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FAKULTA STROJNÍHO INŽENÝRSTVÍ

INSTITUTE OF MATHEMATICS

ÚSTAV MATEMATIKY

MATHEMATICAL MODELLING WITH DIFFERENTIAL EQUATIONS

MATEMATICKÉ MODELOVÁNÍ POMOCÍ DIFERENCIÁLNÍCH ROVNIC

MASTER'S THESIS

DIPLOMOVÁ PRÁCE

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Specification Master's Thesis

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Pursuant to Act no. 111/1998 concerning universities and the BUT study and examination rules, you have been assigned the following topic by the institute director Master's Thesis:

Mathematical modelling with differential equations

Concise characteristic of the task:

Mathematical modelling is an important part of many branches in natural sciences. Common problem is application of theoretical results (models) in real conditions. We are interested in continuous models, which one can describe by system of differential equations or delay differential equations.

Goals Master's Thesis:

Get the experience in general modelling of continuous mathematical problems. To extend the knowledge of the differential equations theory. To study the topic of delay differential equations. The application of the adopted theoretical background onto particular real models.

Recommended bibliography:

ERNEUX, Thomas. Applied Delay Differential Equations. Springer-Verlag. 2009.

KALAS, Josef a Miloš RÁB. Obyčejné diferenciální rovnice. Brno: Masarykova univerzita. 1995.

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MURRAY, J. Mathematical biology. Springer. 3rd ed. Springer-Verlag. 2002.

Deadline for submission Master's Thesis is given by the Schedule of the Academic year 2016/17

In Brno,

L. S.

prof. RNDr. Josef Šlapal, CSc.	doc. Ing. Jaroslav Katolický, Ph.D.
Director of the Institute	FME dean

Abstrakt
Diplomová práce je zaměřena na problematiku nelineárních diferenciálních rovnic. Obsahuje věty důležité k určení chování nelineárního systému pouze za pomoci zlinearizovaného systému, což je následně ukázáno na rovnici matematického kyvadla. Dále se práce zabývá problematikou diferenciálních rovnic se zpožděním. Pomocí těchto rovnic je možné přesněji popsat některé reálné systémy, především systémy, ve kterých se vyskytují časové prodlevy. Zpoždění ale komplikuje řešitelnost těchto rovnic, což je ukázáno na zjednodušené rovnici portálového jeřábu. Následně je zkoumána oscilace lineární rovnice s nekonstantním zpožděním a nalezení podmínek pro koeficienty rovnice zaručující oscilačnost každého řešení.

Summary
The master's thesis is focused on the nonlinear differential equations. It contains theorems important to determine the behaviour of the nonlinear system only by study of the linearized system, which is subsequently shown on the equation of the mathematical pendulum. Furthermore, the thesis deals with differential equations with delay. The delay complicates finding the solution, which is shown on the simplified equation of a gantry crane. Subsequently is investigated the oscillation of the linear equation with non-constant delay. Determining the conditions for the coefficients in the equation, such that every solution is oscillatory.

Klíčová slova
Nelineární diferenciální rovnice, kyvadlo, diferenciální rovnice se zpožděním, portálový jeřáb, oscilace lineární rovnice s nekonstantním zpožděním.

Keywords
Nonlinear differential equations, pendulum, delay differential equations, gantry crane, oscillation of the linear equation with non-constant delay.

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I hereby certify that this thesis is the result of my own work and I have properly cited all sources used in the thesis.

Bc. Lukáš Béreš

I would like to sincerely thank my supervisor doc. Mgr. Zdeněk Opluštil, Ph.D. for his helpful advices, leading in the topic, great patience and other discussions.

Bc. Lukáš Béreš

Contents

1 Introduction

Differential equations have numerous applications in engineering and science disciplines. There exist methods to solve linear differential equations, and also special cases of nonlinear are solvable explicitly. But what to do when the explicit solution does not exist or can be found only numerically, using software? Theory of dynamical systems tells us, that in some cases it is possible to use just the linearized system near the equilibrium, even though the equations are not linear. The first chapter of the thesis gives rough view on these special cases. The survey related to the dynamical systems can be found in books [14].

The thesis shows the application on the equation of pendulum, using not only the linearization (The Hartman - Grobman Theorem, see [9], [11]), but also approximation with higher orders around equilibrium.

The second part of the thesis deals with differential equations with delay, they play an important role in various applied sciences such a control theory, population dynamics, biology, engineering, etc. Mathematical models with delayed differential equations turned out to be useful especially when the system depends not only on the position of the system in the current time, but also in the past. The delayed part may frequently influence the properties of solutions (stability, etc.).

Differential equations with delay were already investigated by Euler in the 18th century, but the systematic study starts at the fifties of 20th century. Theory related to delay differential equation can be found in [1], [4], [5], [7], [8], [12]. The exact solution can be found only in some special cases. There are few methods of solving these special cases, the most known are Method of Characteristics and Method of Steps.

The equation of gantry crane is used as an example of use of the delay differential equation. Specific type of controller is used to control the swinging of the cable using the past. Of course there exist a lot of controllers, in the thesis is used the Pyragas' controller see, [15]) $k(\phi(t-\tau)-\phi(t)),\ \tau \in \mathbb{R}^+$, where $\phi(t-\tau)$ is the known function of the past (on the interval $(-\tau, 0)$).

The last chapter consists of studying the oscillation of the linear differential equation with non-constant delay. There are lemmas and theorems which give conditions on the delay and also on the functions in the equation such that every solution is oscillatory.

