -1$, it is asymptotically equivalent to

$$\frac{t^{1-\alpha} L^{-1}(t)}{\alpha}.$$

Therefore, for $t \to \infty$

$$g(t) \int_t^\infty \frac{\mathrm{d}s}{g(s)} \geq t^{\alpha+1} L(t) \int_t^\infty s^{-\alpha-1} L^{-1}(s)\,\mathrm{d}s \sim \frac{t}{\alpha}$$

and (6.1) follows for any $\alpha \in (0,1]$.

□

Lemma 6.7 *Let us assume that the functions g and h satisfy $g \in RV(\alpha)$ and $h \in RV(\beta)$, where $\alpha > 0$, $\beta > 0$, $\beta - \alpha > -1$, and $c > \tau_0$. Then,*

$$\lim_{t\to\infty} \frac{1}{g(t)} \int_c^t h(s)\,\mathrm{d}s = \infty.$$

Proof: According to Remark 6.3, the functions g and h can be represented as

$$g(t) = t^\alpha L_g(t),\;\; h(t) = t^\beta L_h(t),\; t \in [\tau_0, \infty),$$

where $L_p, L_q \in SV$. Therefore,

$$\frac{1}{g(t)} \int_c^t h(s)\,\mathrm{d}s = t^{-\alpha} L_g^{-1}(t) \int_c^t s^\beta L_h(s)\,\mathrm{d}s. \tag{6.2}$$

Due to Theorem 6.5 (i), the function given by (6.2) is asymptotically equivalent to the function

$$\frac{1}{\beta+1} t^{-\alpha} L_g^{-1}(t) t^{\beta+1} L_h(t) = \frac{1}{\beta+1} t^{\beta-\alpha+1} L(t),$$

where $L(t) = L_g^{-1}(t) L_h(t) \in SV$. Finally, Remark 6.2 and the assumption $\beta - \alpha > -1$ imply

$$\lim_{t\to\infty} \frac{1}{g(t)} \int_c^t h(s)\,\mathrm{d}s = \infty.$$

□

6.2 Asymptotic behaviour

This section focuses on properties of Kneser solutions of problems (2.3), (5.32) and (2.3), (5.33) in the neighbourhood of infinity. Asymptotic formulas for the solutions and for their first derivatives are provided. These asymptotic properties apply to the Kneser solutions of regular differential equations (2.3) with $a > 0$, as well as to the Kneser solutions of singular equation (2.3) with $a = 0$, provided that $u(a) \in [L_0, L]$.

We assume that function f satisfies condition (5.34) and the second conditions in (5.48) and (5.54),

$$\begin{gathered} L_0 < 0 < L,\ f \in C[L_0, L],\ f(L_0) = f(0) = f(L) = 0, \\ xf(x) > 0 \text{ for } x \in (L_0, 0) \cup (0, L). \end{gathered} \tag{6.3}$$

According to Theorem 6.5, condition (5.3), which excludes the existence of damped Kneser solutions, is satisfied when $p \in RV(\alpha)$ with $\alpha < 1$. For $\alpha = 1$, the integral may be convergent (or may not) and hence Kneser solutions of the problem could exist. Therefore, in the asymptotic analysis, we restrict our attention to the case $\alpha \geq 1$.

We first formulate the asymptotic properties of Kneser solutions to problem (2.3), (5.32), or (2.3), (5.33). The next theorem is a corollary of Lemmas 4.4, 6.6 and 6.7.

Theorem 6.8 *Assume that* (6.3) *holds and $a \geq 0$. Moreover, assume that $p \in RV(\alpha) \cap C[a,\infty)$, $q \in RV(\beta) \cap C[a,\infty)$, $\alpha \geq 1$, $\beta > 0$, $\beta - \alpha > -1$. Let u be a Kneser solution of problem* (2.3), (5.32), *or* (2.3), (5.33). *Then,*

$$\lim_{t\to\infty} u(t) = 0, \ \ \lim_{t\to\infty} u'(t) = 0. \tag{6.4}$$

Proof: Although Lemma 4.4 is stated for damped nonoscillatory solutions of singular problem (2.3), (2.4), the arguments can be applied to Kneser solutions of problem (2.3), (5.32), or (2.3), (5.33) with $a \geq 0$. In particular the inequalities for indices of regularly varying functions p and q, $\alpha \geq 1$, $\beta > 0$, $\beta - \alpha > -1$ imply that (4.3) and (4.5) hold, and therefore, due to (5.2), the proof of (6.4) repeats that of Lemma 4.4. □

Finally, we specify the asymptotic behaviour of Kneser solutions in a more precise way.

Theorem 6.9 *Assume that* (6.3) *holds and $a \geq 0$. Moreover, let us assume that $p \in RV(\alpha) \cap C[a,\infty)$, $\alpha \geq 1$, $q \in RV(\beta) \cap C[a,\infty)$, $\beta > 0$, $\beta - \alpha > -1$, and*

$$\exists r > 1 : \liminf_{x\to 0} \frac{|f(x)|}{|x|^r} > 0, \ \limsup_{x\to 0} \frac{|f(x)|}{|x|^r} < \infty. \tag{6.5}$$

Let u be a Kneser solution of problem (2.3), (5.32) *or* (2.3), (5.33). *Then, for any $\varepsilon > 0$*

$$\lim_{t\to\infty} t^{\frac{\beta-\alpha+2}{r-1} - \varepsilon} |u(t)| = 0. \tag{6.6}$$

Proof: Let u be a Kneser solution of problem (2.3), (5.32) or (2.3), (5.33). Consider $t_0 > a$ from (5.2). According to (6.5) and (6.4), there exist $k, K, \delta > 0$ and $t_1 \geq t_0$ such that

$$k < \frac{|f(x)|}{|x|^r} < K, \ x \in (0, \delta) \ \text{ and } 0 < |u(t)| < \delta, \ t \geq t_1.$$

Hence, we have

$$k|u(t)|^r < |f(u(t))| < K|u(t)|^r, \ t \geq t_1. \tag{6.7}$$

We now integrate equation (2.3) from t_1 to $t \geq t_1$ and obtain

$$p(t)u'(t) - p(t_1)u'(t_1) + \int_{t_1}^{t} q(s)f(u(s))\,\mathrm{d}s = 0.$$

Since $u(t)u'(t) < 0$ and $u(t)$ is monotone for $t > t_1$,

$$p(t)|u'(t)| > \int_{t_1}^{t} q(s)|f(u(s))|\,\mathrm{d}s > k|u(t)|^r \int_{t_1}^{t} q(s)\,\mathrm{d}s$$

follows. Therefore,

$$\frac{|u'(t)|}{k|u(t)|^r} > \frac{1}{p(t)} \int_{t_1}^{t} q(s)\,\mathrm{d}s, \ t > t_1.$$

Let L_p and L_q be slowly varying functions such that $p(t) = t^{\alpha} L_p(t)$ and $q(t) = t^{\beta} L_q(t)$. Functions L_p, L_q always exist due to Remark 6.3. According to Theorem 6.5 (i), there exists a sufficiently large $b \geq t_1$ such that

$$\frac{1}{p(t)} \int_{t_1}^{t} q(s)\,\mathrm{d}s = \frac{\int_{t_1}^{t} s^{\beta} L_q(s)\,\mathrm{d}s}{t^{\alpha} L_p(t)} \geq \frac{1}{2(\beta+1)} t^{\beta-\alpha+1} \frac{L_q(t)}{L_p(t)}, \ t > b.$$

Therefore,

$$\frac{|u'(t)|}{k|u(t)|^r} > c_1 t^{\beta-\alpha+1} L(t), \ t$$

where $c_1 = \frac{1}{2(\beta+1)}$ and $L(t) = \frac{L_q(t)}{L_p(t)}$. Again by Theorem 6.5 (i), there exists a sufficiently large $T \geq b$ such that

$$\begin{aligned} \frac{1}{k(r-1)} \left(\frac{1}{|u(t)|^{r-1}} - \frac{1}{|u(T)|^{r-1}} \right) &> c_1 \int_{T}^{t} s^{\beta-\alpha+1} L(s)\,\mathrm{d}s \\ &\geq \frac{c_1}{2(\beta-\alpha+2)} t^{\beta-\alpha+2} L(t) \end{aligned}$$

holds for $t > T$. Consequently,

$$0 < |u(t)| < \left(k(r-1) c_2 t^{\beta-\alpha+2} L(t) \right)^{-\frac{1}{r-1}}, \ t > T,$$

where $c_2 = \frac{c_1}{2(\beta-\alpha+2)}$. Let us define $L_2(t) := (L(t))^{-\frac{1}{r-1}}$ and $c_3 := k(r-1)c_2$, then

$$0 < t^{\frac{\beta-\alpha+2}{r-1}} |u(t)| < c_3 L_2(t),\ t > T. \tag{6.8}$$

Finally, we choose $\varepsilon > 0$ and multiply inequality (6.8) by $t^{-\varepsilon}$. Then, Remark 6.2 yields

$$\lim_{t\to\infty} t^{\frac{\beta-\alpha+2}{r-1} - \varepsilon} |u(t)| = 0$$

which completes the proof. □

We finally focus our attention on the first derivatives of Kneser solutions.

Theorem 6.10 *Let all assumptions of Theorem 6.9 be satisfied. Then, for any $\varepsilon > 0$ the following statements hold:*

(i) If $\beta > r\alpha - r - 1$, then

$$\lim_{t\to\infty} t^{\alpha-\varepsilon} |u'(t)| = 0. \tag{6.9}$$

(ii) If $\beta \le r\alpha - r - 1$, then

$$\lim_{t\to\infty} t^{\frac{\beta-\alpha+r+1}{r-1} - \varepsilon} |u'(t)| = 0. \tag{6.10}$$

Proof: Let u be a Kneser solution of problem (2.3), (5.32) or (2.3), (5.33) and let $a \le t_0 \le t_1$ be the points from the proof of Theorem 6.9. Then, $uu' < 0$ on $[t_1, \infty)$ and (6.7) holds. Let us choose $\varepsilon_1 > 0$. Due to (6.6), for each $c > 0$ there exists $T_1 \ge t_1$ such that

$$0 < t^{\frac{\beta-\alpha+2}{r-1} - \varepsilon_1} |u(t)| < c,\ t > T_1. \tag{6.11}$$

We first integrate equation (2.3) over (T_1, t) and set $A_1 := p(T_1)|u'(T_1)|$. Then, by (6.5),

$$0 < p(t)|u'(t)| = A_1 + \int_{T_1}^{t} q(s)|f(u(s))|\,\mathrm{d}s < A_1 + K\int_{T_1}^{t} q(s)|u(s)|^r\,\mathrm{d}s,\ t > T_1.$$

Let L_p and L_q be slowly varying functions such that $p(t) = t^{\alpha}L_p(t)$ and $q(t) = t^{\beta}L_q(t)$. This implies

$$0 < t^{\alpha}L_p(t)|u'(t)| < A_1 + K\int_{T_1}^{t} s^{\beta}L_q(s)|u(s)|^r\,\mathrm{d}s,\ t > T_1.$$

Due to (6.11),

$$\begin{aligned} 0 < t^{\alpha}L_p(t)|u'(t)| &< A_1 + K\int_{T_1}^{t} s^{\beta-r\mu+r\varepsilon_1}L_q(s)|s^{\mu-\varepsilon_1}u(s)|^r\,\mathrm{d}s \\ &< A_1 + Kc^r\int_{T_1}^{t} s^{\beta-r\mu+\varepsilon_2}L_q(s)\,\mathrm{d}s, \end{aligned}$$

where $\mu = \frac{\beta-\alpha+2}{r-1}, \varepsilon_2 = r\varepsilon_1 > 0$.

(i) Let $\beta > r\alpha - r - 1$. Then,

$$\begin{aligned}\beta - r\mu &= \frac{(r\alpha - r - 1) + 1 - \beta - r}{r-1} \\ &< \frac{\beta - \beta + 1 - r}{r-1} = -1.\end{aligned}$$

At this point, we choose ε_1 and accordingly ε_2 which are sufficiently small for $\beta - r\mu + \varepsilon_2 < -1$ to hold. By the Karamata Integration Theorem 6.5 (ii), there exists $A_2 < \infty$ such that

$$0 < t^{\alpha} L_p(t)|u'(t)| < A_1 + Kc^r \int_{T_1}^{\infty} s^{\beta - r\mu + \varepsilon_2} L_q(s)\,\mathrm{d}s = A_2$$

holds for $t > T_1$. We choose an arbitrary $\varepsilon > 0$ and multiply the above inequality by $t^{-\varepsilon} L_p^{-1}$. Thus,

$$0 < t^{\alpha - \varepsilon}|u'(t)| < A_2 t^{-\varepsilon} L_p^{-1}(t),$$

for $t > T_1$. Due to Remark 6.2, asymptotic formula (6.9) follows.

(ii) Let $\beta \le r\alpha - r - 1$. Then, for arbitrary $\varepsilon_2 > 0$,

$$\begin{aligned}\beta - r\mu + \varepsilon_2 &= \frac{r\beta - \beta}{r-1} - \frac{r}{r-1}(\beta - \alpha + 2) + \varepsilon_2 \\ &= \frac{(r\alpha - r - 1) + 1 - \beta - r}{r-1} + \varepsilon_2 \\ &\ge \frac{\beta - \beta + 1 - r}{r-1} + \varepsilon_2 \ge -1 + \varepsilon_2 > -1.\end{aligned}$$

By the Karamata Integration Theorem 6.5 (i), there exists a sufficiently large $T > T_1$, such that

$$0 < t^{\alpha} L_p(t)|u'(t)| < A_1 + \frac{2Kc^r}{\beta - r\mu + \varepsilon_2 + 1} t^{\beta - r\mu + \varepsilon_2 + 1} L_q(t) = A_1 + A_2 t^{\omega} L_q(t)$$

holds for $t > T$, where $A_2 = \frac{2Kc^r}{\beta - r\mu + \varepsilon_2 + 1}$ and $\omega = \beta - r\mu + \varepsilon_2 + 1 > 0$. Therefore,

$$t^{\alpha - \omega}|u'(t)| < A_1 t^{-\omega} L_p^{-1}(t) + A_2 L(t),$$

where $t > T, L(t) = \frac{L_q(t)}{L_p(t)}$. Finally, we choose an arbitrary $\varepsilon_3 > 0$ and multiply the above inequality by $t^{-\varepsilon_3}$. Consequently, we obtain

$$0 < t^{\frac{\beta - \alpha + r + 1}{r-1} - \varepsilon}|u'(t)| < A_1 t^{-\omega - \varepsilon_3} L_p^{-1}(t) + A_2 t^{-\varepsilon_3} L(t),$$

where $\varepsilon = \varepsilon_2 + \varepsilon_3$. By Remark 6.2, the asymptotic formula (6.10) holds and the result follows.

$\square$

Remark 6.11 In [3], asymptotic formulas for damped nonoscillatory solutions of problem (2.3), (2.4) in the case $q \equiv p$ are derived. It is shown that under the assumptions (2.5), (2.6), (3.4), (3.21), (6.5) and

$$p \in C[0,\infty) \cap C^1(0,\infty),\ p(0) = 0\ \lim_{t\to\infty} \frac{tp'(t)}{p(t)} = \alpha \in [1,\infty) \tag{6.12}$$

each damped nonoscillatory solution u of problem (2.3), (2.4) satisfying

$$\lim_{t\to\infty} u(t) = 0,\ \lim_{t\to\infty} u'(t) = 0$$

has the following properties:

$$\limsup_{t\to\infty} t^{\frac{2}{r-1}} |u(t)| < \infty,$$

I. If $\alpha \in [1, \frac{r+1}{r-1})$, then

$$\limsup_{t\to\infty} p(t)|u'(t)| < \infty.$$

II. If $\alpha \geq \frac{r+1}{r-1}$, then for any $\alpha_0 > \alpha$

$$\limsup_{t\to\infty} p(t) t^{\frac{r+1}{r-1} - \alpha_0} |u'(t)| < \infty.$$

The condition (6.12) and condition $p \in RV(\alpha) \cap C[0,\infty), \alpha > 1$ are connected in the following sense: Any positive and differentiable function p satisfying

$$\lim_{t\to\infty} \frac{tp'(t)}{p(t)} = \alpha$$

is a regularly varying function of index α. On the other hand, if $p \in RV(\alpha), \alpha \geq 0$ has a asymptotically monotone derivative then

$$\lim_{t\to\infty} \frac{tp'(t)}{p(t)} = \alpha.$$

We refer to Proposition 9 and Proposition 10 in [88].

6.3 Numerical simulations

We use the open domain MATLAB Code `bvpsuite` to numerically simulate three model problems in order to illustrate theoretical statements made above. The aim is to give numerical evidence for the existence of Kneser solutions. We focus on the singular problems (2.3), (5.32) and (2.3), (5.33) with $a = 0$ and simulate Kneser solutions on the interval $[0,\infty)$ which contains the singular point $t = 0$. Moreover, asymptotic properties of such solutions are investigated and compared with the analytically derived asymptotic formulas (6.6) and (6.10).

The MATLAB software package `bvpsuite` [67] was developed at the Institute for Analysis and Scientific Computing, Vienna University of Technology, to solve BVPs in ODEs and differential algebraic equations. The solver routine is based on a class of collocation methods whose orders may vary from 2 to 8. Collocation is investigated in the context of singular differential equations of first and second order in [46, 121], respectively. This method could be shown to be robust with respect to *singularities in time* and retains its high convergence order if the analytical solution is appropriately smooth. The code also provides an asymptotically correct estimate for the global error of the numerical approximation. To enhance the efficiency of the method, a mesh adaptation strategy is implemented, which attempts to choose grids related to the solution behaviour, making sure that the tolerance is satisfied with the least possible effort. Error estimate procedure and mesh adaptation work dependably provided that the solution of the problem and its global error are appropriately smooth[1]. Both the code and the manual can be downloaded from `http://www.math.tuwien.ac.at/∼ewa`. For further information see [67]. This software has proved useful for the approximation of numerous singular BVPs important for applications, see for example [28, 44, 64, 98].

Since we intend to solve a scalar second order differential equation, we have to specify two boundary/initial conditions which are correctly posed to guarantee the uniqueness of the solution, at least locally. More precisely, we try to solve problems (2.3), (5.32) and (2.3), (5.33) with $a = 0$, but we do not know the values of $u(0)$. Therefore, we solve the differential equation (2.3),

$$(p(t)u'(t))' + q(t)f(u(t)) = 0, \quad t \in (0,\infty),$$

subject to the boundary conditions

$$u'(0) = 0, \quad u(\infty) := \lim_{t\to\infty} u(t) = 0. \tag{6.13}$$

The first condition in (6.13) is motivated by the results obtained for $p \equiv q$, where *this condition is necessary for any solution to be continuous*, *cf.* Remark 2.2. The second condition in (6.13) has to be satisfied by any Kneser solution under the assumptions of Theorem 6.8. It turns out that from the numerical point of view, the problem is very involved and the numerical treatment is by no means straightforward.

For the first tests, we choose the simplest regularly varying functions p and q,

$$p(t) = t^{\alpha}, \ \ \alpha > 1, \quad q(t) = t^{\beta}, \ \ \beta \geq \alpha, \quad t \in [0,\infty). \tag{6.14}$$

[1]The required smoothness of higher derivatives is related to the order of the used collocation method.

In order to recover the solution asymptotics specified in (6.6), the parameter β has to satisfy $\beta > \alpha - 1$. Here, we restrict our attention to the case $\beta \geq \alpha$.[2]

For p, q from (6.14), we rewrite equation (2.3) and obtain

$$u''(t) + \frac{\alpha}{t}u'(t) + t^{\beta-\alpha} f(u(t)) = 0, \quad t \in (0,\infty), \quad u'(0) = 0, \;\; u(\infty) = 0. \qquad (6.15)$$

To solve this boundary value problem, the differential equation is reduced to the finite interval $[0,1]$. To this aim, we rewrite the problem as follows:

$$\begin{aligned} v_1''(t) + \frac{\alpha}{t}v_1'(t) + t^{\beta-\alpha} f(v_1(t)) = 0, \;\; t \in (0,1], \\ v_2''(t) + \frac{\alpha}{t}v_2'(t) + t^{\beta-\alpha} f(v_2(t)) = 0, \;\; t \in [1,\infty), \end{aligned}$$

and use the transformation $\tau = \frac{1}{t}$ in the equation for v_2. Then, the problem is solved on (0,1] subject to boundary conditions

$$v_1'(0) = 0, \;\; v_2(0) = 0, \;\; v_1(1) = v_2(1), \;\; v_1'(1) = -v_2'(1).$$

Example 1

The first model is used to illustrate the existence of positive and negative Kneser solutions of equation (6.15). The problem data reads:

$$p(t) = t^5, \quad q(t) = t^7, \quad t \in [0,\infty),$$

$$f(x) = \begin{cases} -12 - 2x & \text{for } \; x < -2, \\ x^3 & \text{for } \; x \in [-2,1], \\ 2 - x & \text{for } \; x > 1, \end{cases} \qquad (6.16)$$

and $L_0 = -6, L = 2, r = 3$. As shown in Figure 6.1, we found two different Kneser solutions u_1, u_2, lying in the regions indicated in (5.32) and (5.33), respectively. The solutions satisfy $\lim_{t\to\infty} u_i'(t) = 0$, $i = 1,2$ in correspondence to the theory. According to Theorem 6.9, the asymptotic behaviour of any Kneser solution u of (2.3), (5.32) or (2.3), (5.33) is specified by (6.6),

$$\lim_{t\to\infty} t^{\frac{\beta-\alpha+2}{r-1} - \varepsilon} |u(t)| = 0.$$

Thus, for $\alpha = 5$, $\beta = 7$, and $r = 3$ this formula becomes

$$\lim_{t\to\infty} t^{2-\varepsilon} |u_i(t)| = 0, \quad i = 1,2.$$

[2]For $\beta \in (\alpha - 1, \alpha)$ no Kneser solutions were found.

The first derivative of the Kneser solution behaves asymptotically as specified in (6.10). Therefore,

$$\lim_{t\to\infty} t^{\frac{\beta-\alpha+r+1}{r-1}-\varepsilon}|u_i'(t)| = \lim_{t\to\infty} t^{3-\varepsilon}|u_i'(t)| = 0, \quad i=1,2.$$

We illustrate the asymptotic behaviour of the Kneser solutions using graphs with double logarithmic scales, where the power k in the relation $y = ax^k$ corresponds to the slope of the line. Figure 6.1 clearly indicates that not only solutions u_1, u_2, but also the expressions $t^2 u_i(t)$, $i = 1,2$ tend to zero for $t \to \infty$. Similar observations can be made for the first derivatives of both solutions u_1', u_2', where $t^3 u_i'(t)$, $i = 1,2$ tends to zero for $t \to \infty$.

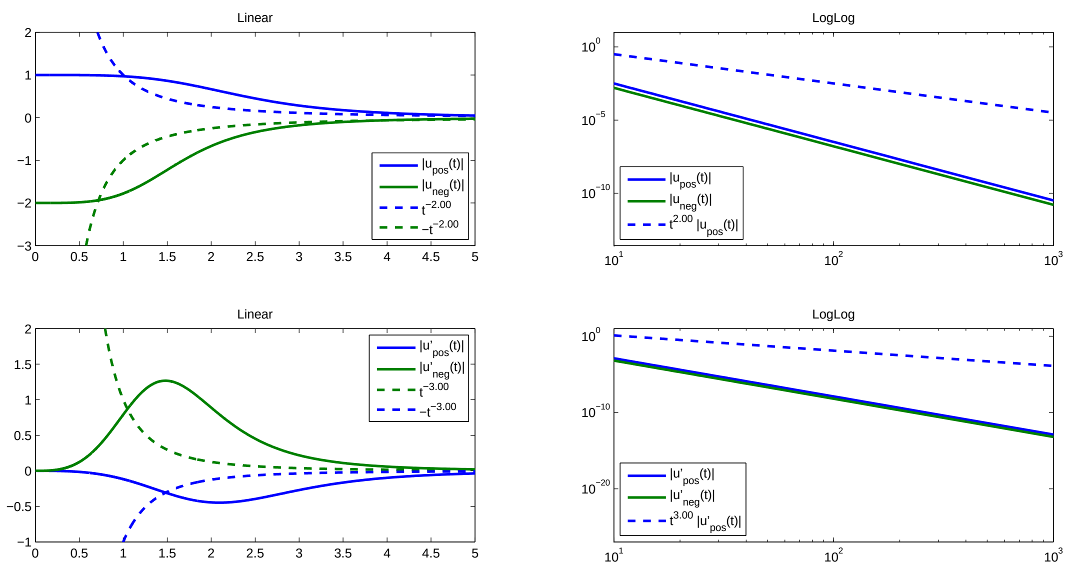

Figure 6.1: Example 1: Solutions u_i, $i = 1,2$, of (6.15), (6.13) corresponding to $\alpha = 5$ and $\beta = 7$ plotted using linear scales, upper graph (left), and double logarithmic scales, upper graph (right). First derivatives of the solutions u_i', $i = 1,2$ are shown in lower graphs.

Numerical simulations for problem (6.15), (6.13) with parameters

$$(\alpha, \beta) \in \{(4,4), (4,5), (5,6)\},$$

and function f satisfying condition (6.5) with $r = 2,3,4$ show similar behaviour and are not discussed here. For more details see [51].

Example 2

Using this example, we illustrate how the difference $\beta - \alpha$ affects the asymptotic behaviour of the Kneser solutions. According to (6.6), if $\beta - \alpha$ grows, we expect that the solution decay towards zero becomes faster. To see this, we consider problem (6.15), (6.13) with f specified in (6.16) and

$$(\alpha,\beta) \in \{(3,3),(4,5),(5,7)\}.$$

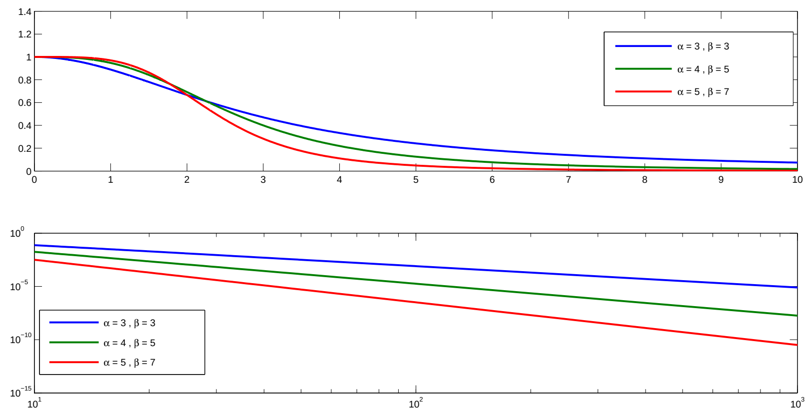

Figure 6.2: Example 2: Comparison of Kneser solutions depending on the difference $\beta-\alpha$. Graph of the Kneser solutions using linear scales (above), double logarithmic scales (below).

We can observe in Figure 6.2 that larger difference $\beta-\alpha$ indeed results in a steeper decline of the solution towards zero.

Example 3

Here, Kneser solutions of problem (2.3), (6.13) with a function f given in (6.16) and

$$p(t) = t^{\alpha} \in RV(\alpha), \quad q(t) = t^{\beta}(1+\exp(-t)) \in RV(\beta), \quad t \in [0,\infty)$$

are discussed. Then, equation (2.3) takes the form

$$u''(t) + \frac{\alpha}{t}u'(t) + t^{\beta-\alpha}(1+\exp(-t))f(u(t)) = 0.$$

Two Kneser solutions and their first derivatives can be found in Figure 6.3 for the parameters $(\alpha,\beta) = (4,5)$ and $(\alpha,\beta) = (5,7)$.

Example 4

We designed this example to illustrate the influence of parameters α and β. We choose

$$p(t)=t^{\alpha}\log(1+t)\in RV(\alpha),\ q(t)=t^{\beta}\in RV(\beta),\ f(x)=\operatorname{sgn}(x)x^4(2+x)(1-x),\ x\in\mathbb{R}.$$

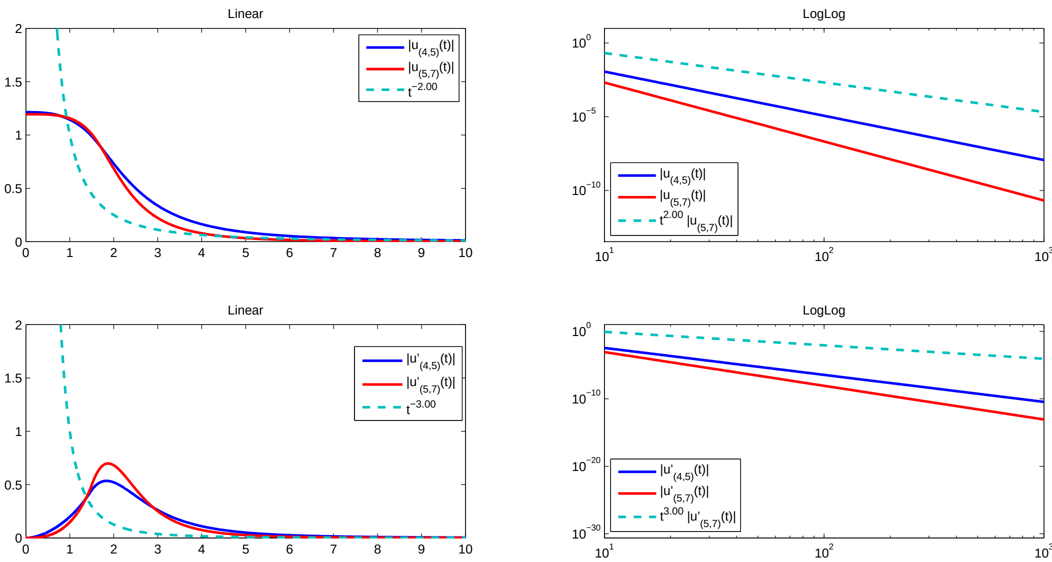

Figure 6.3: Example 3: Solutions corresponding to $\alpha = 4$, $\beta = 5$ and $\alpha = 5$, $\beta = 7$ plotted using linear scales, upper graph (left), and double logarithmic scales, upper graph (right). First derivatives of the solutions are shown in lower graphs.

This yields

$$u''(t)+\left(\frac{\alpha}{t}+\frac{1}{(1+t)\log(1+t)}\right)u'(t)+\frac{t^{\beta-\alpha}}{\log(1+t)}\,\mathrm{sgn}(u(t))u^4(t)(2+u(t))(1-u(t))=0. \tag{6.17}$$

First, we fix $\alpha = 4$ and vary $\beta \in \{4,5,6,7\}$. Numerical results are shown in Figure 6.4. All solutions seem to have a similar asymptotic behaviour. Moreover, we observe that the function $t^{5/3}u(t)$ for u corresponding to $\alpha = 4$, $\beta = 7$ tends to zero for large values of t. The same holds for all other solutions.

We now fix $\beta = 10$ and vary $\alpha \in \{5,6,7,8\}$. Figure 6.5 shows the related Kneser solutions and their first derivatives. A closer look at the solution u with the slowest decay towards zero shows that the limit of $t^{7/3}u$ is zero for $t \to \infty$. Other solutions show faster convergence towards zero.

The above observations mean that the asymptotic formula (6.6) can be applied to all numerical Kneser solutions of problem (6.17), (6.13), but it does not optimally recover the speed of their decay. Asymptotic behaviour of the numerically computed solutions indicates that the second term in equation (6.15),

$$\frac{\alpha}{t}u'(t), \quad t \in [0,\infty),$$

becomes dominant as $t \to \infty$, and therefore, properties of the solutions seem to be mainly controlled by the parameter α.

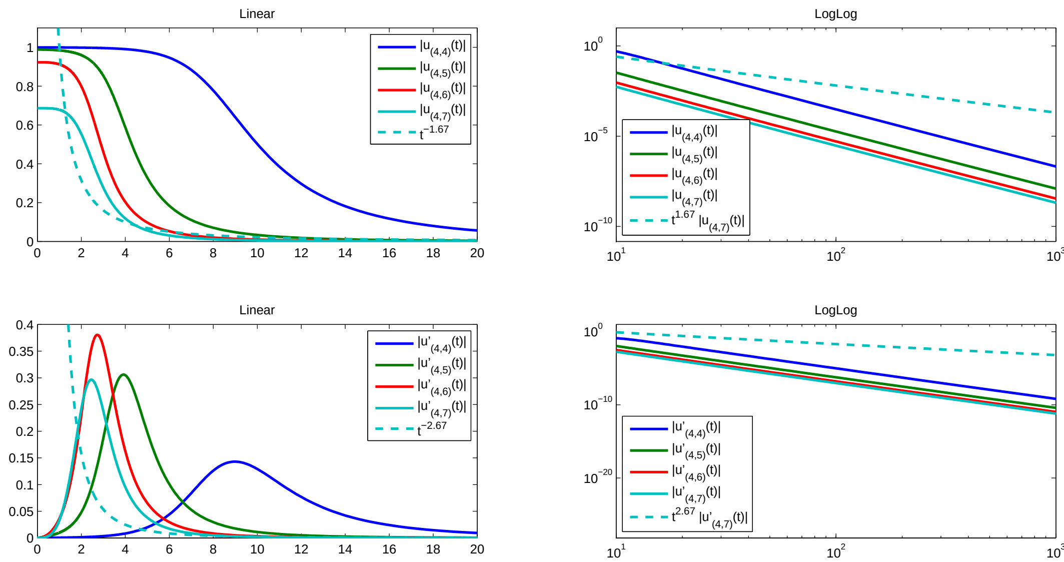

Figure 6.4: Example 4: Comparison of Kneser solutions with fixed α plotted in a graph with linear scales (left above) and double logarithmic scales (right above). First derivatives of solutions are shown in the lower graphs.

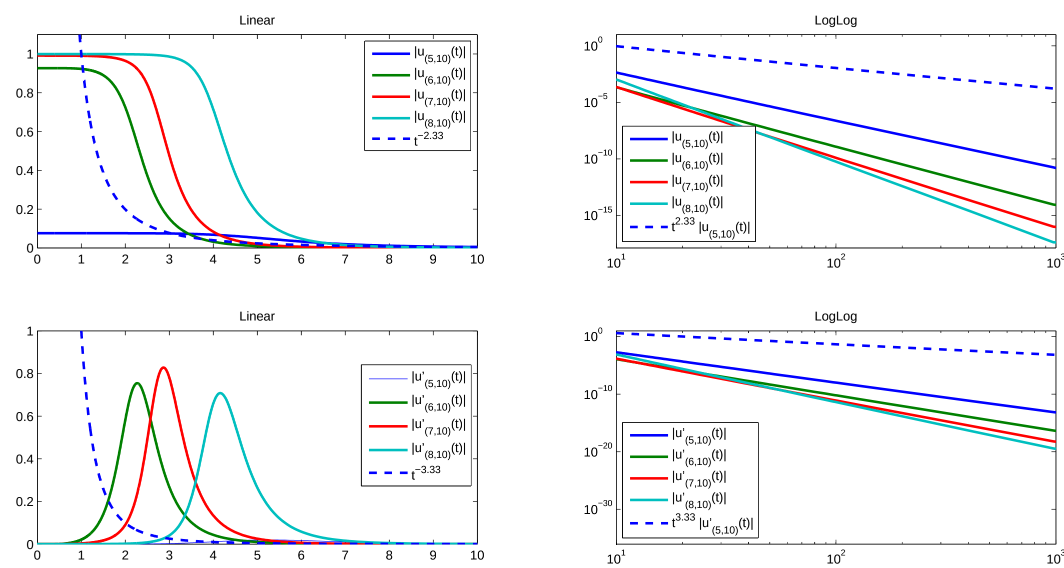

Figure 6.5: Example 4: Comparison of Kneser solutions with fixed β plotted in a graph with linear scales (left above) and double logarithmic scales (right above). First derivatives of the solutions are shown in the lower graphs.

Part II

Linear systems with a singularity of the first kind

7 Introduction

In Part II we investigate analytical properties of systems of linear ordinary differential equations with unsmooth nonintegrable inhomogeneities and a time singularity of the first kind

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t},\ t \in (0,1],$$

subject to boundary conditions

$$B_0 y(0) + B_1 y(1) = \beta,$$

where y is n-dimensional real function, M is $n \times n$ continuous matrix function, f is a n-dimensional function which is at least continuous on $[0,1]$ and $B_0, B_1 \in \mathbb{R}_{m\times n}$, $\beta \in \mathbb{R}^n$. We are especially interested in specifying the structure of general linear two-point boundary conditions guaranteeing the existence and uniqueness of solutions which are continuous on a closed interval including the singular point. Moreover, we study the convergence behaviour of collocation schemes applied to solve the problem numerically. Our theoretical results are supported by numerical experiments. The content of Part II is mostly based on the results published in [4]–[7]. The results concerning problems with constant coefficient matrix are proved in this thesis under less restrictive assumptions than those given in [4].

7.1 Motivation

Singular boundary value problems arise in numerous applications in natural sciences and engineering and therefore they are in focus of extensive investigations. An important class of singular problems takes the form of the following BVP:

$$y'(t) = \frac{M(t)}{t^\alpha}y(t) + f(t,y(t)),\ t \in (0,1], \quad b(y(0),y(1)) = 0, \tag{7.1}$$

where α is positive, $n \times n$ matrix function M and n-dimensional vector functions f, b are continuous. In particular, problems posed on infinite intervals are frequently transformed to a finite domain taking the form of (7.1) with $\alpha > 1$.

The BVP with a singularity of the first kind and unsmooth nonlinearity

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t,y(t))}{t},\ t \in (0,1], \quad b(y(0),y(1)) = 0, \tag{7.2}$$

arises in the modelling of snow avalanche run-up and run-out. Here, we shortly discuss a leading-edge model [70, 92, 93] for the description of the dynamics of dry-flowing avalanches. The aim of the study is to compute the velocity v of the avalanche and to determine the run-out length which is given by integral of v from $t=0$ until $t=t_R$ when the front of the avalanche stops

$$\int_0^{t_R} v(t)\,\mathrm{d}t.$$

In the model, five forces $T_i, i=1,\dots 5$ are combined to give the total force governing the avalanche's dynamics; the driving force, momentum flux, dynamic Coulomb resistive force, turbulent resistive force and passive snow pressure force. The formulation of the resulting differential equation is based on the Newton's second low which yields

$$\frac{\mathrm{d}(\overline{\rho}\overline{h}vx)}{\mathrm{d}t} = T_1+T_2+T_3+T_4+T_5,$$

where $\overline{\rho}$ is the mean density of material in the core of the avalanche, $\overline{h}$ is the mean flow depth along the run-up and x is the distance along the run-up slope. Taking into account the continuity relation

$$\overline{h}x = h_0 v_0 t,$$

where h_0 is the flow depth at $t=0$ and v_0 is the incoming speed, we obtain the singular IVP

$$\begin{aligned} v'(t) &= -\frac{v(t)}{t} - D_0 v^2(t) + \frac{V}{t} - G_0,\ t>0 \\ v(0) &= v_0. \end{aligned}$$

The constants D_0, V, G_0 are determinated by physical parameters of the problem.

The BVPs of type (7.2) also occur when the regular system of ODEs $u'(x) = M_0 u(x) + g(x,u(x))$, posed on the semi-infinite interval $x \in [0,\infty)$, is transformed by $x = -\ln t$ to the finite domain $t \in (0,1]$.

7.2 Statement of the problem

We are interested in analytical and numerical treatment of a certain subclass of BVPs (7.2) with a singularity of the first kind. In particular, we analyse the linear BVP

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t},\ t\in(0,1], \quad y \in C[0,1], \quad B_0 y(0) + B_1 y(1) = \beta, \quad (7.3)$$

where, $f:[0,1]\to\mathbb{R}^n$ and $M:[0,1]\to\mathbb{R}^{n\times n}$ have continuous components. Here $f\in C[0,1]$, but $f(t)/t$ may not be integrable on $[0,1]$. Moreover, $B_0, B_1 \in \mathbb{R}^{m\times n}$ are constant matrices and $\beta\in\mathbb{R}^m$. Note that in general $m\le n$. We focus our attention on the existence and uniqueness of a solution $y \in C[0,1]$. This smoothness requirement results in general in $n-m$ additional initial conditions the solution y has to satisfy. We also specify conditions for f and M which are sufficient for $y \in C^r[0,1]$, $r \in \mathbb{N}$.

Before discussing the most general BVP (7.3), we first consider simpler problems consisting of the ODE system

$$y'(t) = \frac{M(t)}{t} y(t) + \frac{f(t)}{t}, \tag{7.4}$$

subject to initial/terminal conditions. This means that we deal with the IVP,

$$y'(t) = \frac{M(t)}{t} y(t) + \frac{f(t)}{t}, \quad B_0 y(0) = \beta, \tag{7.5}$$

where $B_0 \in \mathbb{R}^{m \times n}$, $\beta \in \mathbb{R}^m$, and $m \leq n$, or with the TVP,

$$y'(t) = \frac{M(t)}{t} y(t) + \frac{f(t)}{t}, \quad B_1 y(1) = \beta, \tag{7.6}$$

where $B_1 \in \mathbb{R}^{n \times n}$, $\beta \in \mathbb{R}^n$.

Particular attention is paid to the structure of the most general two–point boundary conditions which are necessary and sufficient for the existence of a unique continuous solution on the closed interval $[0,1]$. It turns out that the form of such conditions depends on the spectral properties of the coefficient matrix $M(0)$. Therefore, we distinguish between three cases, where all eigenvalues of $M(0)$ have negative real parts, positive real parts, or zero eigenvalues.

8 The three case studies

The form of fundamental solution matrix yields that the cases $Re(\lambda) < 0, \lambda = 0$ and $Re(\lambda) > 0$ where λ is an eigenvalue of $M(0)$ require separate treatment. This is done in the next sections.

8.1 Preliminary results

In this section, we present results derived in [4] for the case of a constant coefficient matrix $M_0 \in \mathbb{R}^{n\times n}$. These results are necessary prerequisites for the investigation of problem (7.3) with a variable coefficient matrix M. More precisely, we analyse the ODE system

$$y'(t) = \frac{M_0}{t}y(t) + \frac{f(t)}{t}, \tag{8.1}$$

subject to initial/terminal conditions. This means that we deal with the IVP,

$$y'(t) = \frac{M_0}{t}y(t) + \frac{f(t)}{t}, \quad B_0 y(0) = \beta, \tag{8.2}$$

where $B_0 \in \mathbb{R}^{m\times n}$, $\beta \in \mathbb{R}^m$, and $m \le n$, and with the TVP,

$$y'(t) = \frac{M_0}{t}y(t) + \frac{f(t)}{t}, \quad B_1 y(1) = \beta, \tag{8.3}$$

where $B_1 \in \mathbb{R}^{n\times n}$, $\beta \in \mathbb{R}^n$.

In the first step of the analysis, we consider the ODE system (8.1) and construct its general solution. We denote by $J \in \mathbb{C}^{n\times n}$ the Jordan canonical form of M_0 and by $E \in \mathbb{C}^{n\times n}$ the associated matrix of the generalized eigenvectors of M_0. Thus, $M_0 = EJE^{-1}$. Moreover, let us introduce new variables, $v(t) := E^{-1}y(t)$ and $g(t) := E^{-1}f(t)$, then we can decouple the system (7.4) and obtain

$$v'(t) = \frac{J}{t}v(t) + \frac{g(t)}{t}. \tag{8.4}$$

By the variation of constant, any general solution of linear equation (8.4) is a complex-valued function of the form

$$v(t) = \Phi(t)d + \Phi(t)\int_1^t \Phi^{-1}(s)\frac{g(s)}{s}\,\mathrm{d}s = t^J d + t^J \int_1^t s^{-J-I} g(s)\,\mathrm{d}s,\ t \in (0,1],$$

where $d \in \mathbb{C}^n$ is an arbitrary vector and

$$\Phi(t) = t^J := \exp(J\ln(t)) = \sum_{j=0}^{\infty} \frac{J^j (\ln t)^j}{j!},$$

is the fundamental solution matrix satisfying

$$\Phi'(t) = \frac{J}{t}\Phi(t), \;\; \Phi(1) = I, \;\; t \in (0,1],$$

see [34, Chapter IV]. In the case that the matrix J consists of l Jordan boxes, $J_1, J_2, \dots, J_l$, the fundamental solution matrix has the form of the block diagonal matrix, $t^J = \mathrm{diag}(t^{J_1}, t^{J_2}, \dots, t^{J_l})$, where

$$J_k = \begin{pmatrix} \lambda_k & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{pmatrix} \in \mathbb{C}^{n_k \times n_k}, \quad k = 1, \dots, l,$$

and

$$t^{J_k} = t^{\lambda_k} \begin{pmatrix} 1 & \ln t & \frac{(\ln t)^2}{2} & \dots & \frac{(\ln t)^{n_k-1}}{(n_k-1)!} \\ 0 & 1 & \ln t & \dots & \frac{(\ln t)^{n_k-2}}{(n_k-2)!} \\ 0 & \ddots & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ln t \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}, \quad t \in (0,1]. \tag{8.5}$$

Here, $\lambda_k = \sigma_k + i\rho_k \in \mathbb{C}$ is an eigenvalue of M_0 and $\dim J_1 + \dim J_2 + \dots + \dim J_l = n$. The general solution of equation (8.1) is then given by

$$y(t) = t^{M_0}c + t^{M_0}\int_1^t s^{-M_0-I} f(s)\,\mathrm{d}s, \; t \in (0,1],$$

where $c \in \mathbb{C}^n$ and $t^{M_0} = Et^J E^{-1} \in \mathbb{C}^{n \times n}$.
Also,

$$\left(t^{M_0}\right)' = M_0 t^{M_0-I}, \; t \in (0,1]$$

and

$$t^{-M_0} = \left(\frac{1}{t}\right)^{M_0} \Rightarrow \left(t^{-M_0}\right)' = -M_0 t^{-M_0-I}, \quad t \in (0,1]. \tag{8.6}$$

From the structure of the matrix t^{J_k} in (8.5), it is obvious that the solution contribution related to the k-th Jordan box may become unbounded for $t = 0$. Apparently, the asymptotic behaviour of the solution depends on the sign of the real part σ_k of the associated eigenvalue λ_k. Therefore, we have to distinguish between three cases, $\sigma_k < 0$, $\lambda_k = 0$, and $\sigma_k > 0$. We assume that M_0 has no purely imaginary eigenvalues to exclude solutions of the form $t^{i\rho} = \cos(\rho \ln t) + i \sin(\rho \ln t)$.

We complete the preliminaries by two technical remarks, which are frequently used in the following analysis. For technical details, we refer the reader to [103].

Remark 8.1 The main focus of our investigation is on correctly posed initial/terminal conditions which guarantee the existence of continuous or even smooth solutions of (7.4) on $[0,1]$. Since logarithm terms occur in the matrix (8.5), the relation

$$\lim_{t\to 0^+} t^{\alpha}(\ln t)^k = 0,\ \forall\alpha\in\mathbb{R}^+,\ \forall k\in\mathbb{N}, \tag{8.7}$$

is essential when discussing the smoothness of y.

Remark 8.2 By integrating (8.6) we obtain

$$M_0\int_t^1 s^{-M_0-I}\,\mathrm{d}s = -s^{-M_0}\big|_t^1 = t^{-M_0}-I,\ t\in(0,1]. \tag{8.8}$$

Moreover, if M_0 has only eigenvalues with negative real parts, then $\lim_{s\to 0^+} s^{-M_0}=0$ due to Remark 8.1, and therefore

$$\int_0^1 s^{-M_0-I}\,\mathrm{d}s = (-M_0)^{-1}. \tag{8.9}$$

8.2 Eigenvalues of $M(0)$ with negative real parts

In this section, we investigate system (7.4), where all eigenvalues of $M(0)$ have negative real parts. Note that system (7.4) is equivalent to

$$y'(t) = \frac{M(0)}{t}y(t) + \frac{(M(t)-M(0))y(t)+f(t)}{t}.$$

It turns out that in this case, it is necessary to prescribe initial conditions of a certain structure to guarantee that the solution is continuous on $[0,1]$. Moreover, this continuous solution of the associated IVP (7.5) is shown to be unique. In the case of constant coefficient matrix, the form of the solution is provided in Theorem 8.5. In the proof of this theorem, we require the following lemmas.

Lemma 8.3 *Let $\gamma\ge 0$ and let the $n\times n$ matrix J be of the form*

$$J = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}, \quad \lambda = \sigma + i\rho, \tag{8.10}$$

where $\sigma\le 0$. For $\sigma = 0$, we assume $\lambda = 0$ and $\gamma > 0$. Then, for $t\in(0,1]$,

$$\int_0^t |s^{-J}|s^{\gamma-1}\,\mathrm{d}s = \sum_{j=0}^{n-1}\sum_{k=0}^{j}\frac{t^{\gamma-\sigma}(-\ln t)^k}{k!(\gamma-\sigma)^{j+1-k}}, \tag{8.11}$$

and in particular,

$$\int_0^1 |s^{-J}|s^{\gamma-1}\,\mathrm{d}s = \sum_{j=0}^{n-1}\frac{1}{(\gamma-\sigma)^{j+1}}.$$

Proof: Due to the form of matrix J, the norm of s^{-J} for $s \in (0,1]$ is

$$|s^{-J}| = |s^{-\lambda}| \sum_{j=0}^{n-1} \frac{|\ln s|^j}{j!} = s^{-\sigma} \sum_{j=0}^{n-1} \frac{(-\ln s)^j}{j!}. \tag{8.12}$$

By repeated integration by parts, we obtain

$$\begin{aligned}\int \frac{(-\ln s)^j}{j!} s^{\gamma-\sigma-1}\,\mathrm{d}s &= \frac{s^{\gamma-\sigma}}{\gamma-\sigma}\frac{(-\ln s)^j}{j!} + \int \frac{s^{\gamma-\sigma-1}}{\gamma-\sigma}\frac{(-\ln s)^{j-1}}{(j-1)!}\,\mathrm{d}s \\ &= \sum_{k=0}^{j} \frac{s^{\gamma-\sigma}(-\ln s)^k}{k!(\gamma-\sigma)^{j+1-k}}.\end{aligned}$$

Therefore, due to (8.7),

$$\int_0^t |s^{-J}| s^{\gamma-1}\,\mathrm{d}s = \int_0^t \sum_{j=0}^{n-1} \frac{(-\ln s)^j}{j!} s^{\gamma-\sigma-1}\,\mathrm{d}s = \sum_{j=0}^{n-1}\sum_{k=0}^{j} \frac{t^{\gamma-\sigma}(-\ln t)^k}{k!(\gamma-\sigma)^{j+1-k}}.$$

Clearly, for $t = 1$

$$\int_0^1 |s^{-J}| s^{\gamma-1}\,\mathrm{d}s = \sum_{j=0}^{n-1} \frac{1}{(\gamma-\sigma)^{j+1}}$$

which completes the proof. $\square$

Lemma 8.4 *Assume that all eigenvalues of the matrix M_0 have negative real parts. Then*

$$\lim_{t\to 0^+} \int_0^t \left|s^{-M_0-I}\right|\,\mathrm{d}s = 0. \tag{8.13}$$

Proof: Let $\lambda_k = \sigma_k + i\rho_k,\ k = 1,\dots,l$, be eigenvalues of the matrix M_0 and $J_k,\ k = 1,\dots,l$, the associated Jordan boxes of M_0. Then $s^{-M_0} = Es^{-J}E^{-1}$, where $s^{-J} = \operatorname{diag}\left(s^{-J_1}, s^{-J_2}, \dots, s^{-J_l}\right)$. Therefore,

$$\lim_{t\to 0^+} \int_0^t \left|s^{-M_0-I}\right|\,\mathrm{d}s \le |E||E^{-1}| \lim_{t\to 0^+} \int_0^t \left|s^{-J}\right| s^{-1}\,\mathrm{d}s.$$

The result follows from (8.7) and (8.11) with $\gamma = 0$. $\square$

Theorem 8.5 *Let us assume that all eigenvalues of M_0 have negative real parts. Then for any $f \in C[0,1]$, IVP (8.2) with the initial condition $M_0 y(0) = -f(0)$ has a unique solution $y \in C[0,1]$. This solution has the form*

$$y(t) = (\mathscr{L}_1 f)(t),\ t \in [0,1],$$

where $\mathscr{L}_1 : C[0,1] \to C[0,1]$ is defined by

$$(\mathscr{L}_1 f)(t) := \int_0^1 s^{-M_0-I} f(st)\,\mathrm{d}s,\ t \in [0,1].$$

The initial condition $M_0 y(0) = -f(0)$ is necessary and sufficient for y to be continuous on $[0,1]$. Moreover, if $f \in C^r[0,1]$, $r \geq 1$, then $y \in C^r[0,1]$ and satisfies

$$(kI - M_0)y^{(k)}(0) = f^{(k)}(0),\ k = 0,\ldots,r.$$

Proof: The general solution of system (8.1) can be split into two parts

$$\begin{aligned} y(t) &= t^{M_0}c + t^{M_0}\int_1^t s^{-M_0-I}f(s)\,\mathrm{d}s \\ &= t^{M_0}\left(c - \int_0^1 s^{-M_0-I}f(s)\,\mathrm{d}s\right) + t^{M_0}\int_0^t s^{-M_0-I}f(s)\,\mathrm{d}s \\ &=: y_h(t) + y_p(t), \quad t \in (0,1]. \end{aligned} \tag{8.14}$$

First, we show that $y_p \in C[0,1]$. Change of variable, $u = s/t$, yields

$$y_p(t) = \int_0^1 u^{-M_0-I}f(ut)\,\mathrm{d}u,\ t \in (0,1].$$

We introduce the functions,

$$z_m(t) := \int_{\frac{1}{m}}^1 s^{-M_0-I}f(st)\,\mathrm{d}s,\ m \in \mathbb{N}, \tag{8.15}$$

$$z_\infty(t) := \int_0^1 s^{-M_0-I}f(st)\,\mathrm{d}s. \tag{8.16}$$

Then, by (8.13),

$$\lim_{m\to\infty}|z_\infty(t) - z_m(t)| = \lim_{m\to\infty}\left|\int_0^{\frac{1}{m}} s^{-M_0-I}f(st)\,\mathrm{d}s\right| \leq \|f\|_{1/m}\lim_{m\to\infty}\int_0^{\frac{1}{m}}\left|s^{-M_0-I}\right|\,\mathrm{d}s = 0.$$

Clearly $z_m(t) \in C[0,1]$, for $m \in \mathbb{N}$, and hence z_∞ is continuous as the uniform limit of continuous functions. Consequently, $y_p(t) \in C[0,1]$.

Since all real parts of eigenvalues are negative, y_h is not continuous at $t = 0$ and it is obvious that $y \in C[0,1]$ if and only if

$$\tilde{c} := c - \int_0^1 s^{-M_0-I}f(s)\,\mathrm{d}s = 0.$$

Thus the unique continuous solution satisfying (7.4) has the form

$$y(t) = (\mathscr{L}_1 f)(t) := \int_0^1 s^{-M_0-I}f(st)\,\mathrm{d}s,\ t \in [0,1]. \tag{8.17}$$

Clearly, $\mathscr{L}_1$ maps $C[0,1]$ to itself. In addition, the estimate

$$|y(t)| \leq const.\,\|f\|,\ t \in [0,1],$$

holds due to Lemma 8.4. This solution is uniquely determined by $\tilde{c} = 0$ and there are no additional conditions to be imposed. Note that $\tilde{c} = 0$ is equivalent to the condition $M_0 y(0) = -f(0)$ which follows from (8.9) and (8.17).

We now examine the smoothness of y. Let $f \in C^1[0,1]$. For the first derivative y', we have from (8.17)

$$y'(t) = \int_0^1 s^{-M_0} f'(ts)\,\mathrm{d}s,\ t \in [0,1],$$

due to Lemma 8.3. Clearly, if $f \in C^r[0,1]$, then

$$y^{(r)}(t) = \int_0^1 s^{(r-1)I-M_0} f^{(r)}(ts)\,\mathrm{d}s,\ t \in [0,1]$$

and by (8.9) the results follow. □

Theorem 8.5 shows that if all eigenvalues of M_0 have negative real parts, then there exists a unique continuous solution y of IVP (8.2) for $B_0 = M_0$, $\beta = -f(0)$, and $m = n$. Clearly, B_0 has to be nonsingular. Note that for this spectrum of M_0 the TVP (8.3) cannot be set up in a reasonable way.

Once the case with a constant coefficient matrix is covered, we can proceed to the IVP (7.5) with a time-dependent matrix M. We prove the existence and uniqueness of a solution in view of the Banach Fixed Point Theorem. Here, no additional assumptions on the variable coefficient matrix M need to be made, provided that all eigenvalues of $M(0)$ have negative real parts. To show the smoothness of the solution $y \in C^r[0,1]$, $r \in \mathbb{N}$, condition (8.18) has to hold.

Theorem 8.6 *Let us assume that all eigenvalues of $M(0)$ have negative real parts and $M \in C[0,1]$. Then for any $f \in C[0,1]$ system* (7.4) *has a unique solution $y \in C[0,1]$. This solution satisfies the initial condition $M(0)y(0) = -f(0)$ which is necessary and sufficient for y to be continuous on $[0,1]$. Moreover, if $f \in C^r[0,1]$ and*

$$M \in C^r[0,1],\ M^{(r)}(0) = 0,\ r \geq 1, \tag{8.18}$$

then $y \in C^r[0,1]$.

Proof: According to Theorem 8.5, any continuous solution of system (7.4) satisfies

$$y(t) = (\mathscr{L}_1 g)(t) = \int_0^1 s^{-M(0)-I} g(st, y(st))\,\mathrm{d}s,\ t \in [0,1],$$

where $g(t,y(t)) = (M(t) - M(0))\,y(t) + f(t)$. In order to show the existence and uniqueness of a continuous solution of (7.4), we choose $\delta \in (0,1]$ and study the fixed point equation

$$y = \mathscr{K} y,\ y \in C[0,\delta],$$

with an operator $\mathscr{K}$ defined by

$$
\begin{aligned}
(\mathscr{K}y)(t) &:= \int_0^1 s^{-M(0)-I} g(st, y(st))\,\mathrm{d}s \\
&= \int_0^1 s^{-M(0)-I} f(st)\,\mathrm{d}s + \int_0^1 s^{-M(0)-I}\,(M(st)-M(0))\,y(st)\,\mathrm{d}s,\ t\in[0,\delta].
\end{aligned}
$$

The proof is carried out in two steps.

Step 1. *Existence and uniqueness of a solution y.*
We prove the existence and uniqueness of a solution $y = \mathscr{K}y$, $y \in C[0,\delta]$ by means of the Banach Fixed Point Theorem. It follows immediately from Theorem 8.5 that the first contribution in $\mathscr{K}y$,

$$\int_0^1 s^{-M(0)-I} f(st)\,\mathrm{d}s,$$

is continuous on $[0,\delta]$. For any function $y \in C[0,\delta]$, the second contribution,

$$\int_0^1 s^{-M(0)-I}\,(M(st)-M(0))\,y(st)\,\mathrm{d}s,$$

is also continuous on $[0,1]$, *cf.* Theorem 8.5. Therefore, the operator $\mathscr{K}$ maps $C[0,\delta]$ to $C[0,\delta]$.

We show that $\mathscr{K}$ is a contracting operator. Let $y_1, y_2 \in C[0,\delta]$, then

$$
\begin{aligned}
&\|\mathscr{K}y_1 - \mathscr{K}y_2\|_\delta = \max_{t\in[0,\delta]} \left| \int_0^1 s^{-M(0)} s^{-1}\,(M(st)-M(0))\,(y_1(st)-y_2(st))\,\mathrm{d}s \right| \\
&\le \max_{t\in[0,\delta]} \left\{ \int_0^1 \left|s^{-M(0)}\right| s^{-1}\,\mathrm{d}s \max_{s\in[0,1]} |M(st)-M(0)| \max_{s\in[0,1]} |y_1(st)-y_2(st)| \right\} \\
&\le \max_{t\in[0,\delta]} \left\{ \int_0^1 \left|s^{-M(0)}\right| s^{-1}\,\mathrm{d}s \max_{s\in[0,t]} |M(s)-M(0)| \max_{s\in[0,t]} |y_1(s)-y_2(s)| \right\} \\
&\le const.\,\|M(\cdot)-M(0)\|_\delta \|y_1-y_2\|_\delta,
\end{aligned}
$$

where $\|M(\cdot)-M(0)\|_\delta = \max_{t\in[0,\delta]} |M(t)-M(0)|$. Since M is continuous on $[0,1]$

$$\lim_{t\to 0} M(t) - M(0) = 0.$$

Note that by Lemma 3 [4], $\int_0^1 |s^{-M(0)}| s^{-1}\,\mathrm{d}s = const$. Therefore there exists a sufficiently small δ such that

$$const.\,\|M(\cdot)-M(0)\|_\delta =: L_N < 1, \tag{8.19}$$

and consequently, the operator $\mathscr{K}$ is a contraction. The Banach Fixed Point Theorem yields the existence of a unique continuous solution y of (7.4) on $[0,\delta]$. By virtue of the classical theory, this solution can be uniquely extended to $t=1$. The initial condition $M(0)y(0)=-f(0)$ follows from the form of $\mathscr{K}$ and Theorem 8.5.

Step 2. *Smoothness of the solution.*
In order to examine the smoothness of y we assume that $r\in\mathbb{N}$ and $f,M\in C^r[0,1]$. The property $\mathscr{K}: C^r[0,\delta]\to C^r[0,\delta]$ follows by arguing as in the proof of Theorem 8.5. We show that $\mathscr{K}$ is a contraction on $C^r[0,\delta]$ for a sufficiently small δ.

Let $r=1$. Then for any $y_1,y_2\in C^1[0,\delta]$,

$$\|\mathscr{K}y_1-\mathscr{K}y_2\|_{C^1[0,\delta]}=\|\mathscr{K}y_1-\mathscr{K}y_2\|_\delta+\|\,(\mathscr{K}y_1)'-(\mathscr{K}y_2)'\,\|_\delta,$$

where $(\mathscr{K}y)'$ is given by

$$\begin{aligned}(\mathscr{K}y)'(t)=&\int_0^1 s^{-M(0)}f'(st)\,\mathrm{d}s+\int_0^1 s^{-M(0)}M'(st)y(st)\,\mathrm{d}s\\&+\int_0^1 s^{-M(0)}\,(M(st)-M(0))\,y'(st)\,\mathrm{d}s,\ t\in[0,\delta],\end{aligned}$$

and according to (8.18) M' tends to zero for $t\to 0$. This together with (8.19) yields:

$$\begin{aligned}&\|\mathscr{K}y_1-\mathscr{K}y_2\|_{C^1[0,\delta]}\le L_N\|y_1-y_2\|_\delta\\&\quad+\max_{t\in[0,\delta]}\left\{\int_0^1\left|s^{-M(0)}\right|\,\mathrm{d}s\max_{s\in[0,1]}|M'(st)|\max_{s\in[0,1]}|y_1(st)-y_2(st)|\right\}\\&\quad+\max_{t\in[0,\delta]}\left\{\int_0^1\left|s^{-M(0)}\right|\,\mathrm{d}s\max_{s\in[0,1]}|M(st)-M(0)|\max_{s\in[0,1]}|y_1'(st)-y_2'(st)|\right\}\\&\le L_N\|y_1-y_2\|_\delta+L_{N2}\|y_1-y_2\|_\delta+L_{N3}\|y_1'-y_2'\|_\delta\le L\|y_1-y_2\|_{C^1[0,\delta]},\end{aligned}$$

where $L=\max\{L_N+L_{N2},L_{N3}\}$ and

$$L_{N2}=\int_0^1\left|s^{-M(0)}\right|\,\mathrm{d}s\|M'\|_\delta,\ L_{N3}=\int_0^1\left|s^{-M(0)}\right|\,\mathrm{d}s\|M(\cdot)-M(0)\|_\delta.$$

For a sufficiently small δ, the value of L is smaller than 1, and therefore, $\mathscr{K}$ is a contraction on $C^1[0,\delta]$.

At this point, let $r=2$. Then for any $y_1,y_2\in C^2[0,\delta]$,

$$\|\mathscr{K}y_1-\mathscr{K}y_2\|_{C^2[0,\delta]}=\|\mathscr{K}y_1-\mathscr{K}y_2\|_{C^1[0,\delta]}+\|\,(\mathscr{K}y_1)''-(\mathscr{K}y_2)''\,\|_\delta,$$

where, by virtue of (8.18),

$$\begin{aligned}(\mathscr{K}y)''(t)=&\int_0^1 s^{-M(0)}sf''(st)\,\mathrm{d}s+\int_0^1 s^{-M(0)}sM''(st)y(st)\,\mathrm{d}s\\&+2\int_0^1 s^{-M(0)}sM'(st)y'(st)\,\mathrm{d}s+\int_0^1 s^{-M(0)}s\,(M(st)-M(0))\,y''(st)\,\mathrm{d}s,\ t\in[0,\delta],\end{aligned}$$

and the following estimate holds due to (8.18):

$$\|\mathscr{K}y_1 - \mathscr{K}y_2\|_{C^2[0,\delta]} \le L\|y_1 - y_2\|_{C^1[0,\delta]} + L_{N4}\|y_1 - y_2\|_\delta \\ + L_{N5}\|y_1' - y_2'\|_\delta + L_{N6}\|y_1'' - y_2''\|_\delta \le L_2\|y_1 - y_2\|_{C^2[0,\delta]}.$$

Here, $L_2 = \max\{L, L_{N4}, L_{N5}, L_{N6}\}$ and

$$L_{N4} = \int_0^1 \left|s^{-M(0)}\right| s\,\mathrm{d}s \|M''\|_\delta, \quad L_{N5} = \int_0^1 \left|s^{-M(0)}\right| s\,\mathrm{d}s \|M'\|_\delta,$$
$$L_{N6} = \int_0^1 \left|s^{-M(0)}\right| s\,\mathrm{d}s \|M(\cdot) - M(0)\|_\delta.$$

For a sufficiently small δ, the value of L_2 is smaller than 1, and thus, $\mathscr{K}$ is a contracting operator in $C^2[0,\delta]$.

Similarly, we can show that $\mathscr{K}$ is a contraction on $C^r[0,\delta]$ for $r > 2$. This yields the existence of a unique solution $y \in C^r[0,\delta]$ such that $M(0)y(0) = -f(0)$. This solution can be uniquely extended to $t = 1$, so $y \in C[0,1] \cap C^r[0,\delta]$. Under the assumption $f, M \in C^r[0,1]$ the classical theory yields a unique solution $z \in C^r(0,1]$ of equation (7.4) satisfying $z(\delta) = y(\delta)$. Consequently, $z = y$ on $[0,1]$ and $y \in C^r[0,1]$.

□

8.3 Eigenvalues of $M(0)$ with positive real parts

In this section, we deal with system (7.4) where eigenvalues of $M(0)$ have only positive real parts. It turns out that in this case there exists a unique continuous solution of problem (7.6). Its smoothness depends not only on the smoothness of f but also on the size of real parts of the eigenvalues of $M(0)$. Before stating the main result of this section formulated in Theorem 8.12, we prove two lemmas.

Lemma 8.7 *Let $\gamma \ge 0$ and let the $n \times n$ matrix J be of the form* (8.10)*, where $\sigma > 0$. Then for $t \in [0,1]$ the function*

$$u(t) = \int_t^1 \left|\left(\frac{t}{s}\right)^J\right| s^{\gamma-1}\,\mathrm{d}s,$$

satisfies the inequalities

$$(i)\ u(t) \le const.t^\gamma, \quad \gamma < \sigma, \tag{8.20}$$

$$(ii)\ u(t) \le const.t^\sigma \sum_{j=0}^{n-1} \frac{(-\ln t)^{j+1}}{j!}, \quad \gamma = \sigma, \tag{8.21}$$

$$(iii)\ u(t) \le const.t^\sigma \sum_{j=0}^{n-1} \frac{(-\ln t)^j}{j!}, \quad \gamma > \sigma. \tag{8.22}$$

Proof: We discuss the cases $\gamma < \sigma$, $\gamma = \sigma$, and $\gamma > \sigma$, separately. Note that according to (8.7) and (8.12)

$$\int_t^1 \left| \left(\frac{t}{s}\right)^J \right| s^{\gamma-1}\,\mathrm{d}s = \int_t^1 \left(\frac{t}{s}\right)^{\sigma} \sum_{j=0}^{n-1} \frac{\left(-\ln\left(\frac{t}{s}\right)\right)^j}{j!} s^{\gamma-1}\,\mathrm{d}s$$

holds.