2 Nonlinear systems

Any autonomous linear system

$$\dot{\mathbf{x}} = A\mathbf{x}, \tag{2.1}$$

where $\mathbf{x} = (x_1, x_2, ..., x_n)^T$, A is an $n \times n$ constant matrix of real numbers, has a unique solution through each point $\mathbf{x}_0$ in the phase space $\mathbb{R}^n$. The solution is given by

$$\mathbf{x}(t) = \mathbf{x}_0 e^{At}$$

and it is defined for all $t \in \mathbb{R}$. This chapter will deal with the autonomous systems of differential equations

$$\dot{\mathbf{x}} = f(\mathbf{x}), \tag{2.2}$$

where $f : E \longrightarrow \mathbb{R}^n$ is a vector function and E is an open subset of $\mathbb{R}^n$. Any nonautonomous system $\dot{\mathbf{x}} = f(\mathbf{x}, t)$, with $\mathbf{x} = (x_1, x_2, ..., x_n)^T$ can be rewritten as an autonomous system (2.2).

Under certain conditions on the function f, the nonlinear system (2.2) has a unique solution through each point $\mathbf{x}_0 \in E$ defined on a maximal interval of existence $(\alpha, \beta) \subseteq \mathbb{R}$.

2.1 Nonlinear systems

In general it is not possible to solve the nonlinear system (2.2), however, we show qualitative information about the local behaviour of the solution. In particular, the Hartman-Grobman theorem which shows that with certain conditions on the function f, the local behaviour of the nonlinear system (2.2) near an equilibrium point $\mathbf{x}_0$ where $f(\mathbf{x}_0) = 0$ is topologically equivalent to the behaviour of the linear system (2.1), with $A = Df(\mathbf{x}_0)$, the derivative of f at $\mathbf{x}_0$. The definitions and theorems in this chapter, and also more information with examples can be found in [14].

Definition 2.1. The function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is *differentiable* at $\mathbf{x}_0 \in \mathbb{R}^n$ if exist a linear transformation $Df(\mathbf{x}_0)$ (the derivative of f at $\mathbf{x}_0$) that satisfies

$$\lim_{\|\mathbf{h}\| \longrightarrow 0} \frac{\|f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0) - Df(\mathbf{x}_0)\mathbf{h}\|}{\|\mathbf{h}\|} = 0,$$

where $\|\mathbf{h}\|$ is the norm of a vector $\mathbf{h} = (h_1, ..., h_n)$, defined by $\|\mathbf{h}\| = \sqrt{h_1^2 + h_2^2 + ... + h_n^2}$.

Theorem 2.2 ([14]). *If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is differentiable at $\mathbf{x}_0$, then the partial derivatives $\frac{\partial f_i}{\partial x_j}, i, j = 1, \ldots, n$, all exist at $\mathbf{x}_0$ and for all $\mathbf{x} \in \mathbb{R}^n$,*

$$Df(\mathbf{x}_0)\mathbf{x} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(\mathbf{x}_0) x_j.$$

If f is differentiable, then the derivative Df is given by the $n \times n$ Jacobian matrix

$$Df = \left[\frac{\partial f_i}{\partial x_j}\right].$$

Theorem 2.3 ([14]). *Suppose that $f : E \longrightarrow \mathbb{R}^n$, where E is an open subset of $\mathbb{R}^n$, and f is differentiable on E. Then $f \in C^1(E)$ if the derivative Df is continuous on E. $f \in C^1(E)$ if and only if all the partial derivatives $\frac{\partial f_i}{\partial x_j}, i, j = 1, ..., n$, exist and are continuous on E.*

The $\mathbf{x}(t)$ is a solution of (2.2) on an interval I, if it is differentiable, and $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t))$, $\forall t \in I$. Given $\mathbf{x}_0 \in E$, $\mathbf{x}(t)$ is a solution of the initial value problem, if it is a solution of (2.2) and $\mathbf{x}(t_0) = \mathbf{x}_0$.

The function $f : E \longrightarrow \mathbb{R}^n$ is said to be *locally Lipschitz* on E if for each point $\mathbf{x}_0 \in E$ there is an ϵ -neighborhood of $\mathbf{x}_0$ (an open ball $B_\epsilon(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^n, |\mathbf{x} - \mathbf{x}_0| < \epsilon\}$), $B_\epsilon(\mathbf{x}_0) \subset E$ and positive constant K such that $\forall \mathbf{x}, \mathbf{y} \in B_\epsilon(x_0)$

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq K|\mathbf{x} - \mathbf{y}|. \tag{2.3}$$

Function f is said to satisfy a Lipschitz condition on E if (2.3) holds for all $\mathbf{x}, \mathbf{y} \in E$. It can be proved that if $f \in C^1(E)$, then f is locally Lipschitz on E.

Theorem 2.4 (The Fundamental Existence - Uniqueness Theorem, [14], p. 74). *Let E be an open subset of $\mathbb{R}^n$ containing $\mathbf{x}_0$ and assume that $f \in C^1(E)$. Then there exist an $a > 0$ such that the initial value problem*

$$\begin{aligned} \dot{\mathbf{x}} &= f(\mathbf{x}) \\ \mathbf{x}(0) &= \mathbf{x}_0 \end{aligned}$$

has a unique solution $\mathbf{x}(t)$ on the interval $[-a, a]$.

Due to translation it also shows that the initial value problem

$$\begin{aligned} \dot{\mathbf{x}} &= f(\mathbf{x}) \\ \mathbf{x}(t_0) &= \mathbf{x}_0 \end{aligned}$$

has a unique solution on some interval $[t_0 - a, t_0 + a]$.