(i) First, let $\gamma < \sigma$. Then there exists a constant $\varepsilon > 0$ such that $\sigma = \gamma + 2\varepsilon$. The term

$$\left(\frac{t}{s}\right)^{\varepsilon} \sum_{j=0}^{n-1} \frac{\left(-\ln\left(\frac{t}{s}\right)\right)^j}{j!}$$

is bounded on $[0,1]$ due to (8.7) and hence

$$\int_t^1 \left| \left(\frac{t}{s}\right)^J \right| s^{\gamma-1}\,\mathrm{d}s \le const.t^{\gamma+\varepsilon} \int_t^1 s^{-\varepsilon-1}\,\mathrm{d}s = const.t^{\gamma}.$$

(ii) For $\gamma = \sigma$ function u can be estimated by

$$\int_t^1 \left| \left(\frac{t}{s}\right)^J \right| s^{\gamma-1}\,\mathrm{d}s \le t^{\sigma} \sum_{j=0}^{n-1} \frac{(-\ln t)^j}{j!} \int_t^1 s^{-1}\,\mathrm{d}s \le const.t^{\sigma} \sum_{j=0}^{n-1} \frac{(-\ln t)^{j+1}}{j!}.$$

(iii) Finally, for $\gamma > \sigma$, we have

$$\int_t^1 \left| \left(\frac{t}{s}\right)^J \right| s^{\gamma-1}\,\mathrm{d}s \le t^{\sigma} \sum_{j=0}^{n-1} \frac{(-\ln t)^j}{j!} \int_t^1 s^{-\sigma+\gamma-1}\,\mathrm{d}s \le const.t^{\sigma} \sum_{j=0}^{n-1} \frac{(-\ln t)^j}{j!}.$$

□

Lemma 8.8 *Let $\gamma \ge 0$ and let all eigenvalues of M_0 have positive real parts. Then the function*

$$u(t) = \int_t^1 \left| \left(\frac{t}{s}\right)^{M_0} \right| s^{\gamma-1}\,\mathrm{d}s,\ t \in [0,1],$$

is bounded on $[0,1]$ and $\lim_{t\to 0^+} u(t) = 0$ for $\gamma > 0$.

Proof: Let all eigenvalues of M_0 have positive real parts. Let J_k, $k = 1,\ldots l$, be the Jordan box of M_0. Then $s^{-M_0} = Es^{-J}E^{-1}$, where $s^{-1} = diag(s^{-J_1}, s^{-J_2}, \ldots, s^{-J_l})$. Therefore,

$$u(t) = \int_t^1 \left| \left(\frac{t}{s}\right)^{M_0} \right| s^{\gamma-1}\,\mathrm{d}s \le |E||E^{-1}| \int_t^1 \left| \left(\frac{t}{s}\right)^J \right| s^{\gamma-1}\,\mathrm{d}s.$$

Estimates (8.20) to (8.22) and property (8.7) imply $u(t) \le const.t^{\sigma_0}$ for $t \in [0,1]$, where $\sigma_0 = \min\left\{\gamma, \frac{\sigma}{2}\right\} \ge 0$. This means that u is bounded in $[0,1]$. If $\gamma > 0$, then $\sigma_0 > 0$ and the result follows.

□

Lemma 8.9 *Let the $n \times n$ matrix J be of the form* (8.10)*, where $\sigma > 0$. Then for $t \in [0,1]$ the function*

$$u(t) = \int_\delta^1 \left| \left(\frac{t}{s}\right)^J \right| s^{-1} \,\mathrm{d}s,$$

satisfies the inequality

$$u(t) \leq const. \left(\frac{t}{\delta}\right)^{\sigma/2}$$

Proof: Note that according to (8.12)

$$\int_\delta^1 \left| \left(\frac{t}{s}\right)^J \right| s^{-1} \,\mathrm{d}s = \int_\delta^1 \left(\frac{t}{s}\right)^\sigma \sum_{j=0}^{n-1} \frac{\left(-\ln\left(\frac{t}{s}\right)\right)^j}{j!} s^{-1} \,\mathrm{d}s$$

holds. The term

$$\left(\frac{t}{s}\right)^{\sigma/2} \sum_{j=0}^{n-1} \frac{\left(-\ln\left(\frac{t}{s}\right)\right)^j}{j!}$$

is bounded on $[0,1]$ due to (8.7) and hence

$$\int_\delta^1 \left| \left(\frac{t}{s}\right)^J \right| s^{-1} \mathrm{d}s \leq const. \int_\delta^1 \left(\frac{t}{s}\right)^{\sigma/2} s^{-1} \mathrm{d}s \leq const.\, t^{\sigma/2} \frac{2}{\sigma} \left(\delta^{-\sigma/2} - 1\right) \leq const. \left(\frac{t}{\delta}\right)^{\sigma/2}.$$

□

Lemma 8.10 *Let all eigenvalues of M_0 have positive real parts. Then the function*

$$u(t) = \int_\delta^1 \left| \left(\frac{t}{s}\right)^{M_0} \right| s^{-1} \,\mathrm{d}s,$$

is bounded on $[0,1]$ *and*

$$\lim_{t \to 0^+} u(t) = 0.$$

Proof: Let all eigenvalues of M_0 have positive real pars. Then

$$u(t) = \int_\delta^1 \left| \left(\frac{t}{s}\right)^{M_0} \right| s^{-1} \,\mathrm{d}s \leq |E||E^{-1}| \int_\delta^1 \left| \left(\frac{t}{s}\right)^J \right| s^{-1} \,\mathrm{d}s.$$

Lemma 8.9 yields that $u(t)$ is bounded for $t \in [0,1]$ by

$$u(t) \leq const. |E||E^{-1}| \left(\frac{t}{\delta}\right)^{\sigma_+/2}.$$

Moreover, $\lim_{t \to 0^+} u(t) = 0$ follows. □

Theorem 8.11 *Let us assume that all eigenvalues of M_0 have positive real parts and let the matrix $B_1 \in \mathbb{R}^{n\times n}$ in* (8.3) *be nonsingular. Then for any $f \in C[0,1]$ and any $\beta \in \mathbb{R}^n$ there exists a unique solution $y \in C[0,1]$ of TVP* (8.3). *This solution has the form*

$$y(t) = (\mathscr{L}_2 f)(t),\ t \in [0,1],$$

where $\mathscr{L}_2 : C[0,1] \to C[0,1]$ is defined by

$$(\mathscr{L}_2 f)(t) := t^{M_0} B_1^{-1}\beta + t^{M_0}\int_1^t s^{-M_0-I} f(s)\,\mathrm{d}s,\ \ t \in [0,1].$$

Moreover, this solution additionally satisfies the initial condition $M_0 y(0) = -f(0)$. Finally, if $f \in C^r[0,1]$, $r \geq 0$ and if $\sigma_+ > r$, where σ_+ is the smallest positive real part of the eigenvalues of M_0, then $y \in C^r[0,1]$.

Proof: The general solution of equation (8.1) can be written in the form

$$y(t) = t^{M_0}c + t^{M_0}\int_1^t s^{-M_0-I} f(s)\,\mathrm{d}s = t^{M_0}c + \int_1^t \left(\frac{t}{s}\right)^{M_0} \frac{f(s)}{s}\,\mathrm{d}s =: y_h(t) + y_p(t). \quad (8.23)$$

Since all eigenvalues have positive real parts, it follows from (8.7) that $y_h(t) = t^{M_0}c$ is continuous on $[0,1]$. Furhter, we need to prove that $\lim_{t\to 0^+} y_p(t)$ exists and therefore $y \in C[0,1]$.

Since f is continuous, there is $\delta(\varepsilon) > 0$ such that $|f(t) - f(0)| < \varepsilon$ whenever $0 \leq t < \delta(\varepsilon) < 1$. Moreover, for

$$\begin{aligned}
&\left|y_p(t) - (-M_0)^{-1} f(0)\right| = \left|\int_1^t \left(\frac{t}{s}\right)^{M_0} s^{-1} f(s)\,\mathrm{d}s - (-M_0)^{-1} f(0)\right| \\
&= \left|\int_1^t \left(\frac{t}{s}\right)^{M_0} s^{-1}\left(f(s) - f(0)\right)\,\mathrm{d}s + \int_1^t \left(\frac{t}{s}\right)^{M_0} s^{-1}\,\mathrm{d}s f(0) + (M_0)^{-1} f(0)\right|
\end{aligned}$$

the following estimate holds:

$$\begin{aligned}
&\left|y_p(t) - (-M_0)^{-1} f(0)\right| \\
&\leq \left|\int_1^\delta \left(\frac{t}{s}\right)^{M_0} s^{-1}\left(f(s) - f(0)\right)\,\mathrm{d}s\right| + \left|\int_\delta^t \left(\frac{t}{s}\right)^{M_0} s^{-1}\left(f(s) - f(0)\right)\,\mathrm{d}s\right| \\
&+ \left|\int_1^t \left(\frac{t}{s}\right)^{M_0} s^{-1}\,\mathrm{d}s f(0) + (M_0)^{-1} f(0)\right|,
\end{aligned}$$

where $0 \leq t \leq \delta < 1$. Furthermore, for $0 \leq t < \delta < 1$

$$\left|\int_\delta^t \left(\frac{t}{s}\right)^{M_0} s^{-1}\left(f(s) - f(0)\right)\,\mathrm{d}s\right| < \varepsilon \int_t^\delta \left|\left(\frac{t}{s}\right)^{M_0}\right| s^{-1}\,\mathrm{d}s \leq \varepsilon\, const. \quad (8.24)$$

according to Lemma 8.8. Lemma 8.10 yields

$$\lim_{t\to 0^+}\left|\int_1^{\delta}\left(\frac{t}{s}\right)^{M_0}s^{-1}\left(f(s)-f(0)\right)\,\mathrm{d}s\right|\le 2\|f\|\lim_{t\to 0^+}\int_{\delta}^{1}\left|\left(\frac{t}{s}\right)^{M_0}\right|s^{-1}\,\mathrm{d}s=0 \quad (8.25)$$

Finally, the integration formula (8.8) yields

$$\int_1^t\left(\frac{t}{s}\right)^{M_0}\frac{f(0)}{s}\,\mathrm{d}s=M_0^{-1}(t^{M_0}-I)f(0),$$

and hence

$$\int_1^t\left(\frac{t}{s}\right)^{M_0}+s^{-1}f(0)\,\mathrm{d}s+(M_0)^{-1}f(0)=M_0^{-1}\left(t^{M_0}-I+I\right)f(0)=M_0^{-1}\left(t^{M_0}\right)f(0).$$

Since all eigenvalues of M_0 have positive real parts, (8.7) implies

$$\lim_{t\to 0^+}\left|(-M_0)^{-1}t^{M_0}f(0)\right|=0. \quad (8.26)$$

By virtue of (8.24), (8.25) and (8.26)

$$\lim_{t\to 0^+}y_p(t)=(-M_0)^{-1}f(0),$$

and $y\in C[0,1]$.

It is clear from (8.23) that the solution y of (8.1) becomes unique if we specify the constant vector $c\in\mathbb{R}^n$. Note that at $t=0$, $y(0)$ satisfies n linearly independent conditions $M_0y(0)=-f(0)$ for any $c\in\mathbb{R}^n$. Therefore, we have to specify c via the terminal conditions given in (8.3). Let $\beta\in\mathbb{R}^n$ and let $B_1\in\mathbb{R}^{n\times n}$ be nonsingular, then it follows from $B_1y(1)=B_1c=\beta$ that the unique solution of TVP (8.3) is given by

$$y(t)=t^{M_0}B_1^{-1}\beta+\int_1^t\left(\frac{t}{s}\right)^{M_0}\frac{f(s)}{s}\,\mathrm{d}s:=\left(\mathscr{L}_2f\right)(t),$$

where $\mathscr{L}_2:C[0,1]\to C[0,1]$.

In order to discuss the smoothness of y, we first study the general solution of the homogeneous problem y_h. Since σ_+ is positive, there always exists a constant $l\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ such that $0\le l<\sigma_+\le l+1$. Then, we have

$$\begin{aligned}
y_h'(t)&=\left(t^{M_0}c\right)'=M_0t^{M_0-I}c,\\
y_h^{(k)}(t)&=\left(t^{M_0}c\right)^{(k)}=M_0(M_0-I)\cdots(M_0-(k-1)I)t^{M_0-kI}c,\ k=1,\dots,l,
\end{aligned}$$

and it is easily seen that $y_h\in C^l[0,1]\cap C^\infty(0,1]$. We now turn to the smoothness of the particular solution of the inhomogeneous problem y_p. We integrate by parts

$$\begin{aligned}
y_p(t)&=t^{M_0}\int_1^t s^{-M_0-I}f(s)\,\mathrm{d}s\\
&=t^{M_0}\left((-M_0)^{-1}t^{-M_0}f(t)-(-M_0)^{-1}If(1)-(-M_0)^{-1}\int_1^t s^{-M_0}f'(s)\,\mathrm{d}s\right)\\
&=(M_0)^{-1}\left(t^{M_0}f(1)-f(t)+t^{M_0}\int_1^t s^{-M_0}f'(s)\,\mathrm{d}s\right).
\end{aligned}$$

Note that t^{M_0} and M_0^{-1} are commutative. We differentiate the above equation and obtain

$$y_p'(t) = (M_0)^{-1}\left(M_0 t^{M_0-I} f(1) - f'(t) + M_0 t^{M_0-I}\int_1^t s^{-M_0} f'(s)\,\mathrm{d}s + t^{M_0} t^{-M_0} f'(t)\right)$$
$$= t^{M_0-I} f(1) + t^{M_0-I}\int_1^t s^{-M_0} f'(s)\,\mathrm{d}s.$$

Let $f \in C^1[0,1]$ and $\sigma_+ > 1$, then we argue as at the beginning of the proof (in context of y and $\sigma_+ > 0$) and conclude that $y_p \in C^1[0,1]$. Similarly, if $f \in C^r[0,1]$ and $\sigma_+ > r$, then $y_p \in C^r[0,1]$. Consequently, we have $y \in C^r[0,1]$ provided that $f \in C^r[0,1]$ and $\sigma_+ > r$. □

We recapitulate the case when all eigenvalues of M_0 have positive real parts: For any $f \in C[0,1]$ and any vector $\beta \in \mathbb{R}^n$ there exists a unique continuous solution y of TVP (8.3) if and only if the matrix $B_1 \in \mathbb{R}^{n\times n}$ is nonsingular. Each continuous solution y of (8.1) satisfies the initial condition $M_0 y(0) = -f(0)$ independently on $c \in \mathbb{R}^n$ from (8.23). Consequently, in this case there exists no IVP with a unique solution.

Once the case with a constant coefficient matrix is completed, we consider the TVP (7.6) with a time-dependent matrix M and use the Banach Fixed Point Theorem to prove the existence of a solution. Again, equation (7.4) is equivalent to

$$y'(t) = \frac{M(0)}{t} y(t) + \frac{(M(t) - M(0))\,y(t) + f(t)}{t}.$$

Theorem 8.12 *Let us assume that all eigenvalues of $M(0)$ have positive real parts. Moreover, let $f \in C[0,1]$, $M \in C[0,1]$, the matrix $B_1 \in \mathbb{R}^{n\times n}$ be nonsingular, and $\beta \in \mathbb{R}^n$. Then there exists a unique solution $y \in C[0,1]$ of TVP* (7.6)*. Moreover, if $f \in C^r[0,1]$, $r \in \mathbb{N}$, M satisfies condition* (8.18)*, and the smallest positive real part of the eigenvalues of $M(0)$ satisfies $\sigma_+ > r$, then $y \in C^r[0,1]$.*

Proof: The existence and uniqueness of solution $z \in C(0,1]$ of problem (7.6) follows from the classical theory because the interval $(0,1]$ does not contain the singular point $t = 0$. Now, we ask the question, if the solution z can be continuously extended to $t = 0$. In particular, we choose $\delta \in (0,1]$ and investigate the terminal value problem

$$y'(t) = \frac{M(t)}{t} y(t) + \frac{f(t)}{t},\ t \in [0,\delta], \quad y(\delta) = z(\delta). \tag{8.27}$$

By a slight modification of Theorem 8.11, we easily see that any solution y of problem (8.27) satisfies

$$y(t) = \left(\frac{t}{\delta}\right)^{M(0)} z(\delta) + t^{M(0)}\int_\delta^t s^{-M(0)-I} g(s, y(s))\,\mathrm{d}s,\ t \in [0,\delta],$$

where $g(t,y(t)) = f(t) + (M(t) - M(0))y(t)$. Therefore, the existence and uniqueness of a continuous solution to problem (8.27) is equivalent to the existence and uniqueness of a fixed point of the operator $\mathscr{K}$ defined on the space $C[0,\delta]$, where

$$(\mathscr{K}y)(t) = \left(\frac{t}{\delta}\right)^{M(0)} z(\delta) + t^{M(0)} \int_\delta^t s^{-M(0)-I} f(s)\,\mathrm{d}s \\ + t^{M(0)} \int_\delta^t s^{-M(0)-I}(M(s) - M(0))y(s)\,\mathrm{d}s,\ t \in [0,\delta].$$

Again, the proof is divided into two steps.

Step 1. *Existence and uniqueness of a solution y.*
In order to use the Banach Fixed Point Theorem to solve

$$y = \mathscr{K}y,\ y \in C[0,\delta],$$

we first show that $\mathscr{K} : C[0,\delta] \to C[0,\delta]$. According to Theorem 8.11 the first contribution to $\mathscr{K}y$,

$$\left(\frac{t}{\delta}\right)^{M(0)} z(\delta) + t^{M(0)} \int_\delta^t s^{-M(0)-I} f(s)\,\mathrm{d}s,$$

is continuous. Moreover, for any $y \in C[0,\delta]$ we conclude that

$$t^{M(0)} \int_\delta^t s^{-M(0)-I}(M(s) - M(0))y(s)\,\mathrm{d}s \in C[0,\delta].$$

Therefore, $\mathscr{K} : C[0,\delta] \to C[0,\delta]$. To see that the operator $\mathscr{K}$ is a contraction, we consider $y_1, y_2 \in C[0,\delta]$. Then,

$$\begin{aligned} \|\mathscr{K}y_1 - \mathscr{K}y_2\|_\delta &= \max_{t\in[0,\delta]} \left| t^{M(0)} \int_\delta^t s^{-M(0)} s^{-1}(M(s) - M(0))(y_1(s) - y_2(s))\,\mathrm{d}s \right| \\ &\le \max_{t\in[0,\delta]} \left\{ t^{M(0)} \int_t^\delta \left|s^{-M(0)}\right| s^{-1}\,\mathrm{d}s \max_{s\in[t,\delta]} |M(s) - M(0)| \max_{s\in[t,\delta]} |y_1(s) - y_2(s)| \right\} \\ &\le \max_{t\in[0,\delta]} \left\{ \int_t^1 \left| \left(\frac{t}{s}\right)^{M(0)} \right| s^{-1}\,\mathrm{d}s \right\} \|M(\cdot) - M(0)\|_\delta \|y_1 - y_2\|_\delta. \end{aligned}$$

According to Lemma 8.8, the function

$$u(t) := \int_t^1 \left| \left(\frac{t}{s}\right)^{M(0)} \right| s^{-1}\,\mathrm{d}s$$

is bounded. Moreover

$$\lim_{t\to 0} M(t) - M(0) = 0.$$

Therefore, there exists a sufficiently small δ such that

$$\max_{t\in[0,\delta]} u(t)\|M(\cdot) - M(0)\|_\delta =: L_S < 1, \tag{8.28}$$

and hence, the operator $\mathscr{K}$ is a contraction for a sufficiently small δ. By the Banach Fixed Point Theorem there exists a unique fixed point of $\mathscr{K}$ in $C[0,\delta]$. Thus, there exists a unique continuous solution y of problem (8.27). Since $y(\delta) = z(\delta)$, we have $y = z$ on $(0,\delta]$. If, we choose $z(0) := y(0)$, $y \in C[0,1]$ follows and this completes the proof of Step 1.

Step 2. *Smoothness of the solution.*
Let $f \in C^r[0,1]$, $r \geq 1$, let M satisfy condition (8.18) with $\gamma > r-1$, and $D \in C^{r-1}[0,1]$. Finally, let $\sigma_+ > r$. By arguments similar to those used in Theorem 8.11, andit follows that $\mathscr{K} : C^r[0,\delta] \to C^r[0,\delta]$.

Let us first assume $r = 1$ and show that $\mathscr{K}$ is a contraction on $C^1[0,\delta]$. Choose $y_1, y_2 \in C^1[0,\delta]$. Then, we can write $(\mathscr{K}y_1 - \mathscr{K}y_2)(t)$ as shown below, after integration by parts was used. With the shorthand notation $N(t) := (M(t)-M(0))(y_1(t)-y_2(t))$, we have for any $t \in [0,\delta]$,

$$
\begin{aligned}
(\mathscr{K}y_1 - \mathscr{K}y_2)(t) &= -t^{M(0)}M(0)^{-1}t^{-M(0)}N(t) \\
&\quad + t^{M(0)}\left(M(0)^{-1}\delta^{-M(0)}N(\delta) + M(0)^{-1}\int_\delta^t s^{-M(0)}N'(s)\,\mathrm{d}s\right) \\
&= M(0)^{-1}\left(-N(t) + t^{M(0)}\delta^{-M(0)}N(\delta) + t^{M(0)}\int_\delta^t s^{-M(0)}N'(s)\,\mathrm{d}s\right).
\end{aligned}
$$

We now differentiate both sides of the above equality and use $N(\delta) = 0$ to obtain

$$
\begin{aligned}
&((\mathscr{K}y_1)' - (\mathscr{K}y_2)')(t) = \\
&= -M(0)^{-1}N'(t) + t^{M(0)-I}\delta^{-M(0)}N(\delta) + t^{M(0)-I}\int_\delta^t s^{-M(0)}N'(s)\,\mathrm{d}s \\
&\quad + M(0)^{-1}t^{M(0)}t^{-M(0)}N'(t) = t^{M(0)-I}\int_\delta^t s^{-M(0)}N'(s)\,\mathrm{d}s \\
&= \int_\delta^t \left(\frac{t}{s}\right)^{M(0)-I} s^{-1}M'(s)(y_1(s)-y_2(s))\,\mathrm{d}s \\
&\quad + \int_\delta^t \left(\frac{t}{s}\right)^{M(0)-I} s^{-1}(M(s)-M(0))(y_1'(s)-y_2'(s))\,\mathrm{d}s.
\end{aligned}
$$

For $\gamma = 0$ and $\sigma_+ > 1$, Lemma 8.8 implies that the function

$$
u(t) := \int_t^1 \left|\left(\frac{t}{s}\right)^{M(0)-I}\right| s^{-1}\,\mathrm{d}s
$$

is bounded. This together with continuity of M' yields that for any $y_1, y_2 \in C^1[0,\delta]$ and for a sufficiently small $\delta > 0$ using (8.18) and (8.28),

$$
\begin{aligned}
\|\mathscr{K}y_1 - \mathscr{K}y_2\|_{C^1[0,\delta]} &= \|\mathscr{K}y_1 - \mathscr{K}y_2\|_\delta + \|(\mathscr{K}y_1)' - (\mathscr{K}y_2)'\|_\delta \leq L_S\|y_1 - y_2\|_\delta \\
&\quad + \max_{t\in[0,\delta]}\left\{u(t)\|M'\|_\delta\|y_1-y_2\|_\delta + u(t)\|M(\cdot)-M(0)\|_\delta\|y_1'-y_2'\|_\delta\right\}.
\end{aligned}
$$

Therefore $\mathscr{K}$ is a contraction on $C^1[0,\delta]$ for a sufficiently small δ. For $r>1$, we can use similar arguments to show that $\mathscr{K}$ is a contraction on $C^r[0,\delta]$. Finally, the Banach Fixed Point Theorem yields the existence and uniqueness of a solution $y \in C^r[0,\delta]$ of problem (8.27). The classical theory implies that for the solution z of TVP (7.6), derived in Step 1, $z \in C^r(0,1]$ holds. Since $z(\delta)=y(\delta)$, $y=z$ on $[0,1]$ and $y \in C^r[0,1]$ follows. □

8.4 Zero eigenvalues of $M(0)$

Finally, we consider the case, when all eigenvalues of $M(0)$ are zero. It turns out that some additional structure in the function f and in the variable coefficient matrix M is necessary for the solution y to be continuous. We begin with a scalar equation (8.1) which for $M_0=\lambda=0$ immediately reduces to

$$y'(t)=\frac{f(t)}{t}, \tag{8.29}$$

and show that additional structure in the function f is necessary to guarantee that the solution y is continuous on $[0,1]$. To see this, assume that f is a constant function, $f(t)\equiv 1$. Then, any solution y of equation (8.29) has the form

$$y(t)=y(1)+\int_1^t \frac{1}{s}\,\mathrm{d}s = y(1)+\ln t,\ t\in(0,1]$$

and, clearly, y is not continuous at $t=0$. Motivated by the scalar case, we require the inhomogeneity f to satisfy additional conditions providing the continuity of the associated solution.

Remark 8.13 Let us denote by R the projection matrix onto the space $X_0^{(e)}$ spanned by eigenvectors of M_0 associated with zero eigenvalues and by $\tilde{R}$ the matrix consisting of the linearly independent columns of R.

The proof of the main result of this section relies on the following lemma.

Lemma 8.14 *Let us assume that all eigenvalues of the matrix M_0 are zero. Then for $\alpha>0$*

$$\lim_{t\to 0^+}\int_0^t \left|s^{-M_0}\right| s^{\alpha-1}\,\mathrm{d}s = 0. \tag{8.30}$$

Proof: Let J_k, $k=1,\dots,l$, be the Jordan boxes of M_0. Then we can write $s^{-M_0} = Es^{-J}E^{-1}$, $s^{-J}=\operatorname{diag}\left(s^{-J_1},\dots,s^{-J_l}\right)$ and thus

$$\lim_{t\to 0^+}\int_0^t \left|s^{-M_0}\right| s^{\alpha-1}\,\mathrm{d}s \le |E||E^{-1}|\int_0^t \left|s^{-J}\right| s^{\alpha-1}\,\mathrm{d}s.$$

After applying (8.11) and (8.7) we obtain (8.30). □

Let us assume that there exist a constant $\alpha > 0$ and a function $h \in C[0,\delta]$, $\delta > 0$ such that

$$f(t) = O(t^{\alpha} h(t)) \text{ for } t \to 0. \tag{8.31}$$

Let us denote

$$\Omega = \{f \in C[0,1] \text{ such that } f \text{ satisfies } (8.31)\}, \tag{8.32}$$

Theorem 8.15 *Let all eigenvalues of matrix M_0 be zero, $m := \dim X_0^{(e)}$, and let the set Ω be given by* (8.32)*. Then for any $B_0 \in \mathbb{R}^{m\times n}$ such that matrix $B_0\tilde{R} \in \mathbb{R}^{m\times m}$ is nonsingular and for any $f \in \Omega$ and $\beta \in \mathbb{R}^m$, there exists a unique solution $y \in C[0,1]$ of IVP* (8.2)*. This solution has the form*

$$y(t) = (\mathscr{L}_3 f)(t),\ t \in [0,1],$$

where $\mathscr{L}_3 : \Omega \to C[0,1]$ is defined by

$$(\mathscr{L}_3 f)(t) := \tilde{R}(B_0\tilde{R})^{-1}\beta + \int_0^1 s^{-M_0} s^{-1} f(st)\,\mathrm{d}s,\ t \in [0,1].$$

This solution satisfies also the initial condition $M_0 y(0) = 0$, which is necessary and sufficient for $y \in C[0,1]$. Moreover, if $f \in C^r[0,1]$, then $y \in C^r[0,1]$.

Proof: We split the general solution of (8.1) into two parts $y(t) = y_h(t) + y_p(t)$ as defined in (8.14). To prove that $y_p \in C[0,1]$, we again use the functions z_m with $m \in \mathbb{N}$ and z_∞ specified in (8.15) and (8.16). Due to (8.11), (8.30), and (8.31), we obtain

$$\lim_{m\to\infty} |z_\infty(t) - z_m(t)| \le \|h\|_\delta t^{\alpha} \lim_{m\to\infty} \int_0^{\frac{1}{m}} \left|s^{-M_0}\right| s^{\alpha-1}\,\mathrm{d}s = 0. \tag{8.33}$$

Therefore, $y_p = z_\infty \in C[0,1]$ and $y_p(0) = 0$ since $f(0) = 0$ due to (8.31).

We now examine the continuity of

$$y_h(t) = t^{M_0}\left(c + \int_1^0 s^{-M_0} s^{-1} f(s)\,\mathrm{d}s\right) =: t^{M_0}\eta,$$

cf. (8.14). The fundamental solution matrix is given by $t^{M_0} = Et^J E^{-1}$, where t^J has the form $t^J = \mathrm{diag}(t^{J_1}, \dots, t^{J_l})$ and

$$E = \left(v_1, h_1^{(1)}, h_1^{(2)}, \dots, h_1^{(n_1-1)}, v_2, h_2^{(1)}, \dots, h_2^{(n_2-1)}, \dots, v_l, h_l^{(1)}, \dots, h_l^{(n_l-1)}\right),$$

where for $k = 1, \dots, l$, v_k are the eigenvectors of M_0, $h_k^{(1)}, \dots, h_k^{(n_k-1)}$ are the associated principal vectors, and n_k are the dimensions of the Jordan boxes J_k. Clearly, because of the logarithmic terms occurring in t^J, see (8.5), y_h is not continuous at $t = 0$ in general. Only when the contributions including the logarithmic terms vanish, y_h becomes continuous on $[0,1]$. It is clear from (8.5) that the only bounded contributions to y_h are linear combinations of the eigenvectors of M_0. Consequently, any linear combination of principal vectors has to vanish. This is the case when $\eta_i = 0, \forall i \neq 1, n_1+1, n_1+n_2+1, \ldots, \sum_{k=1}^{l} n_k + 1$ and arbitrary η_i for all $i = 1, n_1+1, n_1+n_2+1, \ldots, \sum_{k=1}^{l} n_k + 1$. Thus, y_h is continuous on $[0,1]$ if and only if it is a constant linear combination of the eigenvectors of M_0. In other words, by setting $y_h(t) := \eta$, we have

$$y(t) \in C[0,1] \Leftrightarrow M_0 y(0) = M_0 \eta = 0 \Leftrightarrow \eta \in \mathrm{Ker\,M_0}.$$

Consequently, $M_0 y(0) = 0$ is necessary and sufficient for the solution

$$y(t) = \eta + \int_0^1 s^{-M_0 - I} f(ts)\,\mathrm{d}s,\ t \in [0,1] \tag{8.34}$$

to be continuous on $[0,1]$. Note that the regularity requirement $M_0 y(0) = 0$ contains $n-l$ linearly independent conditions and can be equivalently expressed by $Hy(0) = 0$, $y(0) = Ry(0)$ or $y(0) \in \mathrm{Ker} M_0$. The remaining l free constants have to be uniquely specified by appropriately prescribed initial conditions. Let us consider the initial conditions specified in (8.2), where $B_0 \in \mathbb{R}^{m \times n}$ and $\beta \in \mathbb{R}^m$. Since $y_p(0) = 0$ and $y_h(0) = \eta$, the initial condition $B_0 y(0) = \beta$ is equivalent to $B_0 \eta = \beta$. Due to the fact that $\eta \in \mathrm{Im} R$, there exists a unique l-dimensional vector d, $l = \dim X_0^{(e)}$, such that $\eta = \tilde{R} d$, where $\tilde{R}$ is the $n \times l$ matrix containing the linearly independent columns of R. Clearly, the problem is uniquely solvable if and only if $m = l = \dim X_0^{(e)}$ and the $m \times m$ matrix $B_0 \tilde{R}$ is nonsingular. Hence,

$$B_0 \eta = \beta \Leftrightarrow B_0 \tilde{R} d = \beta \Rightarrow\ d = (B_0 \tilde{R})^{-1} \beta \Rightarrow\ \eta = \tilde{R} (B_0 \tilde{R})^{-1} \beta,$$

and the solution y has the form

$$y(t) = \tilde{R}(B_0 \tilde{R})^{-1} \beta + \int_0^1 s^{-M_0} s^{-1} f(st)\,\mathrm{d}s := (\mathscr{L}_3 f)(t),\ t \in [0,1]$$

where $\mathscr{L}_3 : \Omega \to C[0,1]$.

In order to derive the smoothness results, we differentiate y and obtain

$$y'(t) = \int_0^1 s^{-M_0} f'(st)\,\mathrm{d}s,\ t \in [0,1].$$

By Lemma (8.14) for $\alpha = 1$, $y \in C^1[0,1]$ if $f \in C^1[0,1]$. Similarly, if $f \in C^r[0,1]$

$$y^{(r)}(t) = \int_0^1 s^{(r-1)I - M_0} f^{(r)}(st)\,\mathrm{d}s,\ t \in [0,1],$$

and $y \in C^r[0,1]$.

□

Remark 8.16 A purely polynomial inhomogeneity of the form

$$f(t) = (t^{\alpha_1}, \dots, t^{\alpha_n})^\top,$$

where, $\alpha_i \in \mathbb{N}$, for $i = 1, \dots, n$, yields $y \in C^\infty[0,1]$. For the proof see [103].

In Theorem 8.15, we described the unique solvability of IVP (8.2) in case when all eigenvalues of M_0 are zero. The dimension of the corresponding eigenspace $X_0^{(e)}$ was $m < n$ and it turned out that the regularity requirement $M_0 y(0) = 0$ has to be satisfied. If $m = n$, then $M_0 = 0$ and the regularity condition holds. In this case we can also investigate the unique solvability of TVP (8.3) where all conditions are posed at $t = 1$. We address this question in the next lemma.

Lemma 8.17 *Consider system* (8.1) *with the matrix $M_0 = 0$. Let $f \in C[0,1]$ and assume that* (8.31) *is satisfied. Then, for any vector $\beta \in \mathbb{R}^n$ and a nonsingular matrix $B_1 \in \mathbb{R}^{n\times n}$ there exists a unique solution of* (8.3)*,*

$$y(t) = B_1^{-1}\beta + \int_1^t \frac{f(s)}{s}\,\mathrm{d}s,$$

bounded by

$$|y(t)| \le |B_1^{-1}\beta| + const.\,(\|f\| + t^\alpha \|h\|_\delta).$$

Moreover, if $f \in C^r[0,1]$, $h \in C^r[0,\delta]$, and $\alpha \ge r+1$, then $y \in C^{r+1}[0,1]$.