Theorem 2.5 ([14]). *Let $E \subset \mathbb{R}^n$ be open and assume that $f \in C^1(E)$. Then for each $\mathbf{x}_0 \in E$, there exist a maximal interval J on which the initial value problem has a unique solution x(t). If the initial value problem has a solution $\mathbf{y}(t)$ on an interval I, the $I \subset J$ and $\mathbf{y}(t) = \mathbf{x}(t), \forall t \in I$.*

Furthermore, the maximal interval of existence J is open.

Definition 2.6. A point $\mathbf{x}_0 \in \mathbb{R}^n$ is called an *equilibrium point* (or *critical point*) of

$$\dot{\mathbf{x}} = f(\mathbf{x}) \tag{2.4}$$

if $f(\mathbf{x}_0) = 0$. An equilibrium point is called a *hyperbolic equilibrium point* if none of the eigenvalues of the matrix $A = Df(\mathbf{x}_0)$ have zero real part. The linear system $\dot{\mathbf{x}} = A\mathbf{x}$, with $A = Df(\mathbf{x}_0)$, is called *the linearization* of (2.4).

Definition 2.7. An equilibrium point $\mathbf{x}_0$ of (2.4) is called a *sink* if all the eigenvalues of $Df(\mathbf{x}_0)$ have negative real part. If all the eigenvalues of $Df(\mathbf{x}_0)$ have positive real part, then $\mathbf{x}_0$ is a *source*. It is called *saddle* if it is a hyperbolic equilibrium point and $Df(\mathbf{x}_0)$ has at least one eigenvalue with a positive real part and at least one with a negative real part.

Definition 2.8 (The Flow of a differential equation)**.** Let E be an open subset of $\mathbb{R}^n$ and let $f \in C^1(E)$. For $\mathbf{x}_0 \in E$, let $\phi(t, \mathbf{x}_0)$ be the solution of the initial value problem

$$\begin{aligned} \dot{\mathbf{x}} &= f(\mathbf{x}) \\ \mathbf{x}(0) &= \mathbf{x}_0 \end{aligned} \tag{2.5}$$

defined on its maximal interval of existence $I(\mathbf{x}_0)$. Then for $t \in I(\mathbf{x}_0)$, the set of mappings ϕ_t defined by

$$\phi_t(\mathbf{x}_0) = \phi(t, \mathbf{x}_0)$$

is called the *flow of the differential equation* (2.5) or the flow defined by the differential equation (2.5).

Definition 2.9. Two autonomous systems of differential equations are said to be *topologically equivalent* in a neighborhood of the origin if there is a homeomorphism H mapping an open set U containing the origin onto an open set V containing the origin which maps trajectories o the first system in U onto trajectories of the second system in V and preserves their orientation by time in the sense that if a trajectory is directed from $\mathbf{x}_1$ to $\mathbf{x}_2$ in U, then its image is directed from $H(\mathbf{x}_1)$ to $H(\mathbf{x}_2)$ in V.

Theorem 2.10 (The Hartman - Grobman Theorem, [11])**.** *Let E be an open subset of $\mathbb{R}^n$ containing the origin, $f \in C^1(E)$, and let ϕ_t be the flow of the nonlinear system $\mathbf{y} = f(\mathbf{x})$. Suppose that $f(\mathbf{0}) = \mathbf{0}$ and that the matrix $A = Df(\mathbf{0})$ has no eigenvalue with zero real part. Then there exists a homeomorphism H of an open set U containing the origin onto an open set V containing the origin such that for each $\mathbf{x}_0 \in U$, there is an open interval $I_0 \subset \mathbb{R}$ containing zero such that $\forall \mathbf{x}_0 \in U$ and $t \in I_0$*

$$H \circ \phi_t(\mathbf{x}_0) = e^{At} H(\mathbf{x}_0);$$

i.e., H maps trajectories of $\mathbf{y} = f(\mathbf{x})$ onto trajectories of $\mathbf{y} = A\mathbf{x}$, where $A = Df(\mathbf{0})$, near the origin and preserves the parametrization by time.

2.1.1 Equilibrium points

There are several types of equilibrium points of the nonlinear system

$$\dot{\mathbf{x}} = f(\mathbf{x}), \tag{2.6}$$

where $f : E \longrightarrow \mathbb{R}^2$, E is open subset of $\mathbb{R}^2$. We now give the precise geometrical formulation of them. Assume that $\mathbf{x}_0 \in \mathbb{R}^2$ is an isolated equilibrium point of (2.6) which has been translated to the origin.

Definition 2.11 (Center)**.** The origin $\mathbf{0}$ is a **center** for (2.6) if there exist a $\delta > 0$ such that every solution curve in deleted neighborhood $B_\delta(\mathbf{0}) \setminus \{\mathbf{0}\}$ is a closed curve with $\mathbf{0}$ in its interior.

Definition 2.12 (Center-focus)**.** $\mathbf{0}$ is a **center-focus** if there exist a sequence of closed solution curves Γ_n with Γ_{n+1} in the interior of Γ_n such that $\Gamma_n \longrightarrow \mathbf{0}$ as $n \longrightarrow \infty$ and such that every trajectory between Γ_n and Γ_{n+1} spirals toward Γ_n or Γ_{n+1} as $t \longrightarrow \pm\infty$

For planar systems

$$\begin{aligned}\dot{x} &= P(x,y)\\ \dot{y} &= Q(x,y)\end{aligned} \tag{2.7}$$

it is convenient to rewrite the system in polar coordinates (r,φ), which can reveal the behaviour of the system near origin. Letting $r^2 = x^2 + y^2$ and $\varphi = \tan^{-1}(\frac{y}{x})$, we obtain

$$\begin{aligned}r\dot{r} &= x\dot{x} + y\dot{y}\\ r^2\dot{\varphi} &= x\dot{y} - y\dot{x}\end{aligned}$$

Then the nonlinear system (2.7) may be rewritten as

$$\begin{aligned}\dot{r} &= P(r\cos\varphi, r\sin\varphi)\cos\varphi + Q(r\cos\varphi, r\sin\varphi)\sin\varphi\\ r\dot{\varphi} &= Q(r\cos\varphi, r\sin\varphi)\cos\varphi - P(r\cos\varphi, r\sin\varphi)\sin\varphi\end{aligned}$$

The solution of the planar nonlinear system (2.7) with $r(0) = r_0$ and $\varphi(0) = \varphi_0$ will be denoted $r(t, r_0, \varphi_0)$ and $\varphi(t, r_0, \varphi_0)$.

Definition 2.13 (Stable focus)**.** $\mathbf{0}$ is called a **stable focus** for (2.7) if $\exists\delta > 0$ such that for $0 < r_0 < \delta$ and $\varphi_0 \in \mathbb{R}$, $r(t, r_0, \varphi_0) \longrightarrow 0$ and $|\varphi(t, r_0, \varphi_0)| \longrightarrow \infty$ as $t \longrightarrow \infty$.

Definition 2.14 (Stable node)**.** $\mathbf{0}$ is called a **stable node** if $\exists\delta > 0$ such that for $0 < r_0 < \delta$ and $\varphi_0 \in \mathbb{R}$, $r(t, r_0, \varphi_0) \longrightarrow 0$ as $t \longrightarrow \infty$ and $\lim_{t\longrightarrow\infty} \varphi(t, r_0, \varphi_0)$ exists.

Definition 2.15 (Topological saddle)**.** $\mathbf{0}$ is a **topological saddle** for (2.7) if there exist two trajectories Γ_1 and Γ_2 which approach $\mathbf{0}$ as $t \longrightarrow \infty$, and two trajectories Γ_3 and Γ_4 which approach $\mathbf{0}$ as $t \longrightarrow -\infty$. And if $\exists\delta > 0$ such that all other trajectories which start in the deleted neighborhood of the origin $B_\delta(0) \setminus \{0\}$ leave $B_\delta(0)$ as $t \longrightarrow \pm\infty$.

The special trajectories $\Gamma_1, ..., \Gamma_4$ are called **separatrices**.

It was mentioned earlier that in some cases the behaviour of the planar nonlinear system $\dot{\mathbf{x}} = f(\mathbf{x})$ near equilibrium point can be determined using the linearized system $\dot{\mathbf{x}} = A\mathbf{x}$, where $A = Df(\mathbf{0})$. The following theorems give more precise view on such cases.

Theorem 2.16 (Bendixson, [3])**.** *Let $E \subset \mathbb{R}^2$, E is open containing $\mathbf{0}$, and $f \in C^1(E)$. If the origin is an isolated equilibrium point, then every neighborhood of the origin contains closed solution curve or exists a trajectory approaching $\mathbf{0}$ as $t \longrightarrow \pm\infty$.*

Theorem 2.17 ([14], p. 141)**.** *Let $E \subset \mathbb{R}^2$, E is open containing $\mathbf{0}$, and $f \in C^1(E)$. If $\mathbf{0}$ is a hyperbolic equilibrium point, then the origin is a topological saddle for $\dot{\mathbf{x}} = f(\mathbf{x})$ if and only if the origin is a saddle for $\dot{\mathbf{x}} = Df(0)\mathbf{x}$.*

Theorem 2.18 ([14], p. 143)**.** *Let $E \subset \mathbb{R}^2$, E is open containing $\mathbf{0}$, and $f \in C^2(E)$. Suppose that $\mathbf{0}$ is a hyperbolic equilibrium point. Then $\mathbf{0}$ is a **stable (unstable) node** for the nonlinear system $\dot{\mathbf{x}} = f(\mathbf{x})$ if and only if it is **stable (unstable) node** for the linear system $\dot{\mathbf{x}} = Df(\mathbf{0})\mathbf{x}$.*

*And if $f \in C^1(E)$ the origin is **stable (unstable) focus** for the nonlinear system if and only if it is **stable (unstable) focus** for the linear system.*

If the function f is only $C^1(E)$ and $\mathbf{0}$ is a center for $\dot{\mathbf{x}} = Df(\mathbf{0})\mathbf{x}$, then the origin can be either a center, a center-focus or a focus for $\dot{\mathbf{x}} = f(\mathbf{x})$. But when the function f is analytic a center-focus cannot occur.

2.1.2 Stability

The stability of any hyperbolic equilibrium point $\mathbf{x}_0$ of the nonlinear system

$$\dot{\mathbf{x}} = f(\mathbf{x}), \tag{2.8}$$

where $f : E \longrightarrow \mathbb{R}^n, E \subset \mathbb{R}^n$, E is open containing $\mathbf{x}_0$, is determined by the signs of the real parts of the eigenvalues λ_j of the matrix $Df(\mathbf{x}_0)$. The stability of nonhyperbolic equilibrium points is more difficult to determine.