Proof: For $M_0 = 0$ the system (8.1) reduces to $y'(t) = f(t)/t$, and its solution is $y(t) = y(1) + \int_1^t f(s)/s\,\mathrm{d}s$. To show that $y \in C[0,1]$, we follow the arguments given in the proof of Theorem 8.15. The terminal condition $B_1 y(1) = \beta$ yields $y(1) = B_1^{-1}\beta$. Smoothness results for higher derivatives of y follow in an analogous manner. $\square$

Let us proceed with the IVP (7.5), where M is a time-dependent matrix. Matrices $R, \tilde{R}$ and $X_0^{(e)}$ from Remark 8.13 are defined for $M_0 = M(0)$. It turns out that a special structure of M is required in order to successfully apply the Banach Fixed Point Theorem. Consequently, we assume

$$M(t) = M(0) + t^\gamma D(t),\ \gamma > 0, \quad D \in C[0,1],\ t \in [0,1]. \tag{8.35}$$

Then the equation (7.5) is equivalent to

$$y'(t) = \frac{M(0)}{t} y(t) + \frac{t^\gamma D(t) y(t)}{t}.$$

Theorem 8.18 *Let all eigenvalues of matrix $M(0)$ be zero. Let M satisfy condition* (8.35) *and $m := \dim X_0^{(e)}$. Assume that $f \in \Omega$, $B_0 \in \mathbb{R}^{m\times n}$ is such that matrix $B_0\tilde{R} \in \mathbb{R}^{m\times m}$ is nonsingular, and $\beta \in \mathbb{R}^m$. Then there exists a unique solution $y \in C[0,1]$ of IVP* (7.5)*. This solution satisfies the initial condition $M(0)y(0) = 0$, which is necessary and sufficient for $y \in C[0,1]$. Moreover, if $\alpha \ge r+1$, $\gamma \ge r+1$, $r \ge 1$, $f, D \in C^r[0,1]$, and $h \in C^r[0,\delta]$, then $y \in C^{r+1}[0,1]$.*

Proof: By Theorem 8.15, any continuous solution of IVP (7.5) satisfies

$$y(t) = (\mathscr{L}_3 g)(t) = \tilde{R}(B_0\tilde{R})^{-1}\beta + \int_0^1 s^{-M(0)} s^{-1} g(st, y(st))\,\mathrm{d}s,\ t \in [0,1],$$

where $g(t,y(t)) = f(t) + t^{\gamma}D(t)y(t)$. Consequently, we have to study the fixed point equation

$$y = \mathscr{K} y,\ y \in C[0,\delta],$$

to prove the existence and uniqueness of a solution of IVP (7.5). In particular, we choose $\delta \in (0,1]$ and define

$$\begin{aligned}(\mathscr{K} y)(t) = &\tilde{R}(B_0\tilde{R})^{-1}\beta + \int_0^1 s^{-M(0)} s^{-1} f(st)\,\mathrm{d}s \\ &+ t^{\gamma} \int_0^1 s^{-M(0)} s^{\gamma-1} D(st) y(st)\,\mathrm{d}s,\ t \in [0,\delta].\end{aligned}$$

Step 1. *Existence and uniqueness of a solution y.*
We use the Banach Fixed Point Theorem in order to prove the first part of the statement. From Theorem 8.15, we see that

$$\tilde{R}(B_0\tilde{R})^{-1}\beta + \int_0^1 s^{-M(0)} s^{-1} f(st)\,\mathrm{d}s,$$

is continuous on $[0,\delta]$. For $y \in C[0,\delta]$ the function $t^{\gamma}D(t)y(t)$ belongs to Ω, *cf.* (8.32). By virtue of Theorem 8.15 we see that

$$t^{\gamma} \int_0^1 s^{-M(0)} s^{\gamma-1} D(st) y(st)\,\mathrm{d}s \in C[0,1]$$

follows. Thus, $\mathscr{K} : C[0,\delta] \to C[0,\delta]$. Moreover, $\mathscr{K}$ is a contraction due to the following estimates. Let $y_1, y_2 \in C[0,\delta]$, then

$$\begin{aligned}&\|\mathscr{K} y_1 - \mathscr{K} y_2\|_{\delta} = \max_{t\in[0,\delta]} \left| t^{\gamma} \int_0^1 s^{-M(0)} s^{\gamma-1} D(st)\left(y_1(st) - y_2(st)\right)\,\mathrm{d}s \right| \\ &\le \max_{t\in[0,\delta]} \left\{ t^{\gamma} \int_0^1 \left| s^{-M(0)} \right| s^{\gamma-1}\,\mathrm{d}s \max_{s\in[0,1]} |D(st)| \max_{s\in[0,1]} |y_1(st) - y_2(st)| \right\} \\ &\le \max_{t\in[0,\delta]} \left\{ t^{\gamma} \int_0^1 \left| s^{-M(0)} \right| s^{\gamma-1}\,\mathrm{d}s \max_{s\in[0,t]} |D(s)| \max_{s\in[0,t]} |y_1(s) - y_2(s)| \right\} \\ &\le \delta^{\gamma} const.\, \|D\|_{\delta} \|y_1 - y_2\|_{\delta}.\end{aligned}$$

Note that for $\gamma > 0$, $\int_0^1 |s^{-M(0)}| s^{\gamma-1}\,\mathrm{d}s = const.$ holds, see Lemma 3 [4]. Consequently, there exists a sufficiently small δ such that

$$\delta^{\gamma} const. \|D\|_{\delta} =: L_Z < 1, \tag{8.36}$$

and the operator $\mathscr{K}$ is a contraction. The Banach Fixed Point Theorem yields the existence of a unique continuous solution of (7.5) on $[0,\delta]$. This solution can be uniquely extended to the point $t=1$. The initial condition $M(0)y(0)=0$ follows from the form of $\mathscr{K}$ and Theorem 8.15.

Step 2. *Smoothness of the solution.*
Let $\alpha \geq r+1$, $\gamma \geq r+1$, $r \geq 1$, $f, D \in C^r[0,1]$, and $h \in C^r[0,\delta]$. Then, in follows from Theorem 8.15 that $\mathscr{K} : C^r[0,\delta] \to C^r[0,\delta]$. We again use the Banach Fixed Point Theorem, and thus we need to show that $\mathscr{K}$ is a contraction on $C^r[0,\delta]$ for a sufficiently small δ.

Let $r=1$. Then for $y_1, y_2 \in C^1[0,\delta]$ we have

$$\|\mathscr{K}y_1 - \mathscr{K}y_2\|_{C^1[0,\delta]} = \|\mathscr{K}y_1 - \mathscr{K}y_2\|_{C[0,\delta]} + \|\,(\mathscr{K}y_1)' - (\mathscr{K}y_2)'\,\|_{C[0,\delta]},$$

where $(\mathscr{K}y)'$ is given by

$$\begin{aligned}(\mathscr{K}y)'(t) &= \int_0^1 s^{-M(0)} f'(st)\,\mathrm{d}s + \gamma t^{\gamma-1}\int_0^1 s^{-M(0)} s^{\gamma-1} D(st)y(st)\,\mathrm{d}s \\ &+ t^{\gamma}\int_0^1 s^{-M(0)} s^{\gamma} D'(st)y(st)\,\mathrm{d}s + t^{\gamma}\int_0^1 s^{-M(0)} s^{\gamma} D(st)y'(st)\,\mathrm{d}s,\; t \in [0,\delta].\end{aligned}$$

Moreover, by (8.36), the following estimate holds:

$$\begin{aligned}&\|\mathscr{K}y_1 - \mathscr{K}y_2\|_{C^1[0,\delta]} = |\mathscr{K}y_1 - \mathscr{K}y_2\|_{\delta} + \|(\mathscr{K}y_1)' - (\mathscr{K}y_2)'\|_{\delta} \\ &\leq L_Z\|y_1 - y_2\|_{\delta} \\ &\quad + \max_{t\in[0,\delta]}\left\{\gamma t^{\gamma-1}\int_0^1 \left|s^{-M(0)}\right| s^{\gamma-1}\,\mathrm{d}s \max_{s\in[0,1]}|D(st)| \max_{s\in[0,1]}|y_1(st) - y_2(st)|\right\} \\ &\quad + \max_{t\in[0,\delta]}\left\{t^{\gamma}\int_0^1 \left|s^{-M(0)}\right| s^{\gamma}\,\mathrm{d}s \max_{s\in[0,1]}|D'(st)| \max_{s\in[0,1]}|y_1(st) - y_2(st)|\right\} \\ &\quad + \max_{t\in[0,\delta]}\left\{t^{\gamma}\int_0^1 \left|s^{-M(0)}\right| s^{\gamma}\,\mathrm{d}s \max_{s\in[0,1]}|D(st)| \max_{s\in[0,1]}|y_1'(st) - y_2'(st)|\right\} \\ &\leq \left(L_Z + \gamma\delta^{\gamma-1} const.\|D\|_{\delta} + \delta^{\gamma} const.\|D'\|_{\delta}\right)\|y_1 - y_2\|_{\delta} \\ &\quad + \delta^{\gamma} const.\|D\|_{\delta}\|y_1' - y_2'\|_{\delta}.\end{aligned}$$

Therefore, for a sufficiently small δ, $\mathscr{K}$ is a contracting operator on $C^1[0,\delta]$.

Let $r \geq 2$. By similar arguments we obtain a contraction on $C^r[0,\delta]$. This yields a unique solution $y \in C^r[0,\delta]$ of (7.5) on $[0,\delta]$, which can be uniquely extended to $t=1$. For $f, D \in C^r[0,1]$, the classical theory implies the existence of a unique solution $z \in C^r(0,1]$ of system (7.4) subject to the initial condition $z(\delta) = y(\delta)$. Hence, $z=y$ on $[0,1]$ and $y \in C^r[0,1]$ which completes the proof.

□

Remark 8.19 In the case when $M(0) = 0$, $(m = n)$, we can also study the unique solvability of TVP (7.6). The existence and uniqueness of a solution $z \in C(0,1]$ of (7.6) follows from the classical theory. To obtain the respective result for $[0,1]$, we investigate equation (7.4) subject to the terminal condition $y(\delta) = z(\delta)$, where $\delta > 0$ is sufficiently small. The corresponding operator $\mathscr{K} : C[0,\delta] \to C[0,\delta]$ has in this case the form

$$(\mathscr{K} y)(t) = z(\delta) + \int_\delta^t s^{-1} f(s)\,\mathrm{d}s + \int_\delta^t s^{\gamma-1} D(s) y(s)\,\mathrm{d}s.$$

We can show that $\mathscr{K}$ is contractive in the way analogous to the case when $M(0) \neq 0$.

9 General problems

In this chapter, we study general initial (7.5), terminal (7.6) and boundary value problems (7.3).

For the subsequent discussion, we need to introduce the notation to the matrix $M(0)$:

X_+ is the invariant subspace associated with the eigenvalues with positive real parts;
$X_0^{(e)}$ is the space spanned by the eigenvectors associated with eigenvalues $\lambda = 0$;
X_- is the invariant subspace associated with the eigenvalues with negative real parts;
$X_0^{(h)}$ is the space spanned by the generalized eigenvectors associated with $\lambda = 0$;
S is the orthogonal projection onto X_+;
R is the orthogonal projection onto $X_0^{(e)}$;
$P := R + S$ is the projection onto $X_+ \oplus X_0^{(e)}$;
$Q := I - P$ is the projection onto $X_- \oplus X_0^{(h)}$;
Z is the orthogonal projection onto $X_0^{(e)} \oplus X_0^{(h)}$;
N is the orthogonal projection onto X_-;
H is the orthogonal projection onto $X_0^{(h)}$.

All projections are constructed using the generalized eigenbasis of $M(0)$. Later on, we also use $\tilde{R}$, $\tilde{P}$ for the matrices consisting of the maximal set of linearly independent columns of respective projections,

Note that if $M(0)$ is already in the Jordan form, the matrices representing all projections are diagonal matrices whose only nonzero elements are 1.

9.1 General IVPs, TVPs

We first discuss general IVPs (7.5) and TVPs (7.6), where all conditions which are necessary and sufficient to specify a unique solution $y \in C[0,1]$ are posed at only one point, either at $t = 0$ or at $t = 1$. According to the results derived above, restrictions on the spectrum of $M(0)$ need to be made.

A.1 For IVP (7.5) we assume that the matrix $M(0)$ has only eigenvalues with nonpositive real parts and if $\sigma = 0$ then $\lambda = 0$.

A.2 For TVP (7.6) we assume that the matrix $M(0)$ has only eigenvalues with nonnegative real parts and if $\sigma = 0$ then $\lambda = 0$. Additionally, if zero is an eigenvalue of $M(0)$, then the associated invariant subspace is assumed to be the eigenspace of $M(0)$.

Results formulated below without proofs are simple consequences of Theorems 8.5, 8.11, 8.15, and Lemma 8.17.

Lemma 9.1 *Let us assume that $f \in C[0,1]$, $M \in C[0,1]$, Zf satisfies condition* (8.31) *and ZM, satisfy* (8.35).

*(i) Assume A.*1 *to hold. Let y be a continuous solution of IVP* (7.5). *Then*

$$M(0)Ny(0) = -Nf(0), \quad Hy(0) = 0.$$

*(ii) Assume A.*2 *to hold. Let y be a solution of TVP* (7.6). *Then $y \in C[0,1]$ and*

$$M(0)Sy(0) = -Sf(0).$$

In both cases

$$M(0)y(0) = -f(0).$$

The statement of Lemma 9.1 means that the conditions which are necessary for the solution of IVP (7.5) to be continuous are equivalent to

$$\operatorname{rank} M(0) = \operatorname{rank} H + \operatorname{rank} N = \operatorname{rank} Q = n - \operatorname{rank} R$$

initial conditions, which the solution y has to satisfy. In case of TVP (7.6), where A.2 holds, any solution of (7.4))is continuous on $[0,1]$ and no regularity conditions have to be prescribed.

Theorem 9.2 *Let us assume that A.*1 *holds, the $m \times m$ matrix $B_0\tilde{R}$ is nonsingular, and $\beta \in \mathbb{R}^m$. Then, for any $f \in C[0,1]$ such that Zf satisfies* (8.31) *and ZM satisfies* (8.35), *there exists a unique solution $y \in C[0,1]$ of IVP* (7.5). *The solution satisfies initial condition*

$$M(0)Ny(0) = -Nf(0), \quad Hy(0) = 0.$$

which are necessary and sufficient for solution to be continuous on $[0,1]$.

Proof: The existence of a unique continuous solution follows from Theorems 8.6 and 8.18 by applying the Banach Fixed Point Theorem to the linear operator $\mathscr{K} : C[0,\delta] \to C[0,\delta], \delta > 0$, defined by

$$(\mathscr{K}y)(t) = \tilde{R}(B_0\tilde{R})^{-1}\beta + \int_0^1 s^{-M(0)}s^{-1}f(st)\,\mathrm{d}s + t^{\gamma}\int_0^1 s^{-M(0)}s^{\gamma-1}D(st)y(st)\,\mathrm{d}s.$$

For a sufficiently small δ, the operator $\mathscr{K}$ is a contraction and the Banach Fixed Point Theorem yields the existence of a unique continuous solution of (7.5) on the interval $[0,\delta]$. This solution can be uniquely extended to $t = 1$.

□

Theorem 9.3 *Let us assume that A.2 holds, $B_1 \in \mathbb{R}^{n\times n}$ is nonsingular, and $\beta \in \mathbb{R}^n$. Then, for any $f \in C[0,1]$ such that Rf satisfies* (8.31), *and RM satisfies condition* (8.35), *there exists a unique solution $y \in C[0,1]$ of TVP* (7.6).

Proof: The existence and uniqueness of solution $z \in C(0,1]$ of problem (7.6) follows from the classical theory. Therefore, we investigate (7.4) on $[0,\delta]$, $\delta \in (0,1]$, subject to $y(\delta) = z(\delta)$. The result follows from Theorem 8.12 and Remark 8.19, when the Banach Fixed Point Theorem is applied to the operator $\mathscr{K} : C[0,\delta] \to C[0,\delta]$, $\delta \in (0,1]$, defined by

$$
\begin{aligned}
(\mathscr{K}y)(t) :=& t^{M(0)}\delta^{-M(0)}z(\delta) + t^{M(0)}\int_\delta^t s^{-M(0)-I}f(s)\,\mathrm{d}s \\
&+ t^{M(0)}\int_\delta^t s^{-M(0)}s^{\gamma-1}D(s)y(s)\,\mathrm{d}s.
\end{aligned}
$$

Since for a sufficiently small δ the operator $\mathscr{K}$ is a contraction, there exists a unique continuous solution y of system (7.4) on $[0,\delta]$. Since $y(\delta) = z(\delta)$, $y = z$ on $[0,1]$ follows and this completes the proof.

□

9.2 General BVPs

Finally, we study general BVPs of the form

$$
y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t}, \quad B_0y(0) + B_1y(1) = \beta, \tag{9.1}
$$

stated in Section 7.2 as problem (7.3). We point out that matrix $M(0)$ may have an arbitrary spectrum for general BVPs. Before proceeding with the analysis, we show two auxiliary results.

Lemma 9.4 *The projection matrix R fulfils*

$$
t^{M(0)}R = R,\ t \in [0,1].
$$

Proof: Let E denotes the associated matrix of generalized eigenvectors of $M(0)$ and J denotes the Jordan canonical form of $M(0)$. Let $M(0)$ and R be represented using the eigenbasis of $M(0)$, which means $M(0) = EJE^{-1}$ and $R = E\hat{R}E^{-1}$, where $\hat{R}$ is a diagonal matrix with ones at the positions corresponding to the eigenvalues $\lambda = 0$ and zero entries elsewhere. It is sufficient to show $t^J\hat{R} = \hat{R}$ since this implies

$$
t^{M(0)}R = Et^JE^{-1}E\hat{R}E^{-1} = E\hat{R}E^{-1} = R.
$$

The multiplication $t^J\hat{R}$ means an operation on columns of t^J and due to the structure of $\hat{R}$, the result of the matrix multiplication $t^J\hat{R}$ is a matrix whose columns corresponding to the eigenvectors associated with $\lambda = 0$ remain unchanged and all other columns vanish. Clearly, the nontrivial columns of $t^J\hat{R}$ are unit vectors, since $t^\lambda = 1$ for $\lambda = 0$, and the result, $t^J\hat{R} = \hat{R}$, follows. □

Lemma 9.5 *The projection matrices S, Z and N commute with the matrices $t^{M(0)}$ and $M(0)$.*

Proof: We show the result for the projection S, for the other projections the proof is analogous. First, we prove that for the Jordan canonical form $t^J\hat{S} = \hat{S}t^J$ holds for $t \in [0,1]$, where $M(0) = EJE^{-1}$ and $S = E\hat{S}E^{-1}$. The matrix $\hat{S}$ is a diagonal matrix with ones at the positions corresponding to the eigenvalues with positive real parts and zero entries elsewhere. The multiplication $\hat{S}t^J$ means operations on rows of t^J, while the multiplication $t^J\hat{S}$ represents operations on columns of t^J. Being aware of the block structure of matrix t^J we see that the result of $\hat{S}t^J$ or $t^J\hat{S}$ is a matrix containing Jordan boxes corresponding to eigenvalues with positive real parts and $t^J\hat{S} = \hat{S}t^J$ for $t \in [0,1]$. This implies

$$t^{M(0)}S = Et^JE^{-1}E\hat{S}E^{-1} = E\hat{S}E^{-1}Et^JE^{-1} = St^{M(0)}$$

and the statement follows. By similar arguments we can prove that $M(0)S = SM(0)$. $\square$

To specify the boundary conditions which guarantee the unique solvability of BVP (9.1) the following lemma is required.

Lemma 9.6 *Consider the BVP*

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t},\ t \in (0,1], \tag{9.2}$$

$$Hy(0) = 0,\ M(0)Ny(0) = -Nf(0),\ Sy(1) = S\eta,\ Ry(0) = R\eta. \tag{9.3}$$

Let us assume that $f \in C[0,1]$ is such that Zf satisfies (8.31)*. Moreover, let $M \in C[0,1]$ be given in such a way that the projection ZM satisfies condition* (8.35)*, and $\eta \in \mathbb{R}^n$. Then, there exists a unique solution $y \in C[0,1]$ of the BVP* (9.2)*,* (9.3)*.*

Proof: According to the three case studies, we search for the solution y of (9.2) in the form of three contributions which depend on the real parts of eigenvalues of $M(0)$,

$$y = Ny + Sy + Zy.$$

First, we consider system (9.2) posed on an interval $[0,\delta]$, $\delta \in (0,1]$, subject to the boundary conditions

$$Hy(0) = 0,\ M(0)Ny(0) = -Nf(0),\ Sy(\delta) = S\zeta,\ Ry(0) = R\eta, \tag{9.4}$$

where $\zeta \in \mathbb{R}^n$ is specified later. In order to prove the existence of a unique continuous solution of (9.2), (9.4), we apply the Banach Fixed Point Theorem to

$$y = \mathscr{K}y,\ y \in C[0,\delta].$$

The operator $\mathscr{K} : C[0,\delta] \to C[0,\delta]$ is now defined as

$$(\mathscr{K}y)(t) := N(\mathscr{K}y)(t) + S(\mathscr{K}y)(t) + Z(\mathscr{K}y)(t),$$

where

$$\begin{aligned}
N(\mathscr{K}y)(t) :=& N\int_0^1 s^{-M(0)}s^{-1}f(st)\,\mathrm{d}s + N\int_0^1 s^{-M(0)}s^{-1}\left(M(st)-M(0)\right)y(st)\,\mathrm{d}s,\\
S(\mathscr{K}y)(t) :=& \left(\frac{t}{\delta}\right)^{M(0)} S\zeta + t^{M(0)}S\int_\delta^t s^{-M(0)-I}f(s)\,\mathrm{d}s\\
&+ t^{M(0)}S\int_\delta^t s^{-M(0)}s^{-1}\left(M(s)-M(0)\right)y(s)\,\mathrm{d}s,\\
Z(\mathscr{K}y)(t) :=& t^{M(0)}R\eta + Z\int_0^1 s^{-M(0)}s^{-1}f(st)\,\mathrm{d}s\\
&+ t^{\gamma}Z\int_0^1 s^{-M(0)}s^{\gamma-1}D(st)y(st)\,\mathrm{d}s.
\end{aligned}$$

According to Theorems 8.6, 8.12, and 8.18, the operator $\mathscr{K}$ maps $C[0,\delta]$ into itself. Moreover, we can choose δ sufficiently small for

$$L_N < \frac{1}{3},\ L_S < \frac{1}{3},\ L_Z < \frac{1}{3}$$

to hold, *cf.* (8.19), (8.28) and (8.36). Therefore, $L := L_N + L_S + L_Z < 1$ and

$$\begin{aligned}
\|\mathscr{K}y_1 - \mathscr{K}y_2\|_\delta \le& \|N(\mathscr{K}y_1 - \mathscr{K}y_2)\|_\delta + \|S(\mathscr{K}y_1 - \mathscr{K}y_2)\|_\delta + \|Z(\mathscr{K}y_1 - \mathscr{K}y_2)\|_\delta\\
\le& (L_N + L_S + L_Z)\|y_1 - y_2\|_\delta = L\|y_1 - y_2\|_\delta.
\end{aligned}$$

Consequently, $\mathscr{K}$ is a contracting operator on $C[0,\delta]$. According to the Banach Fixed Point Theorem, there exists a unique continuous solution y of problem (9.2), (9.4) on $[0,\delta]$. Since there is no singularity on $[\delta,1]$, we can use the classical theory to extend the solution contributions Ny and Zy to $[0,1]$. Moreover, the existence and uniqueness of the solution contribution $Sz \in C[\delta,1]$ of regular equation (9.2) subject to $Sz(1) = S\eta$ follows from the classical theory. Therefore, we choose ζ such that $S\zeta = Sz(\delta)$ and we put $Sy := Sz$ on $[\delta,1]$. Altogether, $Sy(\delta) = Sz(\delta)$ and Sy is the second solution contribution defined on $[0,1]$.

□

In order to discuss the solvability of (9.1), we first provide equivalent form of representation of the solution of (9.2), (9.3).

Lemma 9.7 *Let us assume that $f \in C[0,1]$ is such that Zf satisfies* (8.31). *Moreover, let $M \in C[0,1]$ be given in such a way that the projection ZM satisfies condition* (8.35), *and $\eta \in \mathbb{R}^n$. Then, BVP* (9.2), (9.3) *is equivalent to* (9.2) *subject to the boundary conditions*

$$Hy(0) = 0,\ M(0)Ny(0) = -Nf(0),\ Py(1) = P\tilde{\eta},$$

where
$$\tilde{\eta} = \eta + R\int_0^1 s^{-M(0)-I} f(s)\,\mathrm{d}s + R\int_0^1 s^{-M(0)-I} M(s) y(s)\,\mathrm{d}s.$$

Proof: Let u be the unique continuous solution of (9.2), (9.3) given by Lemma 9.6. Then $y = Ny + Sy + Zy$, where

$$Ny(t) = N\int_0^1 s^{-M(0)} s^{-1} f(st)\,\mathrm{d}s + N\int_0^1 s^{-M(0)} s^{-1} \left(M(st) - M(0)\right) y(st)\,\mathrm{d}s =$$
$$Nt^{M(0)}\int_0^t s^{-M(0)-I} f(s)\,\mathrm{d}s + Nt^{M(0)}\int_0^t s^{-M(0)-I} \left(M(s) - M(0)\right) y(s)\,\mathrm{d}s$$
$$Sy(t) = t^{M(0)}S\eta + t^{M(0)}S\int_1^t s^{-M(0)-I} f(s)\,\mathrm{d}s + t^{M(0)}S\int_1^t s^{-M(0)-I} \left(M(s) - M(0)\right) y(s)\,\mathrm{d}s,$$
$$Zy(t) = t^{M(0)}R\eta + Z\int_0^1 s^{-M(0)-I} f(st)\,\mathrm{d}s + Z\int_0^1 s^{-M(0)-I} \left(M(st) - M(0)\right) y(st)\,\mathrm{d}s.$$

First, we rewrite terms in $Zy(t)$, $t \in [0,1]$. To this aim, we use the shorthand notation $g(t, y(t)) = f(t) + (M(t) - M(0))\, y(t)$, then

$$\begin{aligned}
Zy(t) &= R\eta + Z\int_0^1 s^{-M(0)} s^{-1} g(st, y(st))\,\mathrm{d}s = R\eta + Zt^{M(0)}\int_0^t s^{-M(0)-I} g(s, y(s))\,\mathrm{d}s \\
&= R\eta + t^{M(0)}R\int_0^t s^{-M(0)-I} g(s, y(s))\,\mathrm{d}s + t^{M(0)}H\int_0^t s^{-M(0)-I} g(s, y(s))\,\mathrm{d}s \\
&= R\tilde{\eta} + t^{M(0)}R\int_1^t s^{-M(0)-I} g(s, (y(s))\,\mathrm{d}s + t^{M(0)}H\int_0^t s^{-M(0)-I} g(s, y(s))\,\mathrm{d}s.
\end{aligned}$$

Finally, we obtain

$$Zy(t) = R\tilde{\eta} + t^{M(0)}R\int_1^t s^{-M(0)-I} f(s)\,\mathrm{d}s + t^{M(0)}H\int_0^t s^{-M(0)-I} f(s)\,\mathrm{d}s$$
$$+ t^{M(0)}R\int_1^t s^{-M(0)-I} \left(M(s) - M(0)\right) y(s)\,\mathrm{d}s + t^{M(0)}H\int_0^t s^{-M(0)-I} \left(M(s) - M(0)\right) y(s)\,\mathrm{d}s,$$

where
$$\tilde{\eta} = \eta + R\int_0^1 s^{-M(0)-I} f(s)\,\mathrm{d}s + R\int_0^1 s^{-M(0)-I} M(s) y(s)\,\mathrm{d}s.$$

Recall that $P = S + R$, $Q = N + H$, $Z = R + H$ and note that $S\tilde{\eta} = S\eta$, $t^{M(0)}R = R$. Consequently, the solution $y \in C[0,1]$ of (9.2), (9.3) has the integral representation:

$$\begin{aligned}
y(t) = {} & t^{M(0)}P\tilde{\eta} + t^{M(0)}P\int_1^t s^{-M(0)-I} f(s)\,\mathrm{d}s + t^{M(0)}Q\int_0^t s^{-M(0)-I} f(s)\,\mathrm{d}s \\
& + t^{M(0)}P\int_1^t s^{-M(0)-I} \left(M(s) - M(0)\right) y(s)\,\mathrm{d}s \\
& + t^{M(0)}Q\int_0^t s^{-M(0)-I} \left(M(s) - M(0)\right) y(s)\,\mathrm{d}s,
\end{aligned}$$

and satisfies the boundary conditions

$$Hy(0) = 0,\ M(0)Ny(0) = -Nf(0),\ Py(1) = P\tilde{\eta}.$$

The reverse statement can be proved analogously.

$\square$

Remark 9.8 The function

$$t^{M(0)}H\int_0^t s^{-M(0)-I}f(s)\,\mathrm{d}s = t^{M(0)}Ht^{-M(0)}\int_0^1 s^{-M(0)-I}f(st)\,\mathrm{d}s,$$

is continuous on $[0,1]$. In order to see this, we again use functions z_m and z_∞ given by (8.15) and (8.16). Due to (8.7), (8.30), (8.31), and (8.33), we have

$$\lim_{m\to\infty}|(\ln t)^k z_\infty(t) - (\ln t)^k z_m(t)| \le \|h\|_\delta \|t^\alpha(\ln t)^k\| \lim_{m\to\infty}\int_0^{\frac{1}{m}} \left|s^{-M}\right| s^{\alpha-1}\,\mathrm{d}s = 0$$

for $k\in\mathbb{N}_0$. Since each entry of the matrix $t^M Ht^{-M}$ is a sum of terms $const.\ (\ln t)^k$, $k\in\mathbb{N}_0$, the continuity result follows. Moreover,

$$t^{M(0)}H\int_0^t s^{-M(0)-I}M(s)y(s)\,\mathrm{d}s$$

is continuous for $y\in C[0,1]$. This can be proved by similar arguments as in the proof of Theorem 8.18 by virtue of Remark 8.1 and (8.35).

By means of results stated in Lemma 9.6 and Lemma 9.7, we can proceed to the general boundary conditions

$$B_0y(0) + B_1y(1) = \beta. \tag{9.5}$$

Let y be a solution of (9.2). Clearly, the conditions

$$Hy(0) = 0,\ M(0)Ny(0) = -Nf(0)$$

are necessary for $y\in C[0,1]$. Furthermore, the $m = \operatorname{rank}P$ conditions

$$Py(1) = P\tilde{\eta}$$

need to be specified to ensure the uniqueness of the solution. The question whether this solution satisfies boundary conditions (9.5) is answered in the next theorem.

Theorem 9.9 *Consider BVP* (9.1)*, where the inhomogeneity f is given in such a way such that $f\in C[0,1]$ and Zf satisfies* (8.31)*. Let the coefficient matrix $M\in C[0,1]$ be such that its projections ZM satisfy condition* (8.35)*. Moreover, let $B_0, B_1\in\mathbb{R}^{m\times n}$, $\beta\in\mathbb{R}^m$, $m = \operatorname{rank}P$, and the $m\times m$ matrix $B_0\tilde{R} + B_1\tilde{P}$ be nonsingular. Then, BVP* (9.1) *has a unique continuous solution $y\in C[0,1]$. This solution satisfies two sets of initial conditions,*

$$Hy(0) = 0, \quad M(0)Ny(0) = -Nf(0)$$

which are necessary and sufficient for $y\in C[0,1]$.

Proof: For the further investigations we provide an alternative representation of a general continuous solution of system (9.2) by means of the superposition principle. Let $\tilde{y}$ be the unique particular solution of (9.2) subject to boundary conditions

$$H\tilde{y}(0) = 0, \quad M(0)N\tilde{y}(0) = -Nf(0), \quad P\tilde{y}(1) = 0.$$

According to Lemma 9.7, the particular solutions $\tilde{y}$ exists, is unique and continuous on $[0,1]$. Moreover, let Y be the unique continuous $n \times m$ fundamental solution matrix of the homogeneous system

$$Y'(t) = \frac{M(t)}{t} Y(t),\ t \in (0,1],$$

with $Y(1) = \tilde{P}$, where $\tilde{P}$ is the $n \times m$ matrix consisting of the linearly independent columns of P. The existence of continuous solution matrix follows from Theorem 9.3. Then a general continuous solution of system (8.1) had the form

$$y(t) = \tilde{y}(t) + Y(t)\alpha,\ t \in [0,1],$$

where $\alpha \in \mathbb{R}^m$.

We turn to the general boundary conditions specified in (9.1), where $B_0, B_1 \in \mathbb{R}^{m\times n}$, $\beta \in \mathbb{R}^m$, and $m = \operatorname{rank} P$. Since $Hy(0) = 0$ and $\lim_{t\to 0} t^M S = 0$, we conclude from the above solution representation $y(0)$:

$$\begin{aligned} y(0) &= (H+P+N)y(0) = (P+N)(\tilde{y}(0)+Y(0)\alpha) = (P+N)\tilde{y}(0) + PY(0)\alpha \\ &= (P+N)\tilde{y}(0) + (R+S)Y(0)\alpha = (P+N)\tilde{y}(0) + \tilde{R}\alpha. \end{aligned}$$

Moreover, from $\tilde{P}y(1) = 0$ and $1^M S = S$, we deduce

$$y(1) = (Q+P)y(1) = (Q+P)(\tilde{y}(1)+Y(1)\alpha) = Q\tilde{y}(1) + (QY(1)+\tilde{P})\alpha = Q\tilde{y}(1) + \tilde{P}\alpha.$$

After substituting $y(0)$ and $y(1)$ into the boundary conditions (9.1), we obtain

$$B_0 y(0) + B_1 y(1) = B_0\left((P+N)\tilde{y}(0) + \tilde{R}\alpha\right) + B_1\left(Q\tilde{y}(1) + \tilde{P}\alpha\right) = \beta.$$

Thus,

$$\left(B_0\tilde{R} + B_1\tilde{P}\right)\alpha = \beta - B_0(P\tilde{y}(0) + N\tilde{y}(0)) - B_1 Q\tilde{y}(1),$$

and the unknown vector α can be uniquely determined if the $m \times m$ matrix

$$B_0\tilde{R} + B_1\tilde{P}$$

is nonsingular. This completes the proof. □

Remark 9.10 The smoothness results $y \in C^r[0,1]$ follow by applying the smoothness results derived separately in Sections 8.2, 8.3, and 8.4 for components of the solutions associated with eigenvalues with negative real parts, positive real parts and zero eigenvalues, respectively.