Definition 2.19. Let ϕ_t denote the flow of the differential equation (2.8) defined $\forall t \in \mathbb{R}$. An equilibrium point $\mathbf{x}_0$ of (2.8) is **stable** if $\forall \epsilon > 0$ $\exists \delta > 0$ such that $\forall \mathbf{x} \in B_\delta(\mathbf{x}_0)$ and $t > 0$ we have

$$\phi_t(\mathbf{x}) \in B_\epsilon(\mathbf{x}_0).$$

$\mathbf{x}_0$ is **unstable** if it is not stable. And it is **asymptotically stable** if it is stable and if $\exists \delta > 0$ such that for all $\mathbf{x} \in B_\delta(\mathbf{x}_0)$ we have

$$\lim_{t \longrightarrow \infty} \phi_t(\mathbf{x}) = \mathbf{x}_0.$$

The following Liapunov method is very useful to determine the stability of the nonhyperbolic equilibrium point.

Definition 2.20. If $f \in C^1(E), V \in C^1(E)$ and ϕ_t is the flow of the differential equation (2.8), then for $\mathbf{x} \in E$ the derivative of the function $V(\mathbf{x})$ along the solution ϕ_t is

$$\dot{V}(\mathbf{x}) = \frac{d}{dt} V(\phi_t(\mathbf{x}))|_{t=0} = DV(\mathbf{x}) f(\mathbf{x}).$$

Theorem 2.21 (Liapunov, [14], p. 131)**.** *Let $E \subset \mathbb{R}^n$, E open containing $\mathbf{x}_0$, suppose $f \in C^1(E)$ and $f(\mathbf{x}_0) = \mathbf{0}$. Suppose further that there exists a Liapunov function V. $V : E \longrightarrow \mathbb{R}$ satisfying $V(\mathbf{x}_0) = 0$ and $V(\mathbf{x}) > 0, \mathbf{x} \neq \mathbf{x}_0$. Then*

*a if $\dot{V}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in E$, then $\mathbf{x}_0$ is **stable**,*

*b if $\dot{V}(\mathbf{x}) < 0$ for all $\mathbf{x} \in E \setminus \{\mathbf{x}_0\}$, then $\mathbf{x}_0$ is **asymptotically stable**,*

*c if $\dot{V}(\mathbf{x}) > 0$ for all $\mathbf{x} \in E \setminus \{\mathbf{x}_0\}$, then $\mathbf{x}_0$ is **unstable**.*

2.2 Pendulum

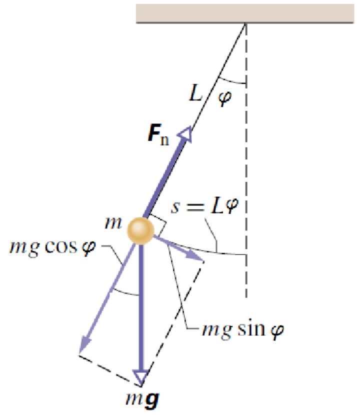

Figure 2.1: Mathematical pendulum with the forces acting on the body [10]

In this section is shown the application of the previous theorems on the simplified differential equation of the pendulum (neglecting friction and air draft)(see [2]).

The equation is obtained by using Newton's law $\vec{F} = m\vec{a}$, where $\vec{F}$ is sum of the forces acting on the body with weight m, and $\vec{a}$ is acceleration.

As can be seen in the Figure 2.1, there are two forces acting on the body:

1. force from the rope $\vec{F}_n$,
2. gravity $\vec{F}_g = m \cdot \vec{g}$.

The position of the pendulum is described with the angle φ, as a deflection from the vertical axis. Then the differential equation for the pendulum of the length l is:

$$\ddot{\varphi} + \frac{g}{l}\sin\varphi = 0 \tag{2.9}$$

Methods of solving this nonlinear equation:

1. The explicit solution of nonlinear of more complicated equation does not exist or it is hard to find analytically. Using numerical methods can help approximately find the solution. The equation (2.9) is such example.

   If we rewrite it as a system of differential equations, where $\varphi_1 = \varphi$ is the position and $\varphi_2 = \dot{\varphi}$ is speed, we obtain:

$$\begin{aligned} \dot{\varphi}_1 &= \varphi_2 \\ \dot{\varphi}_2 &= -\frac{g}{l}\sin\varphi_1 \end{aligned} \tag{2.10}$$

   This system has infinitely many equilibrium points, when $\varphi_2 = 0$ and φ_1 is arbitrary integral multiple of π, $\varphi_1 = k\pi,\ k \in \mathbb{Z}$.