10 Numerical analysis

We are interested in analysing the convergence properties of the polynomial collocation as a numerical approach to solve singular problems

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t},\ t \in (0,1], \tag{10.1}$$

$$B_0 y(0) + B_1 y(1) = \beta, \tag{10.2}$$

stated as problem (7.3) in Section 7.2 and problem (9.1) in Section 9.2. The decision to use collocation is motivated by its advantageous convergence properties for (7.4). The collocation does not suffer from order reduction in the presence of singularities. For problems with smooth solutions, the convergence order is at least equal to the so-called stage order of the method. For the collocation schemes, this convergence results mean that the global error of a collocation scheme with k inner collocation points is $O(h^k)$ uniformly in t.

10.1 Collocation method

We introduce a class of collocation methods applied to approximate solution y of problem (10.1), (10.2) which we assume to be uniquely solvable in $C[0,1]$. We first choose $I, k \in \mathbb{N}$ and discretize problem (10.1), (10.2). To this aim, the interval of integration $[0,1]$ is partitioned by an equidistant mesh Δ,

$$\Delta := \{0 = t_0 < t_1 < \ldots < t_{I-1} < t_I = 1,\ \ t_j = jh,\ j = 0,\ldots,I = 1/h\},$$

and in each subinterval $[t_j, t_{j+1}]$ we introduce k equidistantly spaced collocation nodes $t_{jl} := t_j + u_l h,\ j = 0,\ldots,I-1,\ l = 1,\ldots,k$, where $0 < u_1 < \ldots < u_k \leq 1$. The computational grid including the mesh points and the collocation points is shown in Figure 10.1.

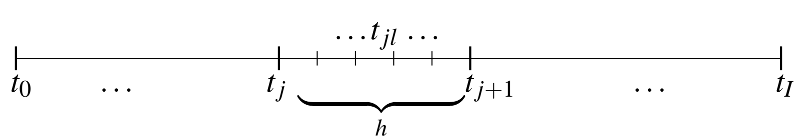

Figure 10.1: The computational grid

By $\mathscr{P}_{k,h}$, we denote the class of piecewise polynomial functions which are globally continuous on $[0,1]$ and reduce in each subinterval $[t_j, t_{j+1}]$ to a polynomial of degree less or equal to k. We approximate the analytical solution y by a piecewise polynomial function $p \in \mathscr{P}_{k,h}$, such that p satisfies system (10.1) at the collocation points,

$$p'(t_{jl}) - \frac{M(t_{jl})}{t_{jl}} p(t_{jl}) = \frac{f(t_{jl})}{t_{jl}}, \;\; l = 1,\dots,k, \;\; j = 0,\dots,I-1,$$

the boundary conditions

$$B_0 p(0) + B_1 p(1) = \beta,$$

and the continuity relations,

$$p_{j-1}(t_j) = p_j(t_j), \;\; j = 1,\dots,I-1,$$

where $p(t) := p_j(t)$, $t \in [t_j, t_{j+1}]$.

For the subsequent analysis, we assume $M \in C^1[0,1]$ which yields

$$M(t) = M(0) + tD(t), \; t \in [0,1], \; D \in C[0,1]. \tag{10.3}$$

Moreover, if $M(0)$ has eigenvalues with positive real parts, we assume that the smallest positive real part $\sigma_+ > 1$. This does not mean a restriction of generality since using the transformation $t = \tau^\mu$, $\mu > 1$, we can enlarge the smallest positive real part according to $\tilde{\sigma}_+ = \mu\sigma_+$, where $\tilde{\sigma}_+$ is the smallest positive real part of the eigenvalues of the transformed system, see Section 10.3, TVP with small positive eigenvalues.

10.2 Convergence results

In this sections, we first discuss the convergence of collocation schemes of IVPs and TVPs. Then, we generalize these results to general BVPs.

Convergence of the collocation scheme for IVPs

Here, we restrict our attention to the class of singular BVPs which can be equivalently expressed as a well-posed IVP, where all boundary conditions are posed at $t = 0$. In this case, we have to assume that the matrix $M(0)$ has only eigenvalues $\lambda := \sigma + i\rho$, with nonpositive real parts, and if $\sigma = 0$ then $\lambda = 0$. These restrictions are necessary to ensure the existence of a well-posed IVP, see assumption A.1 in Section 9.1. Let H and N denote projections onto the subspace spanned by the principal eigenvectors associated with zero eigenvalues and the subspace spanned by the eigenvectors associated with eigenvalues with negative real parts, respectively.

The underlying IVP has the form

$$y'(t) = \frac{M(t)}{t} y(t) + \frac{f(t)}{t}, \;\; B_0 y(0) = \beta, \;\; Hy(0) = 0, \;\; M(0)Ny(0) = -Nf(0), \tag{10.4}$$

where $B_0 \in \mathbb{R}^{m\times n}$, $\beta \in \mathbb{R}^m$, and $\operatorname{rank} R = m \le n$. Note that conditions $Hy(0) = 0$ and $M(0)Ny(0) = -Nf(0)$ are necessary and sufficient for the analytical solution of (7.5) to be continuous, see Theorem 9.2.

Collocation methods for linear problems with a smooth inhomogeneity are studied in [46], where in particular, the unique solvability of the collocation scheme and its convergence properties are shown. For the reader's convenience we recapitulate in the next theorem an important auxiliary result from [46] required in the subsequent investigations.

Lemma 10.1 (Theorem 4.1 in [46]) *Let us consider the collocation scheme,*

$$p'(t_{jl}) - \frac{M(t_{jl})}{t_{jl}} p(t_{jl}) = M^{\mu}(0)\frac{c_{jl}}{t_{jl}^{\nu}}, \;\; l = 1,\dots,k, \;\; j = 0,\dots I-1, \quad p(0) = \delta, \tag{10.5}$$

where $\mu, \nu \in \{0,1\}$, $M \in C^1[0,1]$, $\delta \in R^n$ and c_{jl} are arbitrary constants. Then problem (10.5) *has a unique solution $p \in \mathscr{P}_{k,h}$ provided that h is sufficiently small. This solution satisfies*

$$|p(t)| \le const. \left(|\delta| + |\ln(h)|^d |M(0)\delta| + |\ln(h)|^{(\nu(d-\mu))_+} C_I\right), \; t \in [0,1],$$

where d is the dimension of the largest Jordan box of $M(0)$ associated with the eigenvalue $\lambda = 0$,

$$(x)_+ = \begin{cases} x, & x \ge 0, \\ 0, & x < 0, \end{cases}$$

and

$$C_I = \max_{0\le j\le I-1} \max_{1\le l\le k} |c_{jl}|.$$

By using Lemma 10.1, we can formulate the convergence result for the collocation method applied to IVP (10.4). For the convergence analysis, we rewrite (10.4) to obtain a more convenient form,

$$y'(t) - \frac{M(t)}{t} y(t) = \frac{f(t)}{t}, \quad t \in (0,1], \quad y(0) = \delta, \tag{10.6}$$

where

$$B_0\delta = \beta, \quad H\delta = 0, \quad M(0)N\delta = -Nf(0). \tag{10.7}$$

In Theorems 8.6 and 8.18, conditions on data functions for the existence of a unique solution $y \in C^{k+1}[0,1]$ of problem (10.6), (10.7) are given. To avoid repetition of all these conditions, we simply assume that the analytical solution $y \in C^{k+1}[0,1]$ in the theorem below.

Theorem 10.2 *Let us assume that $y \in C^{k+1}[0,1]$ is the unique solution of problem* (10.6)*,* (10.7) *and $M \in C^1[0,1], f \in C[0,1]$. Let the function $p \in \mathscr{P}_{k,h}$ be the unique solution of the collocation scheme,*

$$p'(t_{jl}) - \frac{M(t_{jl})}{t_{jl}} p(t_{jl}) = \frac{f(t_{jl})}{t_{jl}}, \quad l = 1, \ldots k, \ j = 0, \ldots, I-1, \quad p(0) = \delta.$$

Then

$$\|p - y\| \leq const.h^k.$$

Proof: To prove the convergence of the collocation scheme applied to solve IVP (10.6), (10.7), we first define an *error function* $e \in \mathscr{P}_{k,h}$,

$$e'(t_{jl}) := y'(t_{jl}) - p'(t_{jl}), \quad l = 1, \ldots, k, \ j = 0, \ldots, I-1, \quad e(0) := 0, \quad (10.8)$$

and show that the error function e differs from the global error $p-y$ by $O(h^k)$ terms. Clearly, since the function $e'(t)$ belongs to $\mathscr{P}_{k-1,h}$, it is uniquely determined by its values at k distinct points in each subinterval $[t_j, t_{j+1}]$, $j = 0, \ldots I-1$,

$$e'(t) = \sum_{i=1}^{k} l_i \left(\frac{t - t_j}{h} \right) y'(t_{ji}) - p'(t), \ t \in (t_j, t_{j+1}],$$

where

$$l_i(t) = w(t) / \left((t - u_i) w'(u_i) \right), i = 1, \ldots, k, \ \ w(t) = (t-u_1)(t-u_2) \cdots (t-u_k). \quad (10.9)$$

For $y \in C^{k+1}[0,1]$, the interpolation error is $O(h^k)$ and hence,

$$e'(t) = y'(t) - p'(t) + O(h^k)$$

which by integration on $[0,t]$ yields

$$e(t) = y(t) - p(t) + O(h^k t), \ t \in [0,1],$$

which means that e differs from $y - p$ by $O(h^k)$ terms. Moreover, we see that e satisfies the collocation scheme:

$$\begin{aligned}
e'(t_{jl}) - \frac{M(t_{jl})}{t_{jl}} e(t_{jl}) &= \\
&= y'(t_{jl}) - \frac{M(t_{jl})}{t_{jl}} y(t_{jl}) - \left(p'(t_{jl}) - \frac{M(t_{jl})}{t_{jl}} p(t_{jl}) \right) - \frac{M(t_{jl})}{t_{jl}} O(t_{jl} h^k) \\
&= \frac{f(t_{jl})}{t_{jl}} - \frac{f(t_{jl})}{t_{jl}} - \frac{M(t_{jl})}{t_{jl}} O(t_{jl} h^k) = O(M(0) h^k + t_{jl} D(t_{jl}) h^k) \\
&= O(M(0) h^k), \quad e(0) = 0.
\end{aligned}$$

According to Lemma 10.1, with $\mu = 0$, $\nu = 0$, and $c_{jl} = O(h^k)$, we conclude that the error function $e = O(h^k)$ which together with $e(t) = y(t) - p(t) + O(h^k)$ yields the estimate for the global error $\|p - y\|$. $\square$

The especially attractive property of the collocation is the so-called superconvergence. For regular ODEs and suitably chosen collocation points (Gaussian, Lobatto, Radau), the superconvergence order in the *mesh points* can be considerably higher than k, provided that the solution y is sufficiently smooth. For Gaussian points the superconvergence order is $O(h^{2k})$. For singular problems considered here, the superconvergence order cannot be expected to hold, in general. Counterexamples in [46] show that the superconvergence order does not hold even for singular problems with a smooth inhomogeneity. However, the so-called small superconvergence uniform in t can be shown, see the next theorem. The main prerequisite for the proof is the property

$$\int_0^1 w(s)\,\mathrm{d}s = 0 \tag{10.10}$$

which holds for an appropriate choice of the collocation points and w from (10.9).

Theorem 10.3 *Let us assume that the solution y of* (10.6)*,* (10.7) *satisfies* $y \in C^{k+2}[0,1]$. *If* (10.10) *holds, then the estimate for the global error given in Theorem 10.2 can be replaced by*

$$\|p-y\| \le const.\, h^{k+1}|\ln(h)|^{(d-1)_+}.$$

Proof: Let us consider again the error function e defined in (10.8). Due to the smoothness assumptions made for the solution y, we have for $j = 0,\dots I-1$ and l_i from (10.9),

$$\begin{aligned} e'(t) &= \sum_{i=1}^{k} l_i\left(\frac{t-t_j}{h}\right) y'(t_{ji}) - p'(t) \\ &= y'(t) - p'(t) + \frac{h^k}{k!} w\left(\frac{t-t_j}{h}\right) y^{(k+1)}(t_j) + O(h^{k+1}), \quad t \in (t_j, t_{j+1}]. \end{aligned}$$

We integrate e' on $[0,t]$, $t \in (t_j, t_{j+1}]$, and use (10.10) to obtain

$$\begin{aligned} e(t) &= y(t) - p(t) + \sum_{i=0}^{j-1} \frac{h^k}{k!} y^{(k+1)}(t_i) \int_{t_i}^{t_{i+1}} w\left(\frac{s-t_i}{h}\right)\mathrm{d}s \\ &+ \frac{h^k}{k!} y^{(k+1)}(t_j) \int_{t_j}^{t} w\left(\frac{s-t_j}{h}\right)\mathrm{d}s + O(th^{k+1}) = y(t) - p(t) + O(h^{k+1}). \end{aligned}$$

This implies

$$\begin{aligned} & e'(t_{jl}) - \frac{M(t_{jl})}{t_{jl}} e(t_{jl}) \\ &= y'(t_{jl}) - \frac{M(t_{jl})}{t_{jl}} y(t_{jl}) - \left(p'(t_{jl}) - \frac{M(t_{jl})}{t_{jl}} p(t_{jl})\right) - \frac{M(t_{jl})}{t_{jl}} O(h^{k+1}) \\ &= -\frac{M(t_{jl})}{t_{jl}} O(h^{k+1}) = O\left(\frac{M(0)}{t_{jl}} h^{k+1}\right) + O(D(t_{jl})h^{k+1}) \\ &= O\left(\frac{M(0)}{t_{jl}} h^{k+1}\right), \quad e(0) = 0. \end{aligned}$$

According to Lemma 10.1, with $\mu = 1$, $\nu = 1$, and $c_{jl} = O(h^{k+1})$, we conclude

$$|e(t)| \leq const. \left(|\ln(h)|^{(d-1)_+} h^{k+1} \right),$$

and thus,

$$\|p - y\| \leq const. \left(|\ln(h)|^{(d-1)_+} h^{k+1} \right).$$

□

Convergence of the collocation scheme for TVPs

In order to study convergence for TVPs we assume that all eigenvalues of matrix $M(0)$ have nonnegative real parts and if zero is an eigenvalue of $M(0)$, then the associated invariant subspace is assumed to be the eigenspace of $M(0)$, *cf.* assumption A.2 in Section 9.1. Under these assumptions, we study the TVP

$$y'(t) = \frac{M(t)}{t} y(t) + \frac{f(t)}{t}, \quad t \in (0,1], \quad B_1 y(1) = \beta, \tag{10.11}$$

with $B_1 \in \mathbb{R}^{n \times n}$, $\beta \in \mathbb{R}^n$, stated as problem (7.6) in Section 7.2.

The existence and uniqueness of the respective collocation solution was already studied in [68]. The next lemma covers the case of TVPs with constant matrix $M(0)$ which has only eigenvalues with positive real parts.

Lemma 10.4 (Lemma 3.2 [68]) *Assume that all eigenvalues of $M(0)$ have positive real parts and $M \in C^1[0,1]$. For $\alpha \in \{0,1\}$ and arbitrary constants c_{jl}, there exists a unique polynomial function $p \in \mathscr{P}_{k,h}$ which, for any $0 < b \leq 1$, satisfies*

$$p'(t_{jl}) = \frac{M(0)}{t_{jl}} p(t_{jl}) + \frac{c_{jl}}{t_{jl}^{\alpha}}, \quad p(b) = \gamma, \; j = 0, \ldots I-1, \; l = 1, \ldots k. \tag{10.12}$$

Furthermore,

$$\|p\|_{t_{j+1}} := \max_{0 \leq s \leq t_{j+1}} |p(s)| \leq const. (|\gamma| + t_{j+1}^{1-\alpha} C_I), \; j = 0, \ldots I-1.$$

Remark 10.5 In the case when the matrix $M(0)$ has zero eigenvalues and the associated invariant subspace coincides with the eigenspace of $M(0)$, for $\alpha \in \{0,1\}$, $0 < b \leq 1$, and arbitrary constants c_{jl}, there exists a unique collocation polynomial $p \in \mathscr{P}_{k,h}$ such that

$$p'(t_{jl}) = \frac{c_{jl}}{t_{jl}^{\alpha}}, \quad p(b) = \gamma, \; j = 0, \ldots I-1, \; l = 1, \ldots k,$$

and

$$\|p\|_{t_{j+1}} \leq const. (|\gamma| + t_{j+1}^{1-\alpha} C_I), \; j = 0, \ldots I-1.$$

Consequently, for the matrix $M(0)$ whose spectrum consists of eigenvalues with positive real parts and zero eigenvalues with the same algebraic and geometric multiplicity, there exists a unique polynomial function $p \in \mathscr{P}_{k,h}$ satisfying (10.12). Furthermore,

$$\|p\|_{t_{j+1}} \le const.\,(|\gamma| + t_{j+1}^{1-\alpha} C_I),\ j = 0, \dots I-1.$$

Lemma 10.6 *Assume that $M \in C^1[0,1]$. Then for a sufficiently small h, for $\alpha \in \{0,1\}$, and arbitrary constants c_{jl}, there exists a unique collocation polynomial $p \in \mathscr{P}_{k,h}$ which satisfies*

$$p'(t_{jl}) = \frac{M(t_{jl})}{t_{jl}} p(t_{jl}) + \frac{c_{jl}}{t_{jl}^{\alpha}}, \quad p(1) = \gamma,\ j = 0, \dots I-1,\ l = 1, \dots k.$$

Moreover,

$$\|p\| \le const.\,(|\gamma| + C_I).$$

Proof: First, let us note that the classical theory yields the existence and uniqueness of a collocation solution r on the interval $[t_1, 1]$, $t_1 = h$. In order to show the existence of the solution on $[0, t_1]$ for h small enough, we rewrite the collocation problem as an operator equation $p = \mathscr{K} q$, $\mathscr{K} : \mathscr{P}_{k,h}[0,t_1] \to \mathscr{P}_{k,h}[0,t_1]$, where p is defined for $q \in \mathscr{P}_{k,h}[0,t_1]$ as the solution of the related collocation scheme with the constant coefficient matrix $M(0)$,

$$p'(t_{0l}) = \frac{M(0)}{t_{0l}} p(t_{0l}) + D(t_{0l}) q(t_{0l}) + \frac{c_{0l}}{t_{0l}^{\alpha}}, \quad p(t_1) = r(t_1),\ l = 1, \dots k,$$

where D is specified in (10.3). We now show that for a sufficiently small h, the operator $\mathscr{K}$ is a contraction on $\mathscr{P}_{k,h}[0,t_1]$ and therefore, the Banach Fixed Point Theorem can be used. Let $q_1, q_2 \in \mathscr{P}_{k,h}[0,t_1]$. Then $\mathscr{K} q_1$ and $\mathscr{K} q_2$ are solutions of the collocation schemes with $q = q_1$ and $q = q_2$, respectively. Therefore, $v := \mathscr{K} q_1 - \mathscr{K} q_2$ is implicitly defined as the solution of the collocation scheme,

$$v'(t_{0l}) = \frac{M(0)}{t_{0l}} v(t_{0l}) + D(t_{0l}) \left(q_1(t_{0l}) - q_2(t_{0l})\right), \quad v(t_1) = 0,\ l = 1, \dots k,$$

According to Lemma 10.4 and Remark 10.5,

$$\|\mathscr{K} q_1 - \mathscr{K} q_2\|_{t_1} \le const.\,h \|D\|_{t_1} \|q_1 - q_2\|_{t_1}.$$

For a sufficiently small $h = t_1$, the estimate

$$const.\,t_1 \|D\|_{t_1} =: L < 1$$

holds and thus, $\mathscr{K}$ is a contraction on $\mathscr{P}_{k,h}[0,t_1]$. Consequently, the Banach Fixed Point Theorem ensures the existence of a unique fixed point

$$p = \mathscr{K} p \text{ in } \mathscr{P}_{k,h}[0,t_1].$$

Moreover, the estimate

$$\|p\|_{t_1} \leq const. (|r(t_1)| + t_1 \|p\|_{t_1} \|D\|_{t_1} + t_1^{1-\alpha} C_1),$$

holds, and thus

$$\|p\|_{t_1} \leq \frac{1}{1-L} const. (|r(t_1)| + t_1^{1-\alpha} C_1),$$

where $C_1 := \max_{1 \leq l \leq k} |c_{0l}|$. By using the classical theory, we extend the estimate to the whole interval,

$$\|p\| \leq const. (|\gamma| + C_I).$$

□

We recapitulate the results of this section: Providing that h is sufficiently small, there exists a unique collocation polynomial $p \in \mathscr{P}_{k,h}$ satisfying

$$\begin{aligned} p'(t_{jl}) &= \frac{M(t_{jl})}{t_{jl}} p(t_{jl}) + \frac{(1-t_{jl})c_{jl}}{t_{jl}} \\ &= \frac{M(t_{jl})}{t_{jl}} p(t_{jl}) + \frac{c_{jl}}{t_{jl}} - c_{jl}, \quad p(1) = \gamma,\ j = 0, \ldots I-1,\ l = 1, \ldots k, \end{aligned}$$

and

$$\|p\| \leq const.(|\gamma| + C_I).$$

We are now able to formulate the convergence result for the TVPs. We consider the TVP (10.11) in the form

$$y'(t) = \frac{M(t)}{t} y(t) + \frac{f(t)}{t},\ t \in (0,1]\ y(1) = \delta, \tag{10.13}$$

where $B_1 \delta = \beta$.

Theorems 8.12, 8.18 and Remark 8.19 yield conditions for the existence of a unique solution $y \in C^{k+1}[0,1]$ of problem (10.13). We avoid repetition and assume that $y \in C^{k+1}[0,1]$ holds in the subsequent convergence analysis.

Theorem 10.7 *Let us assume that $M \in C^1[0,1], f \in C[0,1]$ and $y \in C^{k+1}[0,1]$ is the unique solution of* (10.13)*. Let the function $p \in \mathscr{P}_{k,h}$ satisfy the collocation scheme*

$$p'(t_{jl}) = \frac{M(t_{jl})}{t_{jl}} p(t_{jl}) + \frac{f(t_{jl})}{t_{jl}}, \quad p(1) = \delta,\ j = 0, \ldots I-1,\ l = 1, \ldots k.$$

Then, provided that h is sufficiently small,

$$\|p - y\| \leq const. h^k.$$

Proof: Let us define an error function $e \in \mathscr{P}_{k,h}$ as follows:

$$e'(t_{jl}) := y'(t_{jl}) - p'(t_{jl}),\ j = 0,\dots,I-1,\ l = 1,\dots,k, \quad e(1) = 0.$$

Since the function e' belongs to $\mathscr{P}_{k-1,h}$, it is uniquely determined by

$$e'(t) = \sum_{i=1}^{k} l_i \left(\frac{t-t_j}{h}\right) y'(t_{ji}) - p'(t),\ t \in (t_j, t_{j+1}],$$

where l_i are specified in (10.9). For $y \in C^{k+1}[0,1]$ the interpolation error is $O(h^k)$ and hence, $e'(t) = y'(t) - p'(t) + O(h^k)$. By integration over $[t,1]$ we obtain

$$e(t) = y(t) - p(t) + (1-t)O(h^k).$$

Moreover, we see that e satisfies the collocation scheme

$$\begin{aligned}
&e'(t_{jl}) - \frac{M(t_{jl})}{t_{jl}} e(t_{jl}) \\
&= y'(t_{jl}) - \frac{M(t_{jl})}{t_{jl}} y(t_{jl}) - \left(p'(t_{jl}) - \frac{M(t_{jl})}{t_{jl}} p(t_{jl})\right) - \frac{M(t_{jl})}{t_{jl}} (1-t_{jl})O(h^k) \\
&= -\frac{M(t_{jl})}{t_{jl}} (1-t_{jl})O(h^k), \qquad e(1) = 0,
\end{aligned}$$

and Lemma 10.6 finally yields,

$$\|e\| \le const.\, \|M\| O(h^k).$$

Consequently, $\|y-p\| \le const.\, h^k$. $\square$

Convergence of the collocation scheme for BVPs

In this section, we generalize the convergence results derived for IVPs and TVPs to the general BVPs (10.1), (10.2):

$$\begin{aligned}
&y'(t) = \frac{M(t)}{t} y(t) + \frac{f(t)}{t},\ t \in (0,1], \\
&B_0 y(0) + B_1 y(1) = \beta,
\end{aligned}$$

We allow the spectrum of the matrix $M(0)$ to contains both, eigenvalues with nonpositive and nonnegative real parts. By Lemma 9.6 and Theorem 9.9 the above BVP is well-posed if and only if the boundary conditions (10.2) can be equivalently written in a separated fashion,

$$Hy(0) = 0,\ M(0)Ny(0) = -Nf(0),\ Ry(0) = R\eta,\ Sy(1) = S\eta \tag{10.14}$$

and therefore, we can restrict our attention to the problem (10.1), (10.14). First, we examine the existence and uniqueness of a solution of the associated collocation scheme

$$\begin{aligned} p'(t_{jl}) &= \frac{M(t_{jl})}{t_{jl}} p(t_{jl}) + \frac{f(t_{jl})}{t_{jl}}, \; j = 0, \dots I-1, \; l = 1, \dots k, \\ Hp(0) &= 0, \; M(0)Np(0) = -Nf(0), \; Rp(0) = R\eta, \; Sp(1) = S\eta \end{aligned} \tag{10.15}$$

and then we prove that this scheme converges with the classical stage order.

Theorem 10.8 *There exists a unique solution $p \in \mathscr{P}_{k,h}$ of the collocation scheme* (10.15) *provided that h is sufficiently small and $M \in C^1[0,1], f \in [0,1]$. This solution satisfies*

$$\|p\| \leq const. \left(|\gamma| + |\ln(h)|^d |M(0)||\gamma| + (|\ln(h)|^d + 1)\|f\| \right),$$

where d is the dimension of the largest Jordan box of $M(0)$ associated with the eigenvalue $\lambda = 0$.

Proof: In order to show the existence and uniqueness result for p, we study the fixed point equation $p = \mathscr{K}(q)$, $\mathscr{K} : \mathscr{P}_{k,h} \to \mathscr{P}_{k,h}$, where p is defined as a solution of the related collocation scheme with the constant matrix M(0):

$$p'(t_{ij}) = \frac{M(0)}{t_{ij}} p(t_{ij}) + D(t_{ij})q(t_{ij}) + \frac{f(t_{ij})}{t_{ij}}, \; j = 0, \dots I-1, \; l = 1, \dots k,$$

subject to boundary conditions (10.14). In order to decouple the above scheme, we introduce new variables,

$$v(t_{jl}) = E^{-1}p(t_{jl}), \; Q(t_{jl}) = E^{-1}D(t_{jl}), \; g(t_{jl}) = E^{-1}f(t_{jl}),$$

where J is the Jordan canonical form of $M(0)$ and E is the associated matrix of the generalized eigenvectors of $M(0)$. Then, the decoupled system reads:

$$\begin{aligned} &v'(t_{jl}) = \frac{J}{t_{jl}} v(t_{jl}) + Q(t_{jl})q(t_{jl}) + \frac{g(t_{jl})}{t_{jl}}, \; j = 0, \dots I-1, \; l = 1, \dots k, \\ &JV^N v(0) = -V^N g(0), \; V^Z v(0) = E^{-1}R\gamma, \; V^S v(1) = E^{-1}S\gamma, \end{aligned}$$

where

$$J = \begin{pmatrix} J^N & 0 & 0 \\ 0 & J^Z & 0 \\ 0 & 0 & J^S \end{pmatrix}$$

and

$$V^N = \begin{pmatrix} I^N & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; V^Z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I^Z & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; V^S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I^S \end{pmatrix}.$$

Here, J^N is the Jordan block of dimension $\operatorname{rank} N$ associated with the eigenvalues with negative real parts, J^Z is the Jordan block of dimension $\operatorname{rank} H + \operatorname{rank} R$ associated with zero eigenvalues and J^S is the Jordan block of dimension $\operatorname{rank} S$ associated with the eigenvalues with positive real parts. The matrices I^N, I^Z, and I^S are identity matrices of corresponding dimensions.

We can split the discrete system into an IVP, governed by the blocs J^N and J^Z, associated with the negative real parts of eigenvalues of $M(0)$ and zero eigenvalues of $M(0)$, respectively, and into a TVP, governed by the block J^S, associated with the positive real parts of eigenvalues of $M(0)$. After applying Lemma 10.1 to the IVP and Lemma 10.6 to the TVP, we conclude the existence of a unique collocation solution p of (10.15) which satisfies

$$\|p\| \leq const. \Big(|\gamma| + |\ln(h)|^d |M(0)||\gamma| + (|\ln(h)|^d + 1)\|f\| \Big).$$

$\square$

In the next theorem, we formulate the convergence properties of the collocation solution p to the general BVP (10.15). The proof relies on the techniques developed in Theorem 3.1 [68] for nonlinear problems with smooth inhomogeneities. Therefore, we only discuss the main ideas of this technique and give an outline of the proof. We refer to [68] for further technical details. Note that here, the situation is easier than in [68] since we deal with a linear problem.

Theorem 10.9 *Let us assume that $y \in C^{k+2}[0,1]$ is the unique solution of the BVP* (10.1), (10.14)*, $f \in C^{k+1}[0,1]$, $M \in C^{k+2}[0,1]$, and $\sigma_+ > k+2$. Let $p \in \mathscr{P}_{k,h}$ be the unique solution of the collocation scheme* (10.15)*. Then,*

$$\|p - y\| \leq const. h^k.$$

Proof: The main idea of the proof is to derive a representation for the global error $p - y$ of the collocation solution p at all points[1] t_{jl}, $j = 0, \ldots I-1$, $l = 1, \ldots k+1$,

$$p(t_{jl}) = y(t_{jl}) + e(t_{jl})h^k + r(t_{jl}), \tag{10.16}$$

where y is the exact solution of (10.1), (10.14), $e \in C[0,1]$ and $r \in \mathscr{P}_{k,h}$. After some tedious calculation *cf.* Section 3.2 [68], we arrive at the following relation for t_{jl}, $j = 0, \ldots I-1$, $l = 1, \ldots k+1$,

$$\begin{aligned} p'(t_{jl}) &= y'(t_{jl}) + e'(t_{jl})h^k + r'(t_{jl}) \\ &\quad - \frac{1}{(k+1)!}\Omega'(\rho_l) y^{(k+1)}(t_{jl}) h^k + (1 + \|e''\|) O(h^{k+1}), \end{aligned}$$

[1] For technical reasons the mesh is restricted to $u_k < 1$ and $u_{k+1} := 1$.

where $\Omega(t) := \Pi_{i=1}^{k+1}(t-\rho_i)$. We substitute (10.16) into the collocation scheme (10.15),

$$p'(t_{jl}) = \frac{M(t_{jl})}{t_{jl}} p(t_{jl}) + \frac{f(t_{jl})}{t_{jl}},$$

and obtain

$$\begin{aligned} &y'(t_{jl}) + e'(t_{jl})h^k + r'(t_{jl}) - \frac{1}{(k+1)!}\Omega'(\rho_l) y^{(k+1)}(t_{jl}) h^k + (1+\|e''\|)O(h^{k+1}) \\ &= \frac{M(t_{jl})}{t_{jl}} \left(y(t_{jl}) + e(t_{jl})h^k + r(t_{jl}) \right) + \frac{f(t_{jl})}{t_{jl}}, \end{aligned}$$

or equivalently,

$$\begin{aligned} &e'(t_{jl})h^k + r'(t_{jl}) - \frac{1}{(k+1)!}\Omega'(\rho_l) y^{(k+1)}(t_{jl}) h^k + (1+\|e''\|)O(h^{k+1}) \\ &\quad = \frac{M(t_{jl})}{t_{jl}} \left(e(t_{jl})h^k + r(t_{jl}) \right), \end{aligned}$$

since y is the exact solution. To determinate a relation defining e, we collect all terms multiplying h^k and obtain,

$$\begin{aligned} &e'(t_{jl}) = \frac{M(t_{jl})}{t_{jl}} e(t_{jl}) + \frac{1}{(k+1)!}\Omega'(\rho_l) y^{(k+1)}(t_j), \; j = 0, \dots I-1, \; l = 1, \dots k, \\ &He(0) = 0, \; M(0)Ne(0) = 0, \; Re(0) = 0, \; Se(1) = 0, \end{aligned} \tag{10.17}$$

on noting that $y^{(k+1)}(t_{jl})h^k = y^{(k+1)}(t_j)h^k + O(h^{k+1})$ holds.

The relation for r follows by collecting all remaining terms,

$$\begin{aligned} &r'(t_{jl}) = \frac{M(t_{jl})}{t_{jl}} r(t_{jl}) + (1+\|e''\|)O(h^{k+1}), \; j = 0, \dots I-1, \; l = 1, \dots k, \\ &Hr(0) = 0, \; M(0)Nr(0) = 0, \; Rr(0) = 0, \; Sr(1) = 0. \end{aligned} \tag{10.18}$$

We now construct an analytical BVP related to (10.17) whose solution is $e \in C[0,1]$,

$$\begin{aligned} &e'(t) = \frac{M(t)}{t} e(t) + \frac{1}{(k+1)!} g(t), \\ &He(0) = 0, \; M(0)Ne(0) = -Nf(0), \; Re(0) = 0, \; Se(1) = 0, \end{aligned}$$

where $g = g_i(t), \; t \in [t_j, t_{j+1}], \; j = 0, \dots I-1$, is an appropriate, piecewise polynomial function satisfying $g_i(t_{jl}) = \Omega'(\rho_l) y^{(k+1)}(t_j), \; j = 0, \dots I-1, \; l = 1, \dots k$. Therefore, the function e is also piecewisely defined. The fact that the above BVP has a unique solution $e \in C[0,1] \cap C^{k+2}[t_j, t_{j+1}], \; j = 0, \dots I-1$, follows from Theorem 2.2 [68]. Moreover, note that $\|e''\| h^{k+1} = O(h^k)$.