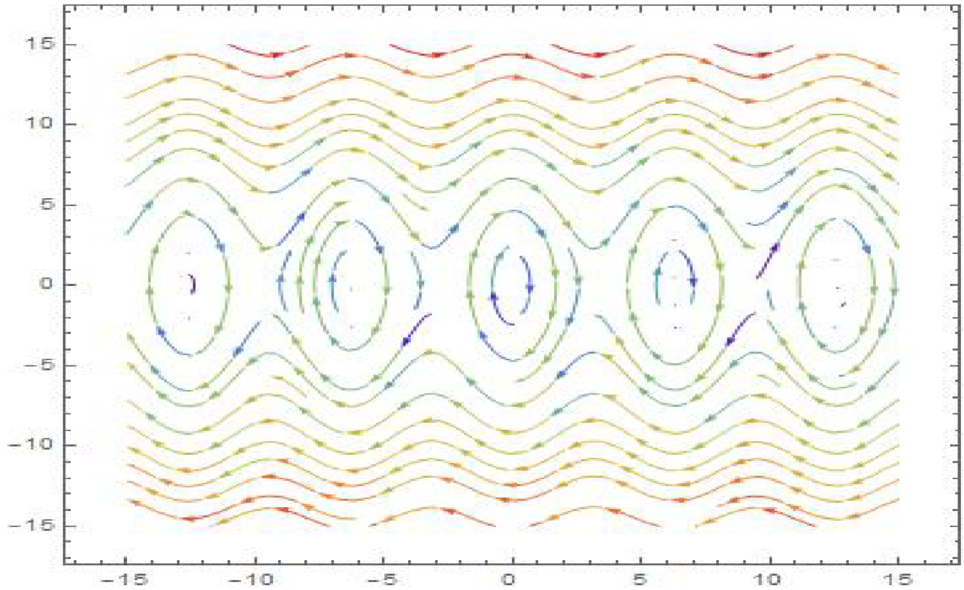

Figure 2.2: The phase portrait for the nonlinear equation of pendulum

The phase portrait for the system of differential equation of the mathematical pendulum (2.10) in the Figure 2.2 shows the behaviour around the equilibrium points. All the equilibrium points are centers or saddles. All the centers are located in even multiples of π, i.e. when $\varphi_2 = 0, \varphi_1 = 2n\pi,\ n \in \mathbb{Z}$ then it is a center, and all the other equilibrium points are saddles.

As can be seen in the phase portrait Figure 2.2 there are four possible cases of curves, let us compute all the solution levels of the equation $\ddot{\varphi} + \frac{g}{l}\sin\varphi = 0$. Multiplying the equation by $\dot{\varphi}$ we obtain

$$\ddot{\varphi} \cdot \dot{\varphi} + \frac{g}{l}\dot{\varphi}\sin\varphi = 0,$$

which can be rewritten as follows

$$\frac{1}{2} \cdot \frac{d}{dt}(\dot{\varphi})^2 + \frac{g}{l} \cdot \frac{d}{dt}(-\cos\varphi) = 0.$$

Integrating the previous equation we get

$$\frac{1}{2}(\dot{\varphi})^2 + \frac{g}{l}(-\cos\varphi + 1) = c,$$

where $c \in \mathbb{R}$. Using $\varphi = \varphi_1$ and $\dot{\varphi} = \varphi_2$, leads to

$$\frac{\varphi_2^2}{2} + \frac{g}{l}(1 - \cos\varphi_1) = c. \tag{2.11}$$

The curves in the Figure 2.2 we obtain by changing the values of the constant c.

There are four cases of curves for different values of c:

- for $c = 0$, from the equation (2.11) we get

$$\varphi_2^2 = \frac{2g}{l}(\cos\varphi_1 - 1),$$

  the value $\varphi_2 = 0$ leads to the equation $\cos\varphi_1 = 1$, which is satisfied for $\varphi_1 = 2k\pi, k \in \mathbb{Z}$. These solutions of φ_1 and φ_2 are the stationary solutions (equilibrium points which are centers).

- for $c = \frac{2g}{l}$, from (2.11) we get

$$\varphi_2 = \pm\sqrt{\frac{2g}{l}(1 + \cos\varphi_1)}$$

  which is defined for all $\varphi_1 \in \mathbb{R}$. For the simplest case when $l = 2g$ we get

$$\varphi_2 = \pm\sqrt{(1 + \cos\varphi_1)},$$

  which are the separatrices for the saddles in the Figure 2.2 (and the curves connecting the saddles).

  From the equation (2.11) we get

$$\varphi_2^2 = 2c - \frac{2g}{l}(1 - \cos\varphi_1).$$

  The left side is always non-negative, consequently $2c \geq \frac{2g}{l}(1 - \cos\varphi_1) \geq 0$, which implies $c \geq 0$.

- for $c > \frac{2g}{l}$ we get that $2c > \frac{2g}{l}(1 - \cos\varphi_1)$ which implies that the square root in the next equation is strictly positive

$$\varphi_2 = \pm\sqrt{2c - \frac{2g}{l}(1 - \cos\varphi_1)}.$$

  The curves generated by the previous equation exist for all $\varphi_1 \in \mathbb{R}$, represent the outer curves in the Figure 2.2.

- for $0 < c < \frac{2g}{l}$

$$\varphi_2 = \pm\sqrt{2c - \frac{2g}{l}(1 - \cos\varphi_1)}.$$

  The term in the square root may be negative for some φ_1, so to the existence of the square root we need condition $c \geq \frac{g}{l}(1 - \cos\varphi_1)$. Hence,

$$\cos\varphi_1 \geq 1 - \frac{cl}{g},$$

  which implies

$$-\arccos\left(1 - \frac{cl}{g}\right) \leq \varphi_1 \leq \arccos\left(1 - \frac{cl}{g}\right).$$

From the periodicity of the goniometric functions we get

$$-\arccos\left(1-\frac{cl}{g}\right) \leq \varphi_1 + 2k\pi \leq \arccos\left(1-\frac{cl}{g}\right) \quad \text{for } k \in \mathbb{Z}.$$

The last type of the curves in the Figure 2.2 is defined by

$$\varphi_2 = \pm\sqrt{2c - \frac{2g}{l}(1-\cos\varphi_1)}$$
$$\text{for } \varphi_1 + 2k\pi \in \left[-\arccos\left(1-\frac{cl}{g}\right), \arccos\left(1-\frac{cl}{g}\right)\right].$$

These curves are the orbits around the equilibrium points which are centers.