It follows from Section 3.1 [68], that there exists a unique solution $r \in \mathscr{P}_{m,h}$ of problem (10.18) such that

$$\|r\|_{t_{j+1}} \le t_{j+1} O(h^k), \; j = 0, \dots I-1.$$

We combine the results for e and r to show the final result. Let E be a piecewise polynomial function of degree less or equal k such that

$$E(t_{jl}) := p(t_{jl}) - y(t_{jl}), \; j = 0, \dots I-1, \; l = 1, \dots k+1.$$

Then,

$$E(t) = p(t) - \sum_{i=1}^{k+1} l_i \left(\frac{t - t_j}{h} \right) y(t_{ji}), \; t \in [t_j, t_{j+1}], \; j = 0, \dots I-1.$$

For $y \in C^{k+2}[0,1]$ the interpolation error is $O(h^{k+1})$ and hence,

$$E(t) = p(t) - y(t) + O(h^{k+1}), \; t \in [t_j, t_{j+1}], \; j = 0, \dots I-1.$$

On the other hand, from the error representation (10.16), $E(t_{jl}) = e(t_{jl})h^k + r(t_{jl})$, $t \in (t_j, t_{j+1}], \; j = 0, \dots I-1$, and therefore,

$$\begin{aligned} E(t) &= \sum_{i=1}^{k+1} l_i \left(\frac{t - t_j}{h} \right) E(t_{ji}) = \sum_{i=1}^{k+1} l_i \left(\frac{t - t_j}{h} \right) \left(e(t_{ji})h^k + r(t_{ji}) \right) \\ &= h^k \sum_{i=1}^{k+1} l_i \left(\frac{t - t_j}{h} \right) e(t_{ji}) + r(t). \end{aligned} \tag{10.19}$$

Since $e \in C^{k+2}(t_j, t_{j+1})$, the interpolation error is $O(h^{k+1})$ and thus,

$$E(t) = h^k \left(e(t) + O(h^{k+1}) \right) + r(t) = h^k \left(O(1) + O(h^{k+1}) \right) + O(h^k) = O(h^k),$$

for $t \in (t_j, t_{j+1}), \; j = 0, \dots, I-1$, For the subinterval endpoints, $t = t_{j+1}$, we have from (10.19)

$$E(t_{j+1}) = h^k \sum_{i=1}^{k+1} l_i(1) e(t_{ji}) + r(t_{ji}) = O(h^k), \; j = 0, \dots I-1,$$

and finally, for $t = 0$, $j = 0$,

$$E(0) := \lim_{t \to 0} E(t) = h^k \sum_{i=1}^{k+1} l_i(0) e(t_{0i}) + r(t_{0i}) = O(h^k).$$

Altogether, $E = O(h^k)$ in $[0,1]$ and the result $\|p - y\| = O(h^k)$ follows. $\square$

Note that the condition $\sigma_+ > k+2$ does not impose a restriction of generality, see Section 10.3, TVP with small positive eigenvalues. Furthermore, we see from Theorem 8.12 that the eigenvalues of $M(0)$ with positive real parts may influence the smoothness properties of a solution and consequently, they may also cause a decrease in the convergence order. Therefore the restriction $\sigma_+ > k+2$ is natural in this context.

10.3 Numerical experiments

In order to illustrate the theoretical results derived in the previous sections, we construct model problems in the IVP, TVP and BVP setting. To calculate the numerical results, we use the MATLAB code `bvpsuite` and run the code on coherently refined meshes to compare the empirically estimated convergence orders with those predicted by the theory.

Initial value problem

We first deal with the IVP

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t},\ t \in (0,1], \quad \begin{pmatrix} 3 & -2 & 1 \\ -2 & 2 & -1 \\ -2 & 1 & 0 \end{pmatrix} y(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Here

$$M(t) = \begin{pmatrix} 3t - 2\sin t - 4 & -2t + \sin t + 2 & t - 1 \\ 3t^2 + 6t - 4\sin t - 8 & -2t^2 - 4t + 2\sin t + 4 & t^2 + 2t - 2 \\ 6t^2 + 6t - 4\sin t - 12 & -4t^2 - 4t + 2\sin t + 8 & 2t^2 + 2t - 4 \end{pmatrix}$$

and

$$f(t) = \begin{pmatrix} -t^2\sin(t) + 2\exp(t) + \sin(t)\cos(t) + 2t\cos^2(t) - t \\ -2t^2\sin(t) - t^2\exp(t) + 4\exp(t) + 2\sin(t)\cos(t) + 4t\cos^2(t) + 2t^2 - 2t \\ -2t^2\sin(t) - 2t^2\exp(t) + 4\exp(t) + 2t\cos^2(t) + 4t^2 - t. \end{pmatrix}.$$

The matrix $M(0)$,

$$M(0) = \begin{pmatrix} -4 & 2 & -1 \\ -8 & 4 & -2 \\ -12 & 8 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} -2 & 1 & \\ & -2 & \\ & & 0 \end{pmatrix} \begin{pmatrix} 3 & -2 & 1 \\ -2 & 2 & -1 \\ -2 & 1 & 0 \end{pmatrix}$$

has double eigenvalue $\lambda_1 = \lambda_2 = -2$, and simple eigenvalue $\lambda_3 = 0$. The exact solution $y \in C^\infty[0,1]$ of the problem is given and has the form

$$y(t) = \begin{pmatrix} \exp(t) + \sin(t)\cos(t) \\ 2\exp(t) + 2\sin(t)\cos(t) + t^2 \\ 2\exp(t) + \sin(t)\cos(t) + 2t^2 \end{pmatrix}.$$

Tables 10.1 – 10.3 illustrate the convergence behaviour for the collocation executed with equidistant and Gaussian collocation points. The number of the collocation points k was chosen to vary from 1 to 8. However, in the simulations shown here, we report only on the values 1 to 4 since the results for 5 to 8 are very similar. The maximal global error is computed either in the mesh points,

$$\|Y_h - Y\|_\Delta := \max_{0 \le j \le I} |p(t_j) - y(t_j)|,$$

or 'uniformly' in t, $\|Y_h - Y\|_u := \max_{0\le i\le 1.000} |p(\tau_i) - y(\tau_i)|$, $\tau_i = ih,\ h = 10^{-3}$. The order of convergence and the error constant c are estimated using two consecutive meshes with the step sizes h and $h/2$. From the estimate of the global error, $\|Y_h - Y\| = O(h^p)$ for $h \to 0$, we have

$$\|Y_h - Y\|_\Delta = ch^p, \quad \|Y_{h/2} - Y\|_\Delta = c\left(\frac{h}{2}\right)^p \Rightarrow p = \ln\left(\frac{\|Y_h - Y\|_\Delta}{\|Y_{h/2} - Y\|_\Delta}\right)\frac{1}{\ln(2)}.$$

Having p, we calculate the error constant from $c = \|Y_{h/2} - Y\|_\Delta / \left(\frac{h}{2}\right)^p$.

According to the experiments, the empirical convergence orders very well reflect the theoretical findings. For Gaussian points, we observe the small superconvergence order $k+1$ in the mesh points. The superconvergence order $2k$ in the mesh points does not hold in general. For uniformly spaced equidistant collocation points we observe the order k uniformly in t as we have proven theoretically.

Table 10.1: IVP: Convergence of the collocation scheme, $k = 2$

	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
h	$\|Y_h - Y\|_\Delta$	c	p	$\|Y_h - Y\|_\Delta$	c	p	$\|Y_h - Y\|_u$	c	p
$1/2$	1.4e-02	6.8e-02	2.23	4.3e-02	1.6e-01	1.94	4.3e-02	1.6e-01	1.94
$1/4$	3.1e-03	1.9e-01	2.96	1.1e-02	2.0e-01	2.08	1.1e-02	2.0e-01	2.08
$1/8$	4.0e-04	2.1e-01	3.02	2.6e-03	1.7e-01	2.01	2.6e-03	1.7e-01	2.01
$1/16$	4.9e-05	2.1e-01	3.01	6.5e-04	1.6e-01	2.00	6.5e-04	1.6e-01	2.00
$1/32$	6.1e-06	2.0e-01	3.01	1.6e-04	1.6e-01	2.00	1.6e-04	1.6e-01	2.00
$1/64$	7.6e-07	2.0e-01	3.00	4.1e-05	1.7e-01	2.00	4.1e-05	1.7e-01	2.00
$1/128$	9.4e-08	2.0e-01	3.00	1.0e-05	1.7e-01	2.00	1.0e-05	1.7e-01	2.00
$1/256$	1.2e-08	2.0e-01	3.00	2.5e-06	1.7e-01	2.00	2.5e-06	1.7e-01	2.00
$1/512$	1.5e-09	–	–	6.4e-07	–	–	6.4e-07	–	–

Table 10.2: IVP: Convergence of the collocation scheme, $k = 3$

	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
h	$\|Y_h - Y\|_\Delta$	c	p	$\|Y_h - Y\|_\Delta$	c	p	$\|Y_h - Y\|_u$	c	p
$1/2$	2.7e-04	2.1e-03	2.99	2.8e-02	5.2e-01	4.19	2.8e-02	5.2e-01	4.19
$1/4$	3.3e-05	2.3e-02	4.70	1.6e-03	4.3e-01	4.05	1.6e-03	4.3e-01	4.05
$1/8$	1.3e-06	3.6e-02	4.93	9.4e-05	4.0e-01	4.01	9.4e-05	4.0e-01	4.01
$1/16$	4.2e-08	4.1e-02	4.98	5.8e-06	3.8e-01	4.00	5.8e-06	3.8e-01	4.00
$1/32$	1.3e-09	4.4e-02	4.99	3.6e-07	3.8e-01	4.00	3.6e-07	3.8e-01	4.00
$1/64$	4.2e-11	4.4e-02	4.99	2.3e-08	3.8e-01	4.00	2.3e-08	3.8e-01	4.00
$1/128$	1.3e-12	1.0e-01	5.17	1.4e-09	3.8e-01	4.00	1.4e-09	3.8e-01	4.00
$1/256$	3.7e-14	2.3e-14	-0.08	8.8e-11	3.8e-01	4.00	8.8e-11	3.8e-01	4.00
$1/512$	3.9e-14	–	–	5.5e-12	–	–	5.5e-12	–	–

Table 10.3: IVP: Convergence of the collocation scheme, $k=4$

	Gaussian, mesh points			equidistant, mesh points			equidistant,uniform		
h	$\|Y_h-Y\|_\Delta$	c	p	$\|Y_h-Y\|_\Delta$	c	p	$\|Y_h-Y\|_u$	c	p
$1/2$	2.2e-05	5.1e-04	4.51	2.1e-03	3.9e-02	4.21	2.1e-03	3.5e-02	4.05
$1/4$	9.7e-07	1.3e-03	5.22	1.1e-04	4.0e-02	4.22	1.3e-04	4.5e-02	4.23
$1/8$	2.6e-08	1.2e-03	5.16	6.1e-06	3.0e-02	4.08	6.8e-06	3.7e-02	4.14
$1/16$	7.3e-10	9.8e-04	5.09	3.6e-07	2.5e-02	4.02	3.8e-07	3.0e-02	4.06
$1/32$	2.1e-11	8.5e-04	5.05	2.2e-08	2.4e-02	4.01	2.3e-08	2.6e-02	4.02
$1/64$	6.5e-13	3.2e-04	4.81	1.4e-09	2.3e-02	4.00	1.4e-09	2.5e-02	4.01
$1/128$	2.3e-14	1.3e-11	1.31	8.7e-11	2.3e-02	4.00	8.8e-11	2.4e-02	4.00
$1/256$	9.3e-15	9.8e-19	-1.65	5.4e-12	2.3e-02	4.00	5.5e-12	2.4e-02	4.00
$1/512$	2.9e-14	–	–	3.4e-13	–	–	3.4e-13	–	–

Terminal value problem

We consider the TVP

$$y'(t)=\frac{M(t)}{t}y(t)+\frac{f(t)}{t},\ t\in(0,1],\qquad \begin{pmatrix} 2 & 1 & -1\\ 0 & 1 & 0\\ -1 & -1 & 1\end{pmatrix} y(1)=\begin{pmatrix} e\\ e\\ 1/15\end{pmatrix},$$

where

$$M(t)=\begin{pmatrix} 24+2t & 12+t & -12-t\\ -26t & 20-12t & 13t\\ 24-24t & 32-11t & -12+12t\end{pmatrix},$$

$$M(0)=\begin{pmatrix} 24 & 12 & -12\\ 0 & 20 & 0\\ 24 & 32 & -12\end{pmatrix}=\begin{pmatrix} 1 & 0 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{pmatrix}\begin{pmatrix} 12 & & \\ & 20 & \\ & & 0\end{pmatrix}\begin{pmatrix} 2 & 1 & -1\\ 0 & 1 & 0\\ -1 & -1 & 1\end{pmatrix}$$

and

$$f(t)=\begin{pmatrix} -12\exp(t)+t^{15}\\ -6t\exp(t)\\ -6t\exp(t)-12\exp(t)+2t^{15}\end{pmatrix}.$$

The eigenvalues of $M(0)$ are simple, $\lambda_1=12$, $\lambda_2=20$, and $\lambda_3=0$ and the exact solution $y\in C^\infty[0,1]$ reads:

$$y(t)=\begin{pmatrix} \exp(t)+\frac{1}{15}t^{15}\\ t\exp(t)\\ \exp(t)+t\exp(t)+\frac{2}{15}t^{15}\end{pmatrix}$$

Tables 10.4 – 10.6 show the same convergence behaviour as observed for IVPs. For equidistant collocation points the convergence order k holds not only in the mesh points, but also uniformly in t. Also, the small superconvergence $k+1$ can be observed in the mesh points, when Gaussian points are used as collocation points.

Table 10.4: TVP: Convergence of the collocation scheme, $k = 2$

	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
h	$\|Y_h - Y\|_\Delta$	c	p	$\|Y_h - Y\|_\Delta$	c	p	$\|Y_h - Y\|_u$	c	p
$1/2$	2.8e-01	1.3e+00	2.22	3.7e-01	1.0e+00	1.42	3.7e-01	1.0e+00	1.42
$1/4$	6.1e-02	4.8e+00	3.15	1.4e-01	1.4e+00	1.64	1.4e-01	1.4e+00	1.64
$1/8$	6.8e-03	9.1e+00	3.46	4.5e-02	2.3e+00	1.89	4.5e-02	2.3e+00	1.89
$1/16$	6.2e-04	7.1e+00	3.37	1.2e-02	2.9e+00	1.98	1.2e-02	2.9e+00	1.98
$1/32$	6.0e-05	4.5e+00	3.24	3.1e-03	3.1e+00	2.00	3.1e-03	3.1e+00	2.00
$1/64$	6.3e-06	3.0e+00	3.14	7.7e-04	3.1e+00	2.00	7.7e-04	3.1e+00	2.00
$1/128$	7.2e-07	2.2e+00	3.07	1.9e-04	3.1e+00	2.00	1.9e-04	3.2e+00	2.00
$1/256$	8.5e-08	1.8e+00	3.04	4.8e-05	3.1e+00	2.00	4.8e-05	3.2e+00	2.00
$1/512$	1.0e-08	–	–	1.2e-05	–	–	1.2e-05	–	–

Table 10.5: TVP: Convergence of the collocation scheme, $k = 3$

	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
h	$\|Y_h - Y\|_\Delta$	c	p	$\|Y_h - Y\|_\Delta$	c	p	$\|Y_h - Y\|_u$	c	p
$1/2$	1.3e-02	1.1e-01	3.03	8.0e-02	1.8e-01	1.18	1.2e-01	2.8e-01	1.29
$1/4$	1.6e-03	5.8e+00	5.90	3.5e-02	2.6e+00	3.11	4.7e-02	1.5e+00	2.48
$1/8$	2.7e-05	2.7e-01	4.41	4.1e-03	1.0e+01	3.76	8.5e-03	6.9e+00	3.22
$1/16$	1.3e-06	6.1e-02	3.88	3.0e-04	1.7e+01	3.94	9.1e-04	2.0e+01	3.61
$1/32$	8.8e-08	5.4e-02	3.85	2.0e-05	2.0e+01	3.98	7.5e-05	4.0e+01	3.80
$1/64$	6.1e-09	8.9e-02	3.97	1.3e-06	2.1e+01	4.00	5.3e-06	6.0e+01	3.90
$1/128$	3.9e-10	1.0e-01	3.99	7.8e-08	2.1e+01	4.00	3.6e-07	7.6e+01	3.95
$1/256$	2.4e-11	9.3e-02	3.98	4.9e-09	2.1e+01	4.00	2.3e-08	8.7e+01	3.98
$1/512$	1.6e-12	–	–	3.1e-10	–	–	1.5e-09	–	–

Table 10.6: TVP: Convergence of the collocation scheme, $k = 4$

	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
h	$\|Y_h - Y\|_\Delta$	c	p	$\|Y_h - Y\|_\Delta$	c	p	$\|Y_h - Y\|_u$	c	p
$1/2$	5.0e-03	3.0e-01	5.93	9.6e-02	3.5e-01	1.87	9.6e-02	3.5e-01	1.87
$1/4$	8.1e-05	4.9e-01	6.28	2.6e-02	2.1e+00	3.16	2.6e-02	2.1e+00	3.16
$1/8$	1.0e-06	9.5e-02	5.49	2.9e-03	7.4e+00	3.77	2.9e-03	7.4e+00	3.77
$1/16$	2.3e-08	3.5e-02	5.13	2.1e-04	1.2e+01	3.94	2.1e-04	1.2e+01	3.94
$1/32$	6.7e-10	2.6e-02	5.04	1.4e-05	1.4e+01	3.99	1.4e-05	1.4e+01	3.99
$1/64$	2.0e-11	2.2e-02	5.00	8.8e-07	1.4e+01	4.00	8.8e-07	1.4e+01	4.00
$1/128$	6.4e-13	6.7e-04	4.28	5.5e-08	1.5e+01	4.00	5.5e-08	1.5e+01	4.00
$1/256$	3.3e-14	2.9e-13	0.39	3.4e-09	1.5e+01	4.00	3.4e-09	1.5e+01	4.00
$1/512$	2.5e-14	–	–	2.1e-10	–	–	2.1e-10	–	–

TVP with small positive eigenvalues

We consider TVPs with a constant coefficient matrix to illustrate the case of eigenvalues with small positive real parts which leads to smoothness reduction of a solution. To overcome this difficulty a suitable transformation is used.

In particular, we consider the model problem:

$$y'(t) = \frac{M_0}{t}y(t) + \frac{f(t)}{t},\ t \in (0,1], \quad B_1 y(1) = \begin{pmatrix} 4 & -1 & 1 \\ 0 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix} y(1) = \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix}, \quad (10.20)$$

where

$$M_0 = \begin{pmatrix} 3.5 & -1 & 1 \\ -14 & 5 & -4 \\ -24.5 & 8 & -7 \end{pmatrix}, \quad f(t) = \begin{pmatrix} 1-4t^2 \\ 4+t^2\ln(t) \\ 1+t^2\ln(t)+14t^2 \end{pmatrix}.$$

The eigenvalues of M_0 are $\lambda_1 = 0.5$, $\lambda_2 = 1$, and $\lambda_3 = 0$ and the solution $y \in C[0,1]$ of (10.20) is

$$y(t) = \begin{pmatrix} 3\sqrt{t}-2t^2-4 \\ 12\sqrt{t}-8+4t+t^2\ln(t)-t^2 \\ 3\sqrt{t}+4t+5+6t^2+t^2\ln(t) \end{pmatrix}.$$

As expected, in Tables 10.7 and 10.8, we observe an order reduction down to 0.5, due to the fact that the first derivative y' is unbounded for $t \to 0$. Moreover, we see that the problem is hard to solve and the convergence is very slow – for h approximately equal to $2\cdot 10^{-3}$ the level of the global error $\|Y_h - Y\|_\infty$ is approximately only 10^{-1}. In [121] convergence order of the collocation scheme for TVPs of the type (1.7), where all eigenvalues of M_0 have positive real parts, is specified. Provided that the Jordan box associated with the eigenvalue whose real part is σ_+ is diagonal, the uniform stage convergence order was shown to be $\min\{\sigma_+, k\}$. This result suggests that also in case of the model (10.20) the convergence order drops down to 0.5, *cf.* Tables 10.7 and 10.8.

Table 10.7: TVP (10.20): Convergence of the collocation scheme, $k = 2$

	equidistant points			Gaussian points					
	uniform			mesh			uniform		
h	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p	$\|Y_h - Y\|_\infty$	c	p
1/2	2.2e+00	–	–	1.7e+00	–	–	1.7e+00	–	–
1/4	1.5e+00	5.4e+00	0.51	1.2e+00	2.4e+00	0.51	1.2e+00	5.4e+00	0.51
1/8	1.1e+00	5.3e+00	0.50	8.5e-01	2.4e+00	0.50	8.5e-01	5.3e+00	0.50
1/16	7.5e-01	5.3e+00	0.50	6.0e-01	2.4e+00	0.50	6.0e-01	5.2e+00	0.50
1/32	5.3e-01	5.2e+00	0.50	4.2e-01	2.4e+00	0.50	4.2e-01	5.2e+00	0.50
1/64	3.8e-01	5.2e+00	0.50	3.0e-01	2.4e+00	0.50	3.0e-01	5.2e+00	0.50
1/128	2.7e-01	5.2e+00	0.50	2.1e-01	2.4e+00	0.50	2.1e-01	5.2e+00	0.50
1/256	1.9e-01	5.2e+00	0.50	1.5e-01	2.4e+00	0.50	1.5e-01	5.2e+00	0.50
1/512	1.3e-01	5.2e+00	0.50	1.1e-01	2.4e+00	0.50	1.1e-01	5.2e+00	0.50

Table 10.8: TVP (10.20): Convergence of the collocation scheme, $k=3$

h	equidistant points			Gaussian points					
	uniform			mesh			uniform		
	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p
1/2	1.7e+00	–	–	1.2e+00	–	–	1.2e+00	–	–
1/4	1.2e+00	4.7e+00	0.50	8.6e-01	1.7e+00	0.50	8.6e-01	5.2e+00	0.50
1/8	8.3e-01	4.7e+00	0.50	6.1e-01	1.7e+00	0.50	6.1e-01	5.1e+00	0.50
1/16	5.8e-01	4.7e+00	0.50	4.3e-01	1.7e+00	0.50	4.3e-01	5.1e+00	0.50
1/32	4.1e-01	4.7e+00	0.50	3.0e-01	1.7e+00	0.50	3.0e-01	5.1e+00	0.50
1/64	2.9e-01	4.7e+00	0.50	2.1e-01	1.7e+00	0.50	2.1e-01	5.1e+00	0.50
1/128	2.1e-01	4.7e+00	0.50	1.5e-01	1.7e+00	0.50	1.5e-01	5.1e+00	0.50
1/256	1.5e-01	4.7e+00	0.50	1.1e-01	1.7e+00	0.50	1.1e-01	5.1e+00	0.50
1/512	1.0e-01	4.7e+00	0.50	7.6e-02	1.7e+00	0.50	7.6e-02	5.1e+00	0.50

The remedy for this lack of smoothness due to the small size of the positive eigenvalues of M_0 is to make a change of the independent variable [29], $t=\tau^\mu$ for some $\mu>1$. Then $\tilde{y}(\tau):=y(\tau^\mu)$ satisfies the transformed ODE system

$$\tilde{y}'(\tau)=\frac{\widetilde{M}_0}{\tau}\tilde{y}(\tau)+\frac{\tilde{f}(\tau)}{\tau},\quad \tau\in(0,1], \tag{10.21}$$

where $\tilde{M}_0=\mu M_0$ and $\tilde{f}(\tau)=\mu f(\tau^\mu)$. The eigenvalues of the matrix $\tilde{M}_0$ become $\tilde{\lambda}=\mu\lambda$ and therefore, the solution $\tilde{y}$ of the transformed equation is smoother than the solution y of the original one. For instance, if $f\in C^{r+2}[0,1]$ and $\sigma_+\le r+1$, then we can choose μ such that $\mu\sigma_+>r+1$. Consequently, by Theorem 8.11 (*ii*), $\tilde{y}(\tau)=y(\tau^\mu)\in C^{r+1}[0,1]$. This allows the use of collocation method with higher order of convergence. One can also interpret the above smoothing in terms of the mesh adaptation – solving the ODE system (10.21) on an equidistant mesh, means solving the original ODE system on a mesh which is adequately refined close to the singularity, where the solution y and its derivatives rapidly change. Consequently, we solve (10.21) subject to terminal conditions given in (10.20), where for $\mu=8$,

$$\widetilde{M_0}=\begin{pmatrix}28 & -8 & 8\\ -112 & 40 & -32\\ -196 & 64 & -56\end{pmatrix},\quad \tilde{f}(\tau)=\begin{pmatrix}8-32\tau^{16}\\ 32+8\tau^{16}\ln(\tau^8)\\ 8+8\tau^{16}\ln(\tau^8)+112\tau^{16}\end{pmatrix}.$$

While the eigenvalues of M_0 are $\lambda_1=0.5$, $\lambda_2=1$, and $\lambda_3=0$, the eigenvalues of $\widetilde{M}_0$ become $\tilde{\lambda}_1=4$, $\tilde{\lambda}_2=8$, and $\tilde{\lambda}_3=0$. The solution of the transformed problem reads:

$$\tilde{y}(\tau)=\begin{pmatrix}3\tau^4-2\tau^{16}-4\\ -8+12\tau^4+4\tau^8+\tau^{16}\ln\tau^8-\tau^{16}\\ 5+6\tau^{16}+4\tau^8+\tau^{16}\ln\tau^8+3\tau^4\end{pmatrix}.$$

Tables 10.9 and 10.10 show the desired effect. For $k=2$ and equidistant collocation points, we observe the $O(h^k)$ behaviour of the global error uniformly in t, as it was the case for a smooth IVP. For the Gaussian points we see the superconvergence $O(h^{2k})$, both, in the mesh points and uniformly in t which is better than expected. However, this very fast convergence for the Gaussian points is put into the right perspective by the data for $k=3$ in Table 10.10. Here, only the expected order $O(h^{k+1})$ uniformly in t can be observed.

Table 10.9: Transformed TVP (10.20): Convergence of the collocation scheme, $k=2$

	equidistant points			Gaussian points					
	uniform			mesh			uniform		
h	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p
1/2	5.4e+00	–	–	2.1e+00	–	–	1.9e+00	–	–
1/4	2.3e+00	5.2e+01	1.26	2.3e-01	1.8e+01	3.17	2.2e-01	2.2e+03	3.14
1/8	6.8e-01	1.7e+02	1.73	1.7e-02	4.2e+01	3.76	1.6e-02	1.3e+04	3.74
1/16	1.8e-01	3.1e+02	1.93	1.1e-03	6.1e+01	3.94	1.1e-03	2.6e+04	3.93
1/32	4.5e-02	3.8e+02	1.98	7.0e-05	6.9e+01	3.98	6.8e-05	3.3e+04	3.98
1/64	1.1e-02	4.1e+02	2.00	4.4e-06	7.2e+01	4.00	4.3e-06	3.5e+04	4.00
1/128	2.8e-03	4.2e+02	2.00	2.7e-07	7.3e+01	4.00	2.7e-07	3.6e+04	4.00
1/256	7.1e-04	4.2e+02	2.00	1.7e-08	7.3e+01	4.00	1.7e-08	3.6e+04	4.00
1/512	1.8e-04	4.2e+02	2.00	1.1e-09	7.4e+01	4.00	1.0e-09	3.6e+04	4.00

Table 10.10: Transformed TVP (10.20): Convergence of the collocation scheme, $k=3$

	equidistant points			Gaussian points					
	uniform			mesh			uniform		
h	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p	$\|Y_h-Y\|_\infty$	c	p
1/2	2.2e+00	–	–	1.5e-01	–	–	1.5e-01	–	–
1/4	2.8e-01	1.1e+03	2.98	3.7e-03	5.7e+00	5.29	4.1e-03	4.5e+05	5.18
1/8	2.1e-02	8.2e+03	3.71	8.3e-05	7.5e+00	5.49	8.3e-05	2.1e+06	5.62
1/16	1.4e-03	1.7e+04	3.93	5.2e-06	3.4e-01	4.00	5.2e-06	2.1e+03	4.00
1/32	8.8e-05	2.2e+04	3.98	3.3e-07	3.4e-01	4.00	3.3e-07	2.1e+03	4.00
1/64	5.5e-06	2.3e+04	4.00	2.0e-08	3.4e-01	4.00	2.0e-08	2.1e+03	4.00
1/128	3.5e-07	2.4e+04	4.00	1.3e-09	3.4e-01	4.00	1.3e-09	2.1e+03	4.00
1/256	2.2e-08	2.4e+04	4.00	8.0e-11	3.3e-01	3.99	8.0e-11	2.0e+03	3.99
1/512	1.4e-09	2.4e+04	4.00	7.0e-12	2.2e-02	3.50	7.0e-12	4.6e+01	3.50

Boundary value problem

Finally, we discuss the BVP

$$y'(t) = \frac{M(t)}{t} y(t) + \frac{f(t)}{t},\ t \in (0,1],\ \begin{pmatrix} -2 & 3 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} y(0) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} y(1) = \begin{pmatrix} 1/2 \\ 0 \\ 4/5 \end{pmatrix},$$

where

$$M(t) = \begin{pmatrix} 1-2t & 3t-3\exp(t) & 3\exp(t)-3 \\ 2-2t & 3t-2\exp(t)-2 & 2\exp(t)-2 \\ 2-2t & 3t-2\exp(t)-12 & 2\exp(t)+8 \end{pmatrix},$$

$$M(0) = \begin{pmatrix} 1 & -3 & 0 \\ 2 & -4 & 0 \\ 2 & -14 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} -2 & & \\ & -1 & \\ & & 10 \end{pmatrix} \begin{pmatrix} -2 & 3 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix},$$

and

$$f(t) = \begin{pmatrix} 3t^{21/2} - t^{-1} + t^{-1}\exp(t) - 3t^{10}\exp(t) + 3t^{10} + \frac{3}{5}\exp(t) - \frac{3}{5} \\ 2t^{21/2} - t^{-1} + t^{-1}\exp(t) - 2t^{10}\exp(t) + 2t^{10} + \frac{2}{5}\exp(t) - \frac{2}{5} \\ 2t^{21/2} - t^{-1} + t^{-1}\exp(t) - 2t^{10}\exp(t) + 2t^{10} + \frac{2}{5}\exp(t) + \frac{8}{5} \end{pmatrix}.$$

The matrix $M(0)$ has both positive and negative eigenvalues $\lambda_1 = 10$, $\lambda_2 = -2$, $\lambda_3 = -1$, and the exact solution has the form

$$y(t) = \begin{pmatrix} t^{-2} + t^{-1}\exp(t) - t^{-2}\exp(t) + \frac{6}{23}t^{21/2} \\ t^{-2} + t^{-1}\exp(t) - t^{-2}\exp(t) + \frac{4}{23}t^{21/2} \\ t^{-2} + t^{-1}\exp(t) - t^{-2}\exp(t) + \frac{4}{23}t^{21/2} + t^{10} - \frac{1}{5} \end{pmatrix}.$$

Here, $y \in C^{10}[0,1]$. In Tables 10.11, 10.12 and 10.13 the convergence order k can be observed uniformly in t. For this model and for the Gaussian points we observe the superconvergence order $O(h^{2k})$. Such high order of convergence can be sometimes observed, but does not hold in general.

Table 10.11: BVP: Convergence of the collocation scheme, $k = 2$

	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
h	$\|Y_h - Y\|_\Delta$	c	p	$\|Y_h - Y\|_\Delta$	c	p	$\|Y_h - Y\|_u$	c	p
1/2	8.1e-02	3.6e-02	-1.18	5.4e-02	1.1e-02	-2.30	6.1e-01	1.4e+00	1.22
1/4	1.8e-01	8.5e+00	2.77	2.6e-01	2.9e+00	1.73	2.6e-01	2.9e+00	1.73
1/8	2.7e-02	6.2e+01	3.72	7.9e-02	5.5e+00	2.04	7.9e-02	5.0e+00	1.99
1/16	2.0e-03	1.7e+02	4.09	1.9e-02	9.3e+00	2.23	2.0e-02	6.8e+00	2.10
1/32	1.2e-04	1.4e+02	4.02	4.1e-03	4.6e+00	2.02	4.6e-03	6.4e+00	2.09
1/64	7.4e-06	1.2e+02	4.00	1.0e-03	4.4e+00	2.01	1.1e-03	5.7e+00	2.06
1/128	4.6e-07	1.2e+02	4.00	2.5e-04	4.1e+00	2.00	2.6e-04	5.1e+00	2.04
1/256	2.9e-08	1.2e+02	4.00	6.2e-05	4.1e+00	2.00	6.4e-05	4.7e+00	2.02
1/512	1.8e-09	–	–	1.6e-05	–	–	1.6e-05	–	–

Table 10.12: BVP: Convergence of the collocation scheme, $k = 3$

	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
h	$\|Y_h - Y\|_\Delta$	c	p	$\|Y_h - Y\|_\Delta$	c	p	$\|Y_h - Y\|_u$	c	p
1/2	2.6e-02	2.1e-02	-0.35	6.7e-03	4.5e-04	-3.91	4.2e-01	1.7e+00	2.04
1/4	3.4e-02	2.4e+01	4.75	1.0e-01	5.6e+00	2.90	1.0e-01	5.6e+00	2.90
1/8	1.2e-03	2.2e+02	5.81	1.4e-02	3.5e+01	3.78	1.4e-02	1.8e+01	3.45
1/16	2.2e-05	4.1e+02	6.04	9.9e-04	8.3e+01	4.09	1.2e-03	2.5e+01	3.57
1/32	3.4e-07	3.8e+02	6.01	5.8e-05	6.6e+01	4.02	1.0e-04	5.0e+01	3.78
1/64	5.2e-09	3.6e+02	6.00	3.6e-06	6.1e+01	4.01	7.6e-06	8.0e+01	3.89
1/128	8.2e-11	3.6e+02	6.00	2.2e-07	6.0e+01	4.00	5.2e-07	1.0e+02	3.94
1/256	1.3e-12	4.2e-10	1.04	1.4e-08	5.9e+01	4.00	3.4e-08	1.2e+02	3.97
1/512	6.2e-13	–	–	8.7e-10	–	–	2.1e-09	–	–

Table 10.13: BVP: Convergence of the collocation scheme, $k = 4$

	Gaussian, mesh points			equidistant, mesh points			equidistant, uniform		
h	$\|Y_h - Y\|_\Delta$	c	p	$\|Y_h - Y\|_\Delta$	c	p	$\|Y_h - Y\|_u$	c	p
1/2	2.8e-03	2.0e-03	-0.45	4.0e-03	5.5e-04	-2.88	2.5e-01	2.1e+00	3.06
1/4	3.8e-03	5.4e+01	6.89	3.0e-02	6.3e+00	3.87	3.0e-02	6.3e+00	3.87
1/8	3.2e-05	4.7e+02	7.94	2.0e-03	1.3e+01	4.23	2.0e-03	8.6e+00	4.02
1/16	1.3e-07	4.7e+02	7.93	1.1e-04	1.3e+01	4.23	1.2e-04	1.6e+01	4.23
1/32	5.3e-10	5.8e+02	8.00	5.7e-06	7.5e+00	4.06	6.6e-06	1.0e+01	4.12
1/64	2.1e-12	4.1e-06	3.49	3.4e-07	6.1e+00	4.01	3.8e-07	9.1e+00	4.08
1/128	1.9e-13	1.1e-16	-1.53	2.1e-08	5.8e+00	4.00	2.3e-08	7.6e+00	4.05
1/256	5.4e-13	1.5e-13	-0.23	1.3e-09	5.7e+00	4.00	1.4e-09	6.6e+00	4.02
1/512	6.3e-13	–	–	8.3e-11	–	–	8.4e-11	–	–

Conclusion

In this thesis new contributions to the theory of singular ordinary differential equations were presented.