2. The most common method for small values of φ is the approximation of the $\sin\varphi$ by the Taylor polynomial at $\varphi = 0$:

   (a) The simplest case is approximation by the Taylor polynomial of the first order $sin\varphi \approx \varphi$, then the equations becomes

   $$\ddot{\varphi} + \frac{g}{l}\varphi = 0 \tag{2.12}$$

   Which is simple ordinary differential equation, it can be solved by method of characteristics. The general solution is

   $$\varphi(t) = c_1\cos\left(\sqrt{\frac{g}{l}}t\right) + c_2\sin\left(\sqrt{\frac{g}{l}}t\right)$$

   Solving with initial conditions $\varphi(0) = \hat{\varphi}$ and $\dot{\varphi}(0) = \bar{\varphi}$, we obtain the solution

   $$\varphi(t) = \hat{\varphi}\cos\left(\sqrt{\frac{g}{l}}t\right) + \bar{\varphi}\cdot\sqrt{\frac{l}{g}}\sin\left(\sqrt{\frac{g}{l}}t\right).$$

   The equation (2.12) is the second order, it can be transformed into the system of ordinary differential equations (where $\varphi_1 = \varphi$ is the angle and $\varphi_2 = \dot{\varphi}$ is speed):

   $$\begin{aligned}\dot{\varphi}_1 &= \varphi_2\\ \dot{\varphi}_2 &= -\frac{g}{l}\varphi_1\end{aligned} \tag{2.13}$$

   It is clear that the only equilibrium point of the system is $(0,0)$. The planar system (2.13) is linear, so the equilibrium point is a center. The phase portrait for the system (2.13) is in the Figure 2.3.

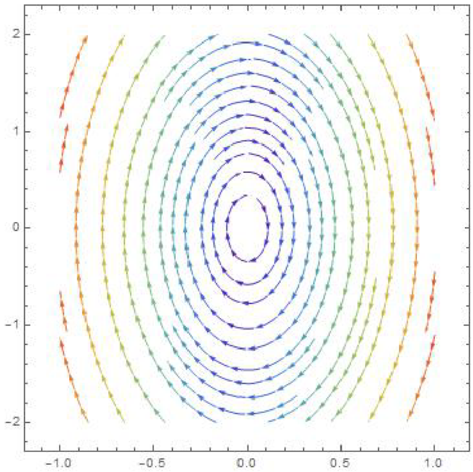

Figure 2.3: Phase portrait of the system (2.13)

(b) To have more precise results it is possible to take the approximation of higher order. Due to he fact that sinus is odd, it is useless to take the polynomial of even order. So, use the approximation of third order $\sin\varphi \approx (\varphi - \frac{\varphi^3}{3!})$, then (2.9) is transformed into the system

$$\begin{aligned} \dot{\varphi}_1 &= \varphi_2 \\ \dot{\varphi}_2 &= -\frac{g}{l}\left(\varphi_1 - \frac{\varphi_1^3}{3!}\right) \end{aligned} \tag{2.14}$$

This case already gives more stationary solutions (equilibrium points) than (2.13), which are of different types. There are 3 equilibrium points: $(0,0)^T$, which is again a center (see p. 16), and using Theorem 2.17 we get that the other two $(\sqrt{6},\ \ 0)^T$, $(-\sqrt{6},\ \ 0)^T$ are saddles for the system (2.14). The phase portrait for the system (2.14) is in the Figure 2.4, it shows the behaviour around the equilibrium points.

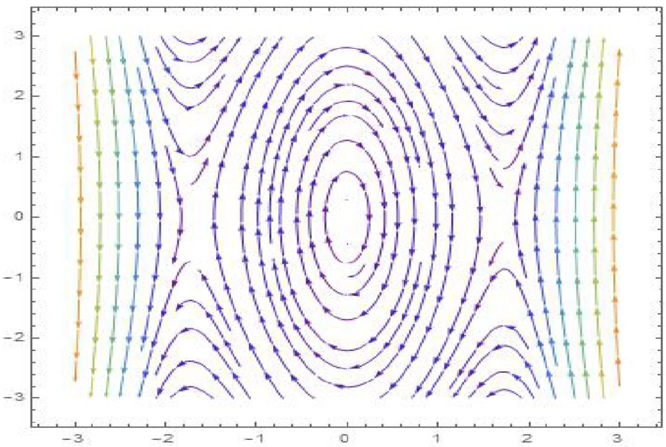

Figure 2.4: The phase portrait for the system (2.14)

(c) It is possible to use higher approximations, but using the approximation of the fifth order gives even less precise results, and using the seventh order changes only the position of the two saddles. Usage of the approximation by ninth order $\sin\varphi \approx (\varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \frac{\varphi^7}{7!} + \frac{\varphi^9}{9!})$, (2.9) is transformed into

$$\begin{aligned} \dot{\varphi}_1 &= \varphi_2 \\ \dot{\varphi}_2 &= -\frac{g}{l}\left(\varphi_1 - \frac{\varphi_1^3}{3!} + \frac{\varphi_1^5}{5!} - \frac{\varphi_1^7}{7!} + \frac{\varphi_1^9}{9!}\right) \end{aligned} \tag{2.15}$$

The system has 5 stationary solutions: $(0,\ \ 0)^T$, $(4.9632,\ \ 0)^T$, $(-4.9632,\ \ 0)^T$, which are centers (see p. 16), and again using Theorem 2.17 $(3.1487,\ \ 0)^T$, $(-3.1487,\ \ 0)^T$ are saddles for the system. The phase portrait in Figure 2.5 gives rough idea of the behaviour of the system near equilibrium points.