In Part I, we studied the solution structure to singular nonlinear second order ODE

$$\big(p(t)u'(t)\big)' + q(t)f((u(t)) = 0,\ \ t \in [0,\infty),$$

subject to the initial conditions

$$u(0) = u_0 \in (L_0, L), \quad u'(0) = 0.$$

New existence results for escape and homoclinic solutions were obtained in Theorem 4.14 and Theorem 4.15 under the assumptions which guarantee that each damped solution of the problem is oscillatory. In addition, the existence of damped solutions and their asymptotic behaviour were derived in a more general settings that have been done before, see Theorem 4.1 and Lemma 4.4. Among damped solutions, the Kneser solutions were of special interest. The existence of Kneser solutions to singular equation (2.3) was proved in Theorems 5.5 and 5.6 provided $p \equiv q$ while for $p \neq q$ this is still an open question.

In the general case $p \neq q$, a regular equation (2.3) on $[a,\infty), a > 0$, was studied. More precisely, two classes of initial value problems were discussed,

$$\begin{gathered}\big(p(t)u'(t)\big)' + q(t)f((u(t)) = 0,\ \ t \in [a,\infty),\\ u(a) = u_0 \in (0,L),\ 0 \le u(t) \le L \text{ for } t \in [a,\infty),\end{gathered} \tag{10.22}$$

and

$$\begin{gathered}\big(p(t)u'(t)\big)' + q(t)f((u(t)) = 0,\ \ t \in [a,\infty),\\ u(a) = u_0 \in (L_0,0),\ L_0 \le u(t) \le 0 \text{ for } t \in [a,\infty).\end{gathered} \tag{10.23}$$

It turned out that there exists a nonoscillatory solution of problem (10.22) (and problem (10.23)), which is either a Kneser solution or a monotonically increasing solution whose limit is L for t tending to infinity, Theorem 5.16 (and a monotonically decreasing solution whose limit is L_0 for t tending to infinity, Theorem 5.17).

In both, the regular and the singular case, we provided new asymptotic formulas in the form of upper bounds for damped Kneser solutions and their first derivatives to the ODE (1.1) with regularly varying coefficients, in Theorem 6.9 and in Theorem 6.10, respectively.

In Part II, we investigated the analytical properties of the singular BVP with a variable coefficient matrix

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t}, \quad B_0 y(0) + B_1 y(1) = \beta$$

and we achieved to generalize results obtained for problems with a smooth inhomogeneity. The structure of the correctly posed boundary conditions which guarantee the existence of a unique solution $y \in C[0,1]$ turned out to depend on the spectral properties of the matrix $M(0)$. Therefore, we had to carry out in full details the following three case studies, the case of only negative real parts of the eigenvalues of $M(0)$ in Theorem 8.6, positive real parts of the eigenvalues of $M(0)$ in Theorem 8.12, and zero eigenvalues of $M(0)$ in Theorem 8.18. The main technical tool used in the analysis was the Banach Fixed Point Theorem which turned out to be very useful in mastering the difficulties caused by the singularity at $t = 0$. Then these case investigations were used to generalize the results to the cases of general IVPs in Theorem 9.2, TVPs in Theorem 9.3, and BVPs in Theorem 9.9.

Moreover, we analysed the collocation method applied to approximate the solution of the above analytical problem and we derived new convergence properties. The convergence behaviour was investigated separately for general IVPs, TVPs and BVPs, in Theorem 10.2, Theorem 10.7 and Theorem 10.9, respectively. We proved that the collocation retains its classical stage order k uniformly in t for a scheme with k collocation points, provided that the analytical solutions are appropriately smooth. Moreover, for Gaussian points the so-called small superconvergence order $k+1$ was shown to hold in context of an IVP in Theorem 10.3, whereas, the superconvergence order in the mesh points, $2k$ for Gaussian points, cannot be expected to hold, in general. The theoretical results are supported by the numerical experiments.

The results contained in this thesis were published in seven papers and presented at six international conferences.

The aim of further investigations is to prove the existence of homoclinic solutions also in the case when damped nonoscillatory solution of (1.1), (1.2) may appear. Next goal is to prove the existence of Kneser solutions to equation (1.1) with a time singularity at $a = 0$ and $p \neq q$, and to more precisely describe their asymptotic behaviour in the form of asymptotic equivalence as t tends to infinity. Furthermore, we intend to generalize problem (1.1), (1.2) to the problem with ϕ-Laplacian

$$\big(p(t)\phi(u'(t))\big)' + q(t)f(u(t)) = 0, \quad u(0) = u_0,\ u'(0) = 0,$$

where ϕ is an increasing homeomorphism with $\phi(\mathbb{R}) = \mathbb{R}$.

The aim of further study concerning the system of first order ODE is to investigate the nonlinear problem

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t,y(t))}{t},\ t \in (0,1], \quad B_0y(0) + B_1y(1) = \beta.$$

and to prove the existence of a solution of the nonlinear BVP which we plan to publish in an upcoming paper.

List of publications

[1] J. BURKOTOVÁ, M. HUBNER, I. RACHŮNKOVÁ, E.B. WEINMÜLLER, Asymptotic properties of Kneser solutions to nonlinear second order ODE with regularly varying coefficients, *submitted*.

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[6] J. BURKOTOVÁ, I. RACHŮNKOVÁ, E.B. WEINMÜLLER, On singular BVPs with unsmooth data. Part 1: Analysis of the linear case with variable coefficient matrix, *submitted*.

[7] J. BURKOTOVÁ, I. RACHŮNKOVÁ, E.B. WEINMÜLLER, On singular BVPs with unsmooth data. Part 2: Convergence of the collocation schemes, *submitted*.

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Contents

Notation **4**

1 Abstract **5**

2 Abstrakt v českém jazyce **6**

3 Introduction **7**

4 Recent state summary **7**

5 Thesis objectives **11**

6 Theoretical framework **12**

7 Applied methods **14**

8 Original results **15**

9 Summary of results **27**

List of publications **29**

References **29**

Notation

Throughout the thesis, the following standard notation is used.

$\mathbb{N}$	set of all natural numbers;
$\mathbb{R}^n$	n-dimensional vector space of real-valued vectors;
$\mathbb{R}^{m\times n}$	$m\times n$-dimensional space of real-valued matrices;
$\lvert x\rvert$	$\lvert x\rvert := \max\{\lvert x_i\rvert : 1\le i\le n\}$; maximum norm for a vector $x\in\mathbb{C}^n$;
$\lvert A\rvert$	$\lvert A\rvert := \max_{1\le i\le m}\sum_{j=1}^{n}\lvert a_{ij}\rvert$; induced operator norm for a matrix $A\in\mathbb{C}^{m\times n}$;
$C[a,b]$	Banach space of real vector-valued functions continuous on $[a,b]$ equipped with the norm $\lVert y\rVert_{C[a,b]} := \max\{\lvert y(t)\rvert : t\in[a,b]\}$;
$C^p[a,b]$	Banach space of real vector-valued functions p-times continuously differentiable on $[a,b]$ equipped with the norm $\lVert y\rVert_{C^p[a,b]} := \sum_{k=0}^{p}\lVert y^{(k)}\rVert_{C[a,b]}$;
$C(I)$	set of real vector-valued functions continuous on the interval I;
$C_{m\times n}[a,b]$	Banach space of real-valued matrix functions continuous on $[a,b]$ equipped with the norm $\lVert y\rVert_{C_{m\times n}[a,b]} := \max\{\lvert y(t)\rvert : t\in[a,b]\}$;
$C^p_{m\times n}[a,b]$	Banach space of real-valued matrix functions p-times continuously differentiable on $[a,b]$ equipped with the norm $\lVert y\rVert_{C^p_{m\times n}[a,b]} := \sum_{k=1}^{p}\lVert M^{(k)}\rVert_\delta$;
$\mathrm{Lip}_{loc}(I)$	set of locally Lipschitz continuous functions on the interval I;
$O(g(x))$	$f(x)=O(g(x))$ for $x\to 0^+$ if $\exists c>0,\delta>0 : \lvert f(x)\rvert\le c\lvert g(x)\rvert$ for $0<x<\delta$;

If it cannot be confusing, we omit the subscripts m and n for simplicity of notation, and write $C[a,b]=C_{m\times n}[a,b]$, $C^p_{m\times n}[a,b]=C^p[a,b]$. We also use a shorthand notation:

$$
\begin{aligned}
\lVert y\rVert &:= \max\{\lvert y(t)\rvert : t\in[0,1]\};\ \text{for } y\in C[0,1];\\
\lVert y\rVert_\delta &:= \max\{\lvert y(t)\rvert : t\in[0,\delta]\};\ \text{norm restricted to the interval } [0,\delta],\ \delta>0;\\
\lVert M\rVert_\delta &:= \max\{\lvert M(t)\rvert : t\in[0,\delta]\};\ \text{norm for } M\in C_{m\times n}[0,\delta];
\end{aligned}
$$

1 Abstract

In modern science and technology, the mathematical description of complex physical processes often leads to differential models. Therefore, the topic of differential equations has been of interest for scientists and engineers for many years. The thesis is devoted to the existence and asymptotic behaviour of solutions to ordinary differential equations with a singularity in the independent variable. Two different singular problems are presented, and therefore, the thesis is divided into two parts.

The first part focusses on the study of second order nonlinear differential equations on unbounded domain $[0,\infty)$ which may have a time singularity at the origin. A set of all solutions is described according to their asymptotic behaviour at infinity. Existence results and properties of damped and escape solutions are derived. By means of these results, the existence of an increasing solution with $u(\infty) = L$, a homoclinic solution, playing an important role in applications is proved. Furthermore, the existence of Kneser solutions is investigated and asymptotic properties of such solutions and their first derivatives are derived in the framework of regularly varying functions. The analytical findings concerning Kneser solutions are illustrated by numerical simulations.

The second part of the thesis investigates analytical and numerical properties of systems of linear ordinary differential equations with an unsmooth nonintegrable inhomogeneity and a time singularity of the first kind on a compact interval. The asymptotic behaviour of solutions at the singular point $t = 0$ is analysed. The focus is on specifying the structure of general linear two-point boundary conditions guaranteeing the existence and uniqueness of solutions which are continuous on a closed interval including the singular point. Moreover, the collocation method which approximates the analytical solution by a continuous piecewise polynomial function is analysed, and its convergence order is derived.

Key words: ordinary differential equations, asymptotic properties, time singularity, Kneser solutions, nonoscillatory solutions, regular variations, collocation method, convergence.

2 Abstrakt v českém jazyce

V moderní vědě a technice vede matematický popis složitých fyzikálních procesů často k diferenciálních modelům. Proto je téma diferenciálních rovnic v zájmu vědců a odborníků již mnoho let. Tato práce je věnována existenci a asymptotickému chování řešení obyčejných diferenciálních rovnic se singularitou v nezávisle proměnné. Uvádíme dva různé singulární problémy, proto je práce rozdělena do dvou částí.

První část se zaměřuje na studium nelineárních diferenciálních rovnic druhého řádu na neomezené oblasti $[0,\infty)$, které mohou mít časovou singularitu v počátku. Množina všech možných řešení je popsána v závislosti na jejich asymptotického chování v nekonečnu. Jsou odvozeny existenční výsledky a vlastnosti tlumených a únikových řešení. Prostřednictvím těchto výsledků je dokázána existence rostoucího řešení s limitou $u(\infty) = L$, homoklinického řešení, které hraje důležitou roli v aplikacích. Dále je zkoumána existence Kneserových řešení a jsou odvozeny asymptotické vlastnosti těchto řešení v rámci teorie regulárně měnících se funkcí. Analytické výsledky týkající se Kneserových řešení jsou znázorněny pomocí numerických simulací.

Druhá část práce zkoumá analytické a numerické vlastnosti systémů lineárních obyčejných diferenciálních rovnic s neintegrovatelnou nehomogenní složkou a časovou singularitou prvního druhu na kompaktním intervalu. Asymptotické chování řešení je analyzováno v singulárním bodě $t = 0$. Důraz je kladen na stanovení struktury obecných lineárních dvoubodových okrajových podmínek zaručujících existenci a jednoznačnost řešení, která jsou spojitá na uzavřeném intervalu zahrnující singulární bod. Dále je studována kolokační metoda, která aproximuje analytické řešení spojitou po částech polynomiální funkcí, a je odvozen její řád konvergence. Teoretické výsledky jsou doloženy numerickými simulacemi.

Klíčová slova: obyčejné diferenciální rovnice, asymptotické vlastnosti, časová singularita, Kneserova řešení, neoscilatorické řešení, regulární variace, metoda kolokace, konvergence.

3 Introduction

The thesis is devoted to the existence and asymptotic behaviour of solutions of ordinary differential equations (ODEs), in particular, of problems which may have a time singularity. The thesis consists of two parts.

Part I is concerned with nonlinear second order differential equation

$$\big(p(t)u'(t)\big)' + q(t)f(u(t)) = 0, \quad t \in [0,\infty), \tag{1}$$

subject to initial conditions

$$u(0) = u_0, \quad u'(0) = 0. \tag{2}$$

Here, $f \in C(\mathbb{R})$, $f(L_0) = f(0) = f(L) = 0$, $L_0 < 0 < L$ and $xf(x) > 0$ for $x \in (L_0,0) \cup (0,L)$. Further, $p,q \in C[0,\infty)$ are positive on $(0,\infty)$ and $p(0) = 0$. Part I deals with the asymptotic behaviour of solutions in a neighbourhood of infinity and collects results from papers [1, 2, 3]. Global existence results on $[0,\infty)$ are obtained and the structure of all possible solutions is described according to their asymptotic properties.

Part II presents results from [4, 5, 6, 7] devoted to boundary value problems (BVPs) of the form

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t}, \quad t \in (0,1], \quad y \in C[0,1], \quad B_0 y(0) + B_1 y(1) = \beta, \tag{3}$$

where y is n-dimensional real function, M is $n \times n$ continuous matrix function, f is a n-dimensional function which is at least continuous on $[0,1]$ and $B_0, B_1 \in \mathbb{R}_{m\times n}$, $\beta \in \mathbb{R}^n$. In Part II, the behaviour of solutions near the singular point is analysed. The stress is laid on the structure of boundary conditions which are necessary and sufficient for the existence of a unique continuous solution on the closed interval $[0,1]$ including the singular point $t = 0$.

4 Recent state summary

In numerous applications from physics, chemistry and mechanics, the mathematical description of complex processes leads to differential models. These models take often the form of systems of time-dependent partial differential equations (PDEs). For the investigation of stationary solutions many of these models can be reduced to singular ODEs of the second order or to singular systems of ODEs of the first order, especially when symmetries in geometry of the problem appear. Therefore, papers providing analytical results on their structural properties, stability and convergence of different numerical methods, and results of numerical simulations are available.

In the thesis we study asymptotic behaviour of solutions of singular second order nonlinear differential equations in Part I and asymptotic behaviour of singular linear systems of first order differential equations in Part II. A problem is denoted as *singular* if the right-hand side does not fulfil the Carathéodory conditions. Otherwise, a problem is called *regular*. In particular, the system of first order ODEs

$$x'(t) = f(t,x),\ t \in I \subset \mathbb{R}, \tag{4}$$

where $x \in \mathbb{R}^n, f : I \times \mathbb{R}^n \to \mathbb{R}^n$, is understood to be *singular* if function f does not fulfil the Carathéodory conditions. For differential equations of the second order we use the terminology corresponding to the equivalent first order system. The solvability of such problems is not covered by the Carathéodory theory and requires a different approach. Two types of singularities are distinguished, time and space singularities. System (4) has a time singularity if $f(\cdot,x) : I \to \mathbb{R}^n$ is not integrable for some $x \in \mathbb{R}^n$. Both problems studied in the thesis are singular with a time singularity at the origin, which means that

$$\int_0^\varepsilon |f(t,x)|\,\mathrm{d}t = \infty$$

for some $x \in \mathbb{R}^n$ and each sufficiently small $\varepsilon > 0$.

In **Part I**, we study the second order differential equation (1):

$$(p(t)u'(t)) + q(t)f(u(t)) = 0,\ t \geq 0.$$

This equation can be assumed as a special case of (4). For $v = pu'$, we obtain

$$u'(t) = \frac{1}{p(t)}v(t), \quad v'(t) = -q(t)f(u(t)), \quad t \geq 0.$$

In general, the case

$$\int_0^1 \frac{\mathrm{d}s}{p(s)} = \infty$$

is also considered. Therefore, the differential equation (1) may have a time singularity at $t = 0$.

Regular equation

There exists an extensive literature which is devoted to a qualitative analysis of solutions of the nonlinear equation (1), or to its quasilinear generalizations in the regular setting, where $p > 0$ on $[0,\infty)$. A general survey concerning asymptotic properties of solutions of nonautonomous ODEs is provided in the monograph [34]. In last decades, great effort has been devoted to the asymptotic analysis in the case that (1) is the Emden-Fowler equation

$$u''(t) + q(t)|u|^\gamma \operatorname{sgn} u = 0, \quad \gamma > 0, \gamma \neq 1.$$

We refer to [76] for a history survey. The subject of oscillation and nonoscillation of the Emden-Fowler equation has been extensively studied, for example see [43, 47, 48, 57]. The Emden-Fowler equations of arbitrary order are thoroughly discussed in [34]. Equation (1) with a nonconstant p and a more general f is investigated in [15, 42], where the equation is regular and the nonlinearity f in all these papers has behaviour which is characterized by the assumption $xf(x) > 0$ for all $x \neq 0$ and is globally monotone.

A possible generalization of the Emden-Fowler equation has the form with p-Laplacian

$$\big(p(t)\Phi_\alpha(u')\big)' + q(t)\Phi_\gamma(u) = 0,\ \alpha > 0, \gamma > 0,$$

where $\Phi_\alpha(u) := |u|^\alpha \operatorname{sgn} u$. This generalized Emden-Fowler equation is called sub-half-linear, half-linear or super-half-linear if $\alpha > \gamma$, $\alpha = \gamma$ or $\alpha < \gamma$, respectively. Existence results for solutions with certain asymptotic behaviour for sub-half-linear case can be found in [31, 44, 46], and those for the half-linear case in [21, 32, 45], whereas the super-half-linear case is studied in [22, 53].

The theory of regular variations [17] provides a powerful tool for the asymptotic analysis. The systematic study of differential equations by means of regularly varying functions can be found in [52]. Asymptotic results for related equations or systems which are characterized by regularly varying functions are obtained in [25, 30, 54]. Oscillation and nonoscillation criteria for related two-dimensional systems of linear and nonlinear ODEs are established in [50] and in [24, 55], respectively. We also refer to [23, 35], where Kneser solutions of two-dimensional systems of ODEs are studied.

Singular equation

The singular equation

$$(p(t)u'(t))' + p(t)f(u(t)) = 0, \tag{5}$$

which is a special case of (1) with $p \equiv q$, is studied in [58, 59], [62]–[68]. The dynamical system approach and the lower and upper function method are used in [64, 65] where the existence of an escape solution and the existence of a homoclinic solutions of this problem is reached. All types of possible solutions of (5), (2) with conditions which guarantee their existence and specify their asymptotic behaviour are described in [62, 63, 66]. In particular, the existence of damped oscillatory solutions with decreasing amplitudes of problem (5), (2) is described in [68]. Sufficient conditions for convergence of such oscillatory solutions to zero are given in [59]. Problem (5), (2) can be transformed into a problem of the existence of a positive solutions on the half-line. For $p(t) = t^k, k \in \mathbb{N}$, and for $p(t) = t^k, k \in (1,\infty)$, such problem is solved by variational methods in [16] and [18], respectively.

Moreover, the existence and uniqueness of damped solution to (1), (2) is studied in [69] provided that the derivative of function p is continuous and positive and satisfies another more restrictive assumptions. In addition, conditions guaranteeing that all damped solutions are oscillatory with decreasing amplitudes are derived.

For other problems with singularities we refer to [8, 9, 56, 60, 61], where in [56] the existence theory for singular two-point BVPs on finite and semi-infinite intervals is introduced. Existence theory for a variety of singular problems based on regularization and sequential technique is presented in [60, 61]. For further development see [8, 9] and references therein.

Part II is devoted to singular systems of first order ordinary differential equations. A popular model class for theoretical investigations is the singular BVP

$$y'(t) = \frac{M(t)}{t^\alpha} y(t) + f(t, y(t)),\ t \in (0,1], \quad b(y(0), y(1)) = 0,$$

where $\alpha \geq 1$, $n \times n$ matrix function M and n-dimensional vector functions f and b are continuous. Two types of time singularities are distinguished depending on the value of α. For $\alpha = 1$ the singularity is denoted as a *singularity of the first kind* and for $\alpha > 1$ the problem is called singular with an *essential singularity* or a *singularity of the second kind*. Part II deals with the linear systems of the form

$$y'(t) = \frac{M(t)}{t} y(t) + \frac{f(t)}{t},\ t \in (0,1], \tag{6}$$

where in general $\frac{M(t)}{t}$ and $\frac{f(t)}{t}$ may not be integrable on $[0,1]$ which yields a time singularity of the first kind at $t = 0$.

Analytical properties of problems with a continuous inhomogeneity

$$y'(t) = \frac{M(t)}{t} y(t) + f(t),\ t \in (0,1], \quad B_0 y(0) + B_1 y(1) = \beta, \tag{7}$$

are discussed in [27, 75], where the attention is focused on the existence, uniqueness and smoothness of solutions. In particular, the structure of the boundary conditions which are necessary and sufficient for (7) to have a unique continuous solution on $[0,1]$ is of special interest. Problems of type (7), where f may additionally depend on the space variable y and have a space singularity are studied for example in [10, 51, 72]. The equation (6) with unsmooth inhomogeneity is investigated in [73] where the existence only of a unique continuous particular solution without any boundary conditions is examined. The nonlinear equation of the form (6) is studied in [74].

To compute the numerical solution of (7) the polynomial collocation [19] is proposed in [26]. See also [75] for second order systems. The choice of the collocation is motivated by its advantageous convergence properties for (7) while in the presence of a singularity other high order methods show order reductions and become inefficient [29]. Consequently, for singular BVPs two open domain MATLAB codes based on collocation have been implemented [14, 36]. The code `sbvp` can be applied to explicit first order ODEs [14], while `bvpsuite` can be used to solve arbitrary mixed order problems in implicit formulation. Its scope also includes the differential algebraic equations [36]. Both codes have been used to numerically simulate singular BVPs important for applications and they have proved to work dependably and efficiently [20, 37]. This is our motivation to propose the polynomial collocation for the approximation of (3).

Due to very advantageous properties of the collocation method, this approach has been used in a variety of other openly available programs. We enclose some examples for the existing software packages designed to deal with regular and singular ODEs: the standard MATLAB code `bvp4c` [70] and the related solver `bvp5c` [33], two FORTRAN codes, `BVP_SOLVER` specified in [71], and COLNEW described in [12] and based on one of the best established BVP solvers COLSYS [11]. For most of the basic solvers, error estimation routines and grid adaptation strategies implemented in these codes, analytical justification in context of singular systems is given. Typically, to enhance the efficiency of the code, the order of the basic solver varies depending on the tolerances specified by the user. Collocation has proved to be also a useful tool to treat other problem classes, dynamical system in ODEs [41] and algebraic-differential equations, see [49].

5 Thesis objectives

The thesis has several objectives, above all to obtain new contributions to the theory of singular differential equation and to extend results for a more general class of problems.

The first aim of Part I is to investigate the existence of all three types of solutions of problem (1), (2) from Definitions 8.3 and 8.4. This means to generalize the existence result about damped solutions from [69], and moreover, to prove the existence of escape and homoclinic solutions to problem (1), (2). The effort to show the existence of a homoclinic solution is emphasised by its important role in applications. The second goal is to describe in more details the asymptotic behaviour of damped nonoscillatory solutions of problem (1), (2). In particular, investigation of the existence of damped Kneser solutions of equation (1) is of special interest. The aim is also to derive asymptotic formulas for such solutions and their first derivatives in the framework of regularly varying functions which has shown to be a powerful tool for the study of asymptotic properties of solutions of differential equations. We confront our results with numerical simulations to provide a helping insight into the analysis.

The aim of Part II is to investigate analytical properties of linear BVPs (7) with a variable coefficient matrix and an unsmooth inhomogeneity. We are interested to recover solution y which is at least continuous, $y \in C[0,1]$. In particular, our attention is focused on specifying the structure of general linear two-point boundary condition guaranteeing the existence and uniqueness of solutions which are continuous on the closed interval including the singular point $t = 0$. We also specify conditions for f and M which are sufficient for $y \in C^r[0,1]$, $r \in \mathbb{N}$. The motivation for the above analysis of the variable coefficient case is twofold:

First of all, in order to investigate the nonlinear case one can choose to study the properties of its linearization, see [27]. In this context a related linear BVP with a variable coefficient matrix has to be studied. More precisely, the technique applied in [27] is based on the *assumption that a solution to the nonlinear problem exists*. Next, the nonlinear problem is linearized at the exact solution and the well-posedness of this linearization is studied. However, we are not going to follow this technique and plan in an upcoming paper to *show the existence of the solution of the nonlinear BVP*, instead of assuming its existence.

Secondly, for us, the investigation of the structural properties of (3) is necessary and interesting in its own right as a prerequisite for numerical analysis. A knowledge of qualitative properties such as the existence, uniqueness and smoothness of a solution in the linear case is required for the convergence theory of the collocation method. The aim is to prove that the convergence order of the collocation is at least equal to the stage order of the method as predicted by numerical simulations.

6 Theoretical framework

In this section, we present the theoretical framework for the subsequent analysis. In Part I we continue the research initiated by Rachůnková, Tomeček et al. in [58, 59], [62]–[68] and [69]. In particular, we extend the results reached for equation (5) to the more general equation (1). All these papers are establish on the basic assumptions: Function f is Lipschitz continuous on the domain where the solution is searched for, f has prescribed sign conditions and two or three zeros. The coefficient function p is assumed to fulfil:

$$p \in C[0,\infty) \cap C^1(0,\infty),\ p(0) = 0,\ p' > 0 \text{ on } (0,\infty),\ \lim_{t\to\infty} \frac{p'(t)}{p(t)} = 0. \tag{8}$$

In [64, 65] the existence of three types of solutions (damped, homoclinic and escape) in the sense of Definitions 8.3, 8.4 is proved under additional conditions posed on p and f. We are mainly motivated by papers [62, 63, 66, 67] where another technique based on differential inequalities is applied. The existence and asymptotic behaviour of all types of possible solutions of problem (5), (2) are derived without additional conditions. Problem (5), (2) with f having just two zeros in $f(0) = f(L) = 0$ is studied in [62] and [67]. In particular, in [62] the problem (5), (2) is investigated provided either f has a sublinear behaviour near $-\infty$ or

$$\int_0^1 \frac{\mathrm{d}s}{p(s)} < \infty.$$

The case when f has a superlinear behaviour near $-\infty$ is studied [67]. In paper [63], the existence of an escape solution and also of a homoclinic solution of (5), (2) is proved under the assumptions that f has at least two zeros $f(0) = f(L) = 0$ and a sublinear or linear behaviour near $-\infty$. In [66], sufficient conditions for the existence of at least one escape solution of problem (5), (2) which is fundamental for the existence of a homoclinic solution, are presented. The existence and asymptotic properties of oscillatory solutions of problem (5), (2) are described in [58, 59, 68].

The investigation of the more general problem (1), (2) has started in [69], where the existence and uniqueness of damped solution to (1), (2) is studied under the basic assumptions on function f which are specified at the beginning of this chapter, under assumption (8) on function p and under assuming q to be continuous on $[0,\infty)$ and positive on $(0,\infty)$. Moreover, the existence and uniqueness of damped oscillatory solutions with decreasing amplitudes is proved under other additional conditions.

In Part II we continue the investigation initiated by de Hoog and Weiss [26]–[29] and later followed by Weinmüller et al. [36]–[40], [75] where the boundary value problem (7) is studied. Here we summarize results obtained in the framework given in [27, 40], which are extended in Part II to the more general problem (3) with unsmooth $f(t)/t$. We discuss the case of negative real parts of eigenvalues, the case of positive real parts of eigenvalues and zero eigenvalues of matrix $M(0)$ separately, since this is the key to understand the rest of the theory. Here, $M \in C^1[0,1]$ which yields $M(t) = M(0) + tD(t)$, $D \in C[0,1]$.

- Let all eigenvalues of $M(0)$ have negative real parts. Then, $y \in C[0,1]$ if and only if $y(0) = 0$. Therefore, the initial value problem (IVP)

$$y'(t) = \frac{M(t)}{t}y(t) + f(t), \quad y(0) = 0,$$

  has a unique solution. Moreover, $y \in C^{r+1}[0,1]$ if $f \in C^r[0,1], D \in C^r[0,1], r \geq 0$.

- In the case of only positive real parts of eigenvalues of $M(0)$, we have to specify the boundary conditions at $t = 1$ and solve a terminal value problem (TVP). In particular, the TVP

$$y'(t) = \frac{M}{t}y(t) + f(t), \quad B_1 y(1) = \beta,$$

  where $B_1 \in \mathbb{R}^{n\times n}$ is nonsingular and $\beta \in \mathbb{R}^n$, has a unique solution $y \in C[0,1]$. This solution satisfies $y(0) = 0$. If $f \in C^r[0,1]$, $D \in C^r[0,1]$ and $\sigma_+ > r+1$ then $y \in C^{r+1}[0,1]$.

- Let all eigenvalues of $M(0)$ be zero. Consider the IVP which takes the form

$$y'(t) = \frac{M(t)}{t}y(t) + f(t), \quad M(0)y(0) = 0, \quad B_0 y(0) = \beta,$$

where the $m \times m$ matrix $B_0\tilde{R}$ is nonsingular, $\beta \in \mathbb{R}^m$, and $m = \dim X_0^{(e)}$. The initial condition $M(0)y(0) = 0$ is necessary and sufficient for the solution to by continuous. The remaining m conditions necessary for its uniqueness are specified by $B_0 y(0) = \beta$. For $f \in C^r[0,1], D \in C^r[0,1], r \geq 0$, $y \in C^{r+1}[0,1]$.

In the numerical part, the polynomial collocation is used to approximate the analytical solution by a continuous piecewise polynomial function. For regular problems with appropriately smooth data a convergence order $O(h^k)$ can be guaranteed [13], where k denotes the number of collocation points. Moreover, for special choice of the collocation nodes, for example the Gaussian points, superconvergence order $O(h^{2k})$ holds at the mesh points.

The collocation for singular problems (7) with only nonpositive real parts of eigenvalues of $M(0)$ is discussed in [26]. For smooth solutions of such problems, the polynomial collocation converges with at least the stage order $O(h^k)$ uniformly in t when polynomials of degree k are used to define the basic numerical scheme. Similar results hold in context of TVP with nonnegative real parts of eigenvalues. The counterexamples given in [26] show that the superconvergence breaks down in general for singular problems. However, for problems with nonpositive real parts of eigenvalues of $M(0)$ the small superconvergence $O(h^{k+1})$ is proved to hold for certain choice of collocation points. In the case of a general spectrum of $M(0)$ the approach to convergence analysis in [26] is not feasible. In [39] a new representation of the global error of collocation method applied to (7) is derived and the global error is shown to be equal at least to the stage order $O(h^k)$ for sufficiently smooth data.

In [74] local existence and uniqueness analysis is provided for a certain class of nonlinear differential equation of the type $tu'(t) = g(t, u(t))$, see also [73]. The boundary conditions are disregarded and the problem is solved numerically by collocation applied to the integral equation resulting after the integration of the ODE system. It turns out that the global error of the collocation scheme is $O(h^k|\ln h|)$ provided that the problem data are appropriately smooth and h is sufficiently small.

7 Applied methods

For nonlinear differential equations and linear equations with variable coefficients only seldom the explicit solution formula is known. There are two approaches to solve such equations. The aim of the first one is to obtain qualitative properties of solutions, such as existence, uniqueness, asymptotic behaviour and other characterisations. The second approach, which heavily relies on the first one, is the use of numerical methods to solve the problem. In the thesis both, analytical and numerical methods are applied.

Many problems in differential equations can be reduced to operator equations in Banach spaces where methods of functional analysis are effective, especially those based on the fixed point theory. The Schauder Fixed Point Theorem plays an important role for existence results. We apply this theorem in Part I with the help of the Arzelà-Ascoli Theorem which yields the compactness of operators. By means of the methods of a priori estimates we can apply general existence principles to wide range of problems even when all assumptions are not fulfilled. The Banach Fixed Point Theorem is especially useful as it both guarantees the existence and the uniqueness of a solution. We apply the Banach Fixed Point Theorem for linear system in Part II since it turns out to be very helpful to deal with difficulties caused by the singularity at $t = 0$.

From the great variety of numerical methods we decided to use the polynomial collocation to compute a numerical solution. This method shows advantageous convergence properties compared to other direct high order methods which may be affected by order reductions and become inefficient in the presence of a singularity.

8 Original results

In the thesis new contributions to the theory of singular nonlinear ordinary differential equations of second order (1) and linear first order boundary value problems with a singularity of the first kind and unsmooth inhomogeneities (3) are presented. The thesis is based on the results published in [1]–[7].

Singular nonlinear second order differential equation

Main original results of Part I are the existence of a homoclinic solution of problem (1), (2) published in [2], the existence of a Kneser solution to problem (1), (2) published in [3], and asymptotic formulas for Kneser solutions published in [1].

We investigate the equation (1):

$$\big(p(t)u'(t)\big)' + q(t)f(u(t)) = 0$$

with the initial conditions (2):

$$u(0) = u_0, \quad u'(0) = 0, \quad u_0 \in [L_0, L],$$

where

$$L_0 < 0 < L, \quad f(L_0) = f(0) = f(L) = 0, \tag{9}$$
$$f \in C(\mathbb{R}), \quad xf(x) > 0 \text{ for } x \in (L_0, L) \setminus \{0\}, \tag{10}$$
$$p \in C[0,\infty), \quad p(0) = 0, \quad p(t) > 0 \text{ for } t \in (0,\infty), \tag{11}$$
$$q \in C[0,\infty), \quad q(t) > 0 \text{ for } t \in (0,\infty). \tag{12}$$

We would like to stress that $p(0) = 0$ and the case $\int_0^1 \frac{\mathrm{d}t}{p(t)} = \infty$ is also considered here. Hence, the differential operator in equation (1) may have a singularity at $t = 0$. This is a fundamental difference from the papers investigating problem (1), (2) in regular settings. Moreover, we would like to point out that for our results, the nonlinearity f does not have globally monotonous behaviour and that the condition $xf(x) > 0$ need not be fulfilled for all $x \neq 0$.

At the beginning, we specify smoothness of solutions that we are interested in. Further, we define different types of solution according to their asymptotic behaviour.

Definition 8.1 *Let $c \in (0,\infty)$. A function $u \in C^1[0,c]$ with $pu' \in C^1[0,c]$ which satisfies equation* (1) *for every $t \in [0,c]$ and which satisfies the initial conditions* (2) *is called a solution of problem* (1), (2) *on $[0,c]$. If u is a solution of problem* (1), (2) *on $[0,c]$ for every $c > 0$, then u is called a solution of problem* (1), (2).

Definition 8.2 *A solution u of problem* (1), (2) *is said to be oscillatory if $u \not\equiv 0$ in any neighbourhood of ∞ and if u has a sequence of zeros tending to ∞. Otherwise, u is called nonoscillatory.*

Definition 8.3 *Consider a solution of problem* (1), (2) *with $u_0 \in (L_0, L)$ and denote*

$$u_{sup} = \sup\{u(t)\colon t \in [0,\infty)\}.$$

If $u_{sup} = L$, then u is called a homoclinic solution of problem (1)), (2).
If $u_{sup} < L$, then u is called a damped solution of problem (1), (2).

Note that if we extend functions p and q in equation (1) from the half-line onto $\mathbb{R}$ as even functions, then a homoclinic solution of (1), (2) has the same limit L as $t \to -\infty$ and $t \to \infty$. This is a motivation of Definition 8.3.

Definition 8.4 *Let u be a solution of problem* (1), (2) *on $[0,c]$, where $c \in (0,\infty)$. If u satisfies*

$$u(c) = L, \quad u'(c) > 0,$$

then u is called an escape solution of problem (1), (2) *on $[0,c]$.*

Definition 8.5 *A solution u of equation* (1) *on $[a,\infty), a \geq 0$, is called a Kneser solution if there exists $t_0 > a$ such that*

$$u(t)u'(t) < 0 \, for \, t \in [t_0,\infty).$$

Different types of solutions with respect to their asymptotic behaviour are illustrated in Figure 1.

Our first goal is to study properties of solutions determined by Definitions 8.3, 8.4, 8.5 and to find conditions guaranteeing their existence.

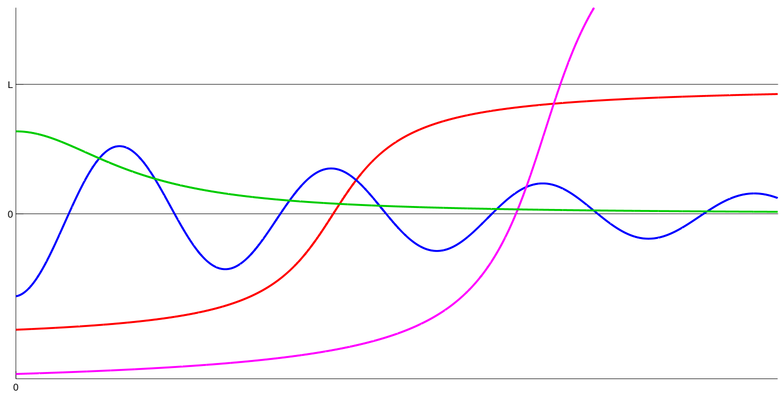

Figure 1: Different types of solutions

In order to prove the existence of damped solution to problem (1), (2) stated in Theorem 8.6 we need besides the basic assumptions (9)–(12) that there exists $\bar{B} \in (L_0, 0)$ such that

$$F(\bar{B}) = F(L), \text{ where } F(x) = \int_0^x f(s)\,\mathrm{d}s, \tag{13}$$

and

$$pq \text{ is nondecreasing on } [0,\infty), \tag{14}$$

$$f \in \mathrm{Lip}_{loc}[L_0, L] \setminus \{0\}, \tag{15}$$

$$\lim_{t\to 0^+} \frac{1}{p(t)} \int_0^t q(s)\,\mathrm{d}s = 0. \tag{16}$$

Theorem 8.6 (Existence of damped solutions of problem (1), (2)) *Assume that the assumptions* (9)–(12)*,* (13)*,* (14)*,* (15) *and* (16) *are fulfilled. Then for each* $u_0 \in (\bar{B}, L)$*, problem* (1)*,* (2) *has a solution u. The solution u is damped and satisfies* $u(t) > \bar{B}$ *for* $t \in (0,\infty)$.

The next theorem guarantees the existence of at least one escape solution to problem (1), (2).

Theorem 8.7 (Existence of an escape solution of problem (1), (2)) *Assume that* (9)–(12)*,* (13) *and* (16)*,*

$$f \in \mathrm{Lip}_{loc}[L_0, L] \tag{17}$$

and moreover

$$(pq)' > 0 \text{ on } (0,\infty),\ \lim_{t\to\infty} \frac{(p(t)q(t))'}{q^2(t)} = 0, \liminf_{t\to\infty} q(t) > 0,\ \lim_{t\to\infty} \frac{p}{q} > 0. \tag{18}$$

Then if either

$$\liminf_{t\to 0^+}\frac{f(x)}{x}>0,\ \liminf_{t\to 0^-}\frac{f(x)}{x}>0, \tag{19}$$

$$\int_1^\infty \frac{\mathrm{d}s}{p(s)}<\infty, \quad \int_1^\infty \left(\int_s^\infty \frac{\mathrm{d}\tau}{p(\tau)}\right)^2 q(s)\,\mathrm{d}s=\infty, \tag{20}$$

or

$$\int_1^\infty \frac{\mathrm{d}s}{p(s)}=\infty, \quad \liminf_{t\to\infty} p(t)>0. \tag{21}$$

Then there exist constant $c\in(0,\infty)$ *and function* u *such that* u *is an escape solution of problem* (1)*,* (2) *on* $[0,c]$.

In Theorem 8.8, we succeeded in generalizing the existence results of homoclinic solutions to problem (1), (2) where $p\neq q$. The existence of such solution has great importance in applications and is derived by means of properties of the sets of initial values of damped and escape solutions which are nonempty and open.

Theorem 8.8 (Existence of a homoclinic solution of problem (1), (2)) *Assume that* (9)–(12)*,* (13) *and* (16)*,* (17)*,* (18) *and either* (19)*,* (20) *or* (21) *hold. Then there exists a homoclinic solution of problem* (1)*,* (2)*.*

Furthermore, the existence of Kneser solutions is investigated and asymptotic properties of such solutions and their first derivatives are derived in the framework of regularly varying functions. In particular, the new existence results about Kneser solutions of singular problem (5), (2) for $p\equiv q$:

$$\begin{gathered}\big(p(t)u'(t)\big)'+p(t)f(u(t))=0\\ u(0)=u_0\in[L_0,L],\ u'(0)=0,\end{gathered}$$

given here in Theorem 8.9 and Theorem 8.10 are generalizations of those published in [3]. For the existence of Kneser solutions the additional condition need to be satisfied:

$$p\in C^1(0,\infty). \tag{22}$$

Here we use the notation

$$P(t)=\int_0^t p(s)\,\mathrm{d}s, \quad F(x)=\int_0^x f(s)\,\mathrm{d}s.$$

Theorem 8.9 *(On the existence of Kneser solutions I.)*
Assume (9)–(11)*,* (13)*,* (14)*,* (17)*,* (22)*,*

$$\lim_{t\to 0^+}\frac{P(t)}{p(t)}=0, \tag{23}$$

and that there exist $c > \frac{1}{2}$ and $A_0 \in (0,L)$ such that the following inequalities

$$\begin{aligned} \frac{p'(t)P(t)}{p^2(t)} &\geq c, \ t \in (0,\infty), \\ \frac{xf(x)}{F(x)} &\geq \frac{2}{2c-1}, \ x \in (0,A_0] \end{aligned} \tag{24}$$

hold. Then for each $u_0 \in (0,A_0]$ there exists a unique solution u of problem (5), (2)*. The solution u is damped and satisfies $u(t) > 0, \ u'(t) < 0, \ t \in (0,\infty)$.*

Theorem 8.10 *(On the existence of Kneser solutions II.)*
Assume (9)–(11), (13), (14), (17), (22) *and* (23) *hold. Let condition* (24) *hold with a constant $c > \frac{1}{2}$ and assume that there exists $B_0 \in (L_0,0)$ such that the inequality*

$$\frac{xf(x)}{F(x)} \geq \frac{2}{2c-1}, \ x \in [B_0,0)$$

is satisfied. Then for each $u_0 \in [B_0,0)$ there exists a unique solution u of problem (5), (2)*. The solution u is damped and satisfies $u(t) < 0, \ u'(t) > 0, \ t \in (0,\infty)$.*

By our knowledge, for $p \neq q$, the existence of Kneser solutions of singular problem (1), (2) remains to be an open question.

In general case $p \neq q$, we study the solvability of equation

$$(p(t)u'(t))' + q(t)f(u(t)) = 0 \quad t \in [a,\infty), a > 0. \tag{25}$$

Since $a > 0$, equation (25) is regular. We investigate (25) together with one of the following conditions:

$$u(a) = u_0 \in (0,L), \ 0 \leq u(t) \leq L \text{ for } t \in [a,\infty), \tag{26}$$

or

$$u(a) = u_0 \in (L_0,0), \ L_0 \leq u(t) \leq 0 \text{ for } t \in [a,\infty). \tag{27}$$

Problems (25), (26) and (25), (27) are studied under the assumptions:

$$L_0 < 0 < L, \ f \in C[L_0,L], \ f(L_0) = f(0) = f(L) = 0, \tag{28}$$

$$p \in C[a,\infty), p > 0 \text{ on } [a,\infty), \tag{29}$$

$$q \in C[a,\infty), \ q > 0 \text{ on } (a,\infty). \tag{30}$$

In the next theorem, the existence of a nonoscillatory solution of problem (25), (26) is proved. This solution is s either a Kneser solution or a monotonically increasing solution whose limit is L for t tending to infinity.

Theorem 8.11 *Let us assume that $a > 0$, conditions* (28)–(30) *hold, and*

$$f \in \mathrm{Lip}_{loc}(0,L], \quad f(x) > 0 \textit{ for } x \in (0,L).$$

Then problem (25), (26) *has a solution u, such that* $0 < u(t) < L \quad$ *for* $t \in [a,\infty)$. *If in addition*

$$\lim_{t\to\infty} \frac{1}{p(t)} \int_a^t q(s)\,\mathrm{d}s = \infty, \tag{31}$$

with

$$\liminf_{t\to\infty} p(t) > 0, \tag{32}$$

then either

$$u'(t) > 0 \textit{ for } t \geq a \textit{ and } \lim_{t\to\infty} u(t) = L,$$

or

u is a Kneser solution.

The dual theorem formulated below yields the existence of a nonoscillatory solution of problem (25), (27) which is either a Kneser solution or a monotonically decreasing solution whose limit is L_0 for t tending to infinity.

Theorem 8.12 *Let* (28)–(30) *hold and $a > 0$. Moreover, we assume that*

$$f \in \mathrm{Lip}_{loc}[L_0,0), \quad f(x) < 0 \textit{ for } x \in (L_0,0).$$

Then problem (25), (27) *has a solution u, such that*

$$L_0 < u(t) < 0 \quad \textit{for } t \in [a,\infty).$$

If in addition (31) *and* (32) *hold, then either*

$$u'(t) < 0 \textit{ for } t \geq a \textit{ and } \lim_{t\to\infty} u(t) = L_0,$$

or

u is a Kneser solution.

In both, the regular and the singular case, the new asymptotic formulas in the form of upper bounds for damped Kneser solutions to the ODE (1) and their first derivatives are derived in Theorem 8.14 and in Theorem 8.15, respectively. The results hold for function p and q which are regularly varying at infinity.

Definition 8.13 *A function g, which is positive and measurable on $[\tau_0,\infty)$, $\tau_0 > 0$, is called regularly varying of index $\alpha \in \mathbb{R}$ if for each $\lambda > 0$*

$$\lim_{t\to\infty} \frac{g(\lambda t)}{g(t)} = \lambda^\alpha.$$

The set of all regularly varying functions of index α is denoted by $RV(\alpha)$.

Theorem 8.14 *Assume that* (9), (10) *holds and* $a \geq 0$. *Moreover, let us assume that* $p \in RV(\alpha) \cap C[a,\infty)$, $\alpha \geq 1$, $q \in RV(\beta) \cap C[a,\infty)$, $\beta > 0$, $\beta - \alpha > -1$, *and*

$$\exists r > 1 : \liminf_{x \to 0} \frac{|f(x)|}{|x|^r} > 0,\ \limsup_{x \to 0} \frac{|f(x)|}{|x|^r} < \infty.$$

Let u *be a Kneser solution of problem* (25), (26) *or* (25), (27). *Then, for any* $\varepsilon > 0$

$$\lim_{t \to \infty} t^{\frac{\beta-\alpha+2}{r-1}} - \varepsilon |u(t)| = 0.$$

Theorem 8.15 *Let all assumptions of Theorem 8.14 be satisfied. Then, for any* $\varepsilon > 0$ *the following statements hold.*

(i) If $\beta > r\alpha - r - 1$, *then*

$$\lim_{t \to \infty} t^{\alpha - \varepsilon} |u'(t)| = 0.$$

(ii) If $\beta \leq r\alpha - r - 1$, *then*

$$\lim_{t \to \infty} t^{\frac{\beta-\alpha+r+1}{r-1}} - \varepsilon |u'(t)| = 0.$$

According to the terminology concerning the Emden-Fowler equation

$$\left(p(t)|u'(t)|^{\alpha} \operatorname{sgn}(u'(t))\right)' + q(t)|u(t)|^{\gamma} \operatorname{sgn}(u(t)) = 0, \quad \alpha, \gamma \geq 0,$$

equation (25) can be treated as a super-linear equation since here $\alpha = 1$ and $\gamma = r > 1 = \alpha$, holds for $f(x)$ with x in a neighbourhood of the origin. The super-half-linear Emden-Fowler equation was studied in [22, 53] where a different sign condition was posed on the nonlinear term when compared to results presented in the thesis. Therefore, the results in [22, 53] cannot be used for our problem.

Linear systems with a singularity of the first kind

In Part II original contributions to BVPs (3) are described. The main analytical result is the existence and uniqueness of a continuous or even smooth solution of problem (3) published in [4] for the case with a constant coefficient matrix, and in [6] for the case with a variable coefficient matrix. The new numerical result is the convergence of a numerical solution obtained by a collocation method. The numerical analysis is published in [4] for IVPs with a constant coefficient matrix and it is completed in [7] for the general BVPs with a variable coefficient matrix.

We are interested in analysis and numerical treatment of linear systems of first order differential equations with a singularity of the first kind and with a generally nonintegrable inhomogeneity subject to general linear two-point boundary conditions:

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t},\ t \in (0,1], \quad B_0 y(0) + B_1 y(1) = \beta,$$

where, $f : [0,1] \to \mathbb{R}^n$ and $M : [0,1] \to \mathbb{R}^{n\times n}$ have continuous components. Here $f \in C[0,1]$ but $f(t)/t$ may not be integrable on $[0,1]$. Moreover, $B_0, B_1 \in \mathbb{R}^{m\times n}$ are constant matrices and $\beta \in \mathbb{R}^m$. Note that in general $m \le n$. We focus our attention on the existence and uniqueness of a solution $y \in C[0,1]$. This smoothness requirement results in general, in $n-m$ additional initial conditions the solution y has to satisfy. We also specify conditions for f and M which are sufficient for $y \in C^r[0,1]$, $r \in \mathbb{N}$.

While for BVP (7) with smooth inhomogeneity and its applications, comprehensive literature is available, this is not the case for problem (3). The ODE system (6) was investigated in [73], where only particular solutions without boundary conditions are considered, whereas in Part II a general structure of linear two–point boundary conditions is of interest.

Before discussing the most general BVP (3), we first consider simpler problems consisting of the ODE system (6)

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t},$$

subject to initial/terminal conditions. This means that we deal with the initial value problem (IVP),

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t}, \quad B_0 y(0) = \beta, \tag{33}$$

where $B_0 \in \mathbb{R}^{m\times n}$, $\beta \in \mathbb{R}^m$, and $m \le n$, or with the terminal value problem (TVP),

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t}, \quad B_1 y(1) = \beta, \tag{34}$$

where $B_1 \in \mathbb{R}^{n\times n}$, $\beta \in \mathbb{R}^n$, respectively.

Particular attention is paid to the structure of the most general two–point boundary conditions which are necessary and sufficient for the existence of a unique continuous solution on the closed interval $[0,1]$. It turns out that the form of such conditions depends on the spectral properties of the coefficient matrix $M(0)$. Therefore, we distinguish between three key cases, where all eigenvalues of $M(0)$ have negative real parts, positive real parts, or they are zero, and we discuss three cases separately in Theorem 8.16, Theorem 8.17 and Theorem 8.18, respectively.

If all eigenvalues of $M(0)$ have negative real parts, it is necessary to prescribe initial conditions of a certain structure to guarantee that the solution is continuous on $[0,1]$. Moreover, this continuous solution of the associated IVP (33) is shown to be unique.

Theorem 8.16 *Let us assume that all eigenvalues of $M(0)$ have negative real parts and $M \in C[0,1]$. Then for any $f \in C[0,1]$ system* (6) *has a unique solution $y \in C[0,1]$. This solution satisfies the initial condition $M(0)y(0) = -f(0)$ which is necessary and sufficient for y to be continuous on $[0,1]$. Moreover, if $f \in C^r[0,1]$and*

$$M \in C^r[0,1],\ M^{(r)}(0) = 0,\ r \geq 1, \tag{35}$$

then $y \in C^r[0,1]$.

If all eigenvalues of $M(0)$ are assumed to have positive real part, it turns out that there exists a unique continuous solution of problem (34). Its smoothness depends not only on the smoothness of f and M but also on the size of real parts of the eigenvalues of $M(0)$.

Theorem 8.17 *Let us assume that all eigenvalues of $M(0)$ have positive real parts. Moreover, let $f \in C[0,1]$, $M \in C[0,1]$, the matrix $B_1 \in \mathbb{R}^{n\times n}$ be nonsingular, and $\beta \in \mathbb{R}^n$. Then there exists a unique solution $y \in C[0,1]$ of TVP* (34)*. Moreover, if $f \in C^r[0,1]$, $r \in \mathbb{N}$, M satisfies condition* (35)*, and the smallest positive real part of the eigenvalues of $M(0)$ satisfies $\sigma_+ > r$, then $y \in C^r[0,1]$.*

In the case that all eigenvalues of $M(0)$ are equal zero, we have to assume some special structure in f close to the singularity. In particular, let us assume that there exist a constant $\alpha > 0$ and a function $h \in C[0,\delta]$, $\delta > 0$ such that

$$f(t) = O(t^\alpha h(t)) \text{ for } t \to 0. \tag{36}$$

and denote

$$\Omega = \{f \in C[0,1] \text{ such that } f \text{ satisfies (36)}\},$$

Moreover, we assume

$$M(t) = M(0) + t^\gamma D(t),\ \gamma > 0, \quad D \in C[0,1],\ t \in [0,1]. \tag{37}$$

Theorem 8.18 *Let all eigenvalues of the matrix $M(0)$ be zero. Let M satisfy condition* (37) *and $m := \dim X_0^{(e)}$. Assume that $f \in \Omega$, $B_0 \in \mathbb{R}^{m\times n}$ is such that the matrix $B_0\tilde{R} \in \mathbb{R}^{m\times m}$ is nonsingular, and $\beta \in \mathbb{R}^m$. Then there exists a unique solution $y \in C[0,1]$ of IVP* (33)*. This solution satisfies the initial condition $M(0)y(0) = 0$, which is necessary and sufficient for $y \in C[0,1]$. Moreover, if $\alpha \geq r+1$, $\gamma \geq r+1$, $r \geq 1$, $f, D \in C^r[0,1]$, and $h \in C^r[0,\delta]$, then $y \in C^{r+1}[0,1]$.*

Here, the matrix $\tilde{R}$ consists of the linearly independent columns of the projection R onto the m-dimensional space $X_0^{(e)}$ spanned by eigenvectors associated with zero eigenvalues of $M(0)$.

In the most general case, the spectrum of $M(0)$ is arbitrary, contains zero eigenvalues, and both eigenvalues with positive and negative real parts. In order to state the main analytic results for BVPs in Theorem 8.19, the following projections are needed. Let S, R, H and N denote the projection onto the subspace spanned by the eigenvectors associated with eigenvalues of $M(0)$ with positive real parts, the subspace spanned by eigenvectors associated with zero eigenvalues of $M(0)$, the subspace spanned by principal eigenvectors associated with zero eigenvalues of $M(0)$, and the subspace spanned by eigenvectors associated with eigenvalues of $M(0)$ with negative real parts, respectively. Moreover, we define $Z := R + H$, $P := R + S$. We also use $\tilde{P}$, $\tilde{R}$ to denote the matrices consisting of the maximal set of linearly independent columns of the respective projections.

Theorem 8.19 *Consider BVP* (3)*, where the inhomogeneity f is given in such a way such that $f \in C[0,1]$ and Zf satisfies* (36)*. Let the coefficient matrix $M \in C[0,1]$ be such that its projections ZM satisfy condition* (37)*. Moreover, let $B_0, B_1 \in \mathbb{R}^{m \times n}$, $\beta \in \mathbb{R}^m$, $m = \operatorname{rank} P$, and the $m \times m$ matrix $B_0\tilde{R} + B_1\tilde{P}$ be nonsingular. Then, BVP* (3) *has a unique continuous solution $y \in C[0,1]$. This solution satisfies two sets of initial conditions,*

$$Hy(0) = 0, \quad M(0)Ny(0) = -Nf(0)$$

which are necessary and sufficient for $y \in C[0,1]$.

The smoothness result $y \in C^r[0,1]$ follows by applying the smoothness results derived separately for components of the solutions associated with eigenvalues of $M(0)$ with negative real parts, positive real parts and zero eigenvalues, respectively.

In the numerical part, we analysed the collocation method applied to approximate the solution of the analytical problem (3). On the equidistant mesh $\Delta = \{0 - t_0 < t_1 < \ldots < t_I = 1\}$ with a step h the analytical solution is approximated by a continuous piecewise polynomial function p of degree less or equal k which satisfies the problem at k collocation points $t_{jl} = t_j + u_l h$, $l = 1, \ldots k$ where $0 < u_1 < \ldots < u_k \leq 1$. The decision to use collocation is motivated by its advantageous convergence properties for (7). The collocation do not suffer from order reduction in the presence of singularities. The convergence behaviour was investigated separately for general IVPs, TVPs and BVPs in Theorem 8.20, in Theorem 8.22 and in Theorem 8.23, respectively. It turned out that the collocation retains its classical stage order k uniformly in t for a scheme with k collocation points, provided that the analytical solutions are appropriately smooth.

In context of an IVPs with appropriately smooth solution, polynomial collocation method executed with k arbitrary collocation points retains its classical stage order $O(h^k)$ uniformly in t. This convergence result is derived in Theorem 8.20. For the convergence analysis, we rewrite the IVP to obtain a more convenient form,

$$y'(t) - \frac{M(t)}{t}y(t) = \frac{f(t)}{t}, \quad y(0) = \delta, \tag{38}$$

where

$$B_0\delta = \beta, \quad H\delta = 0, \quad M(0)N\delta = -Nf(0). \tag{39}$$

Theorem 8.20 *Let us assume that $y \in C^{k+1}[0,1]$ is the unique solution of problem* (38), (39) *and $M \in C^1[0,1], f \in C[0,1]$. Let the function $p \in \mathscr{P}_{k,h}$ be the unique solution of the collocation scheme,*

$$p'(t_{jl}) - \frac{M(t_{jl})}{t_{jl}}p(t_{jl}) = \frac{f(t_{jl})}{t_{jl}}, \quad l = 1,\ldots k,\ j = 0,\ldots,I-1, \quad p(0) = \delta.$$

Then

$$\|p - y\| \leq const.h^k.$$

For certain choice of collocation points satisfying

$$\int_0^1 w(s)\,\mathrm{d}s = 0,\ \ w(t) = (t-u_1)(t-u_2)\cdots(t-u_k) \tag{40}$$

the so-called small superconvergence order $O(h^{k+1})$ was proved to hold uniformly in t; only in the case where $M(0)$ has multiple zero eigenvalues the order of convergence is affected by logarithm terms:

Theorem 8.21 *Let us assume that the solution y of* (38), (39) *satisfies $y \in C^{k+2}[0,1]$. If* (40) *holds, then the estimate for the global error given in Theorem 8.20 can be replaced by*

$$\|p - y\| \leq const.h^{k+1}|\ln(h)|^{(d-1)_+},$$

d is the dimension of the largest Jordan box of $M(0)$ associated with the eigenvalue $\lambda = 0$ and

$$(x)_+ = \begin{cases} x, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

Similarly, for TVP the convergence of collocation with stage order $O(h^k)$ is proved. We consider the TVP (34) in the form

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t},\ y(1) = \delta, \tag{41}$$

where $B_1\delta = \beta$.

Theorem 8.22 *Let us assume that $M \in C^1[0,1], f \in C[0,1]$ and $y \in C^{k+1}[0,1]$ is the unique solution of* (41). *Let the function $p \in \mathscr{P}_{k,h}$ satisfy the collocation scheme*

$$p'(t_{jl}) = \frac{M(t_{jl})}{t_{jl}} p(t_{jl}) + \frac{f(t_{jl})}{t_{jl}}, \quad p(1) = \delta,\ j = 0, \dots I-1,\ l = 1, \dots k.$$

Then, provided that h is sufficiently small,

$$\|p - y\| \le const. h^k.$$

The main result for BVPs from the numerical point of view is the existence and convergence of collocation solution of the corresponding numerical scheme to (3). According to the detailed analysis BVP (3) is well-posed if and only if the boundary conditions given in (3) can be equivalently written in a separated fashion,

$$Hy(0) = 0,\ M(0)Ny(0) = -Nf(0),\ Ry(0) = R\eta,\ Sy(1) = S\eta,\ \eta \in \mathbb{R}^n. \tag{42}$$

Therefore, we can restrict our attention to the problem (6), (42). The existence and uniqueness of solution p to the associated collocation scheme,

$$\begin{gathered} p'(t_{jl}) = \frac{M(t_{jl})}{t_{jl}} p(t_{jl}) + \frac{f(t_{jl})}{t_{jl}},\ j = 0, \dots I-1,\ l = 1, \dots k, \\ Hp(0) = 0,\ M(0)Np(0) = -Nf(0),\ Rp(0) = R\eta,\ Sp(1) = S\eta \end{gathered} \tag{43}$$

is proved for $f \in C[0,1], M \in C^1[0,1]$ provided that h is sufficiently small.

Results obtained for IVPs and TVPs cannot be easily modified to BVPs. In this case, a new representation of the global error is needed in the proof of convergence.

Theorem 8.23 *Let us assume that $y \in C^{k+2}[0,1]$ is the unique solution of the BVP* (6), (42), *$f \in C^{k+1}[0,1]$, $M \in C^{k+2}[0,1]$, and $\sigma_+ > k+2$. Let $p \in \mathscr{P}_{k,h}$ be the unique solution of the collocation scheme* (43). *Then,*

$$\|p - y\| \le const. h^k.$$

9 Summary of results

In the thesis new contributions to the theory of singular ordinary differential equations were presented.

In Part I, we studied the solution structure to singular nonlinear second order ODE

$$\big(p(t)u'(t)\big)' + q(t)f((u(t)) = 0, \ \ t \in [0,\infty),$$

subject to the initial conditions

$$u(0) = u_0 \in (L_0, L), \quad u'(0) = 0.$$

New existence results for escape and homoclinic solutions were obtained in Theorem 8.7 and Theorem 8.8 under the assumptions which guarantee that each damped solution of the problem is oscillatory. In addition, the existence of damped solutions and their asymptotic behaviour were derived in a more general settings that have been done before, see Theorem 8.6. Among damped solutions, the Kneser solutions were of special interest. The existence of Kneser solutions to singular equation (1) was proved in Theorems 8.9 and 8.10 provided $p \equiv q$ while for $p \neq q$ this is still an open question.

In general case $p \neq q$, a regular equation (1) on $[a,\infty), a > 0$, was studied. More precisely, two classes of initial value problems were discussed, (25), (26) and (25), (27). It turned out that there exists a nonoscillatory solution of problem (25), (26) (and problem (25), (27)), which is either a Kneser solution or a monotonically increasing solution whose limit is L for t tending to infinity, Theorem 8.11 (and a monotonically decreasing solution whose limit is L_0 for t tending to infinity, Theorem 8.12).

In both, the regular and the singular case, we have provided new asymptotic formulas in the form of upper bounds for damped Kneser solutions and their first derivatives to the ODE (1) with regularly varying coefficients, Theorem 8.14, Theorem 8.15, respectively.

In Part II, we investigated the analytical properties of the singular BVP with a variable coefficient matrix

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t)}{t}, \quad B_0 y(0) + B_1 y(1) = \beta$$

and we achieved to generalize results obtained for problems with a smooth inhomogeneity. The structure of the correctly posed boundary conditions which guarantee the existence of a unique solution $y \in C[0,1]$ turned out to depend on the spectral properties of the matrix $M(0)$. Therefore, we had to carry out in full details the following three case studies, the case of only negative real parts of the eigenvalues of $M(0)$ in Theorem 8.16, positive real parts of the eigenvalues of $M(0)$ in Theorem 8.17, and zero eigenvalues of $M(0)$ in Theorem 8.18. The main technical tool used in the analysis is the Banach Fixed Point Theorem which turned out to be very useful in mastering the difficulties caused by the singularity at $t = 0$. These case investigations were then used to generalize the results to the case boundary value problems in Theorem 8.19.

Moreover, we analysed the collocation method applied to approximate the solution of the above analytical problem and we derived new convergence properties. The convergence behaviour was investigated separately for general IVPs, TVPs and BVPs, in Theorem 8.20, Theorem 8.22 and Theorem 8.23, respectively. We proved that the collocation retains its classical stage order k uniformly in t for a scheme with k collocation points, provided that the analytical solutions are appropriately smooth. Moreover, for Gaussian points the so-called small superconvergence order $k+1$ was shown to hold in context of an IVP in Theorem 8.21, whereas, the superconvergence order in the mesh points, $2k$ for Gaussian points, cannot be expected to hold, in general. The theoretical results are supported by the numerical experiments.

The aim of further investigations is to prove the existence of homoclinic solutions also in the case when damped nonoscillatory solution of (1), (2) may appear. This case was excluded in Theorem 8.8. Next goal is to prove the existence of Kneser solutions to equation (1) with a time singularity at $a = 0$ and $p \neq q$, and to more precisely describe their asymptotic behaviour in the form of asymptotic equivalence as t tends to infinity. Furthermore, we intend to generalize problem (1), (2) to the problem with ϕ-Laplacian

$$\big(p(t)\phi(u'(t))\big)' + q(t)f(u(t)) = 0, \quad u(0) = u_0,\ u'(0) = 0,$$

where ϕ is an increasing homeomorphism with $\phi(\mathbb{R}) = \mathbb{R}$.

The aim of further study concerning the system is to investigate the nonlinear problem

$$y'(t) = \frac{M(t)}{t}y(t) + \frac{f(t,y(t))}{t},\ t \in (0,1], \quad B_0 y(0) + B_1 y(1) = \beta.$$

and to prove the existence of a solution of the nonlinear BVP which we plan to publish in an upcoming paper.

List of publications

[1] J. BURKOTOVÁ, M. HUBNER, I. RACHŮNKOVÁ, E.B. WEINMÜLLER, Asymptotic properties of Kneser solutions to nonlinear second order ODE with regularly varying coefficients, *submitted*.

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[6] J. BURKOTOVÁ, I. RACHŮNKOVÁ, E.B. WEINMÜLLER, On singular BVPs with unsmooth data. Part 1: Analysis of the linear case with variable coefficient matrix, *submitted*.

[7] J. BURKOTOVÁ, I. RACHŮNKOVÁ, E.B. WEINMÜLLER, On singular BVPs with unsmooth data. Part 2: Convergence of the collocation schemes, *submitted*.

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