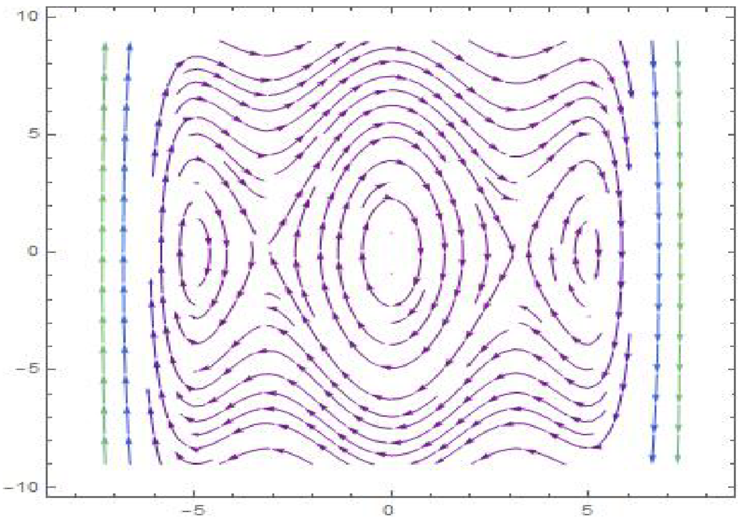

Figure 2.5: The phase portrait for the system (2.15).

From the previous calculations can be seen that the number of stationary solutions arise. The centers alternate with the saddles and they converge to the multiples of π, as expected.

3 Delayed differential equations

3.1 Differential equations with delay

As it is known, an *ordinary differential equation* (ODE), is an equation which connects the values of an unknown function and its derivatives for one and the same argument. For example, the equation $F(t, x(t), \dot{x}(t), \ddot{x}(t)) = 0$, where the dots indicate derivatives $\dot{x} = \frac{dx}{dt}$.

A *functional equation* (FE) involves an unknown function for different arguments. The equation $x(t) = 3x(t-2)+4$, is the example of FE. The differences between the arguments of an unknown function and t are called *argument deviations*.

Differential equations with delay (DDEs) can be considered as combined functional and differential equations.

$$x^{(n)}(t) = f(t, x(t - h_1(t)), ..., x^{(n-1)}(t - h_k(t))),$$

where $n \in \mathbb{N}$, and $h_i(t) \geq 0$ are continuous functions. We will focus on one type, $h_i = c_i$, $\forall i = 1, ...n$, where c_i are some nonnegative constants.

It is a large and important class of dynamical systems. They are very useful in describing either natural or technological problems. Delay arises between the observation and the control action.

3.1.1 Solving DDEs

There are several methods of finding the unique solution $y(t)$ (if exist) of the system

$$\dot{y}(t) = a_1(t)y(t) + a_2(t)y(t-d), \ t \in [0, d], \tag{3.1}$$
$$y(t) = g(t), \ t \in [-d, 0], \tag{3.2}$$

where $a_1(t), a_2(t)$ are class C^1 functions, $d > 0$, $g(t)$ is class C^1 function.

As it is customary, the continuous function $y : [-d, d] \longrightarrow \mathbb{R}$ is called a solution of (3.1), if $y \in C^1(0, d)$ and satisfies the equation (3.1) on the interval $[0, d]$. If the function y is given on the interval $[-d, 0]$, then the problem cannot have more than one solution. By contradiction, if we assume there are two different solutions $u(t), v(t)$, and let $w(t)$ be their difference $u(t) - v(t)$. Then also $w(t)$ has to satisfy equation (3.1). Since $u(t)$ and $v(t)$ both satisfy the same equation, the (3.2) becomes $w(t) \equiv 0$, on $[-d, 0]$, which makes $w(t - d) = 0$, on $[0, d]$. The result is first order ODE $\dot{w}(t) = a_1(t)w(t)$ on $[0, d]$, with $w(0) = 0$. Which proves $w(t) \equiv 0$ on $[0, d]$, which gives $u(t) = v(t)$ on $[-d, d]$.

Of the several methods used for solving problem (3.1),(3.2), we state two:

- The Method of Characteristics
- The Method of Steps

The Method of Characteristics is suitable for solving the simplest case, the equation with constant coefficients. It is necessary to know the Lambert w-function, W(z), namely the *inverse of the equation*

$$z(w) = we^w.$$

The Method of Characteristics

As was mentioned above, the method is suitable for solving the delayed differential equation with constant coefficients. Consider equation (3.1) with $a_1(t) = a_1, a_2(t) = a_2$, where a_1 and a_2 are real constants. Assume that the solution of (3.1) is of the form $y(t) = Ce^{mt}$, for some value $m \in \mathbb{C}$ and with $C \neq 0$ arbitrary constant. Substituting the expected solution into the equation (3.1) we get

$$Cme^{mt} = a_1Ce^{mt} + a_2Ce^{m(t-d)}.$$

After division by Ce^{mt} we obtain the characteristic equation

$$(m - a_1)e^{md} - a_2 = 0 \tag{3.3}$$

If $a_2 = 0$ then the equation (3.1) is an ODE with $m = a_1$. On the other hand, if $a_1 = 0$ and $a_2 \neq 0$, then (3.1) is pure delay equation.

Multiplying the equation (3.3) by de^{-a_1d} gives

$$(dm - da_1)e^{(dm-da_1)} = de^{-a_1d}a_2.$$

Which is the inverse Lambert function, $d(m - a_1) = W(de^{-a_1d}a_2)$. From where

$$m = \frac{1}{d}W(de^{-a_1d}a_2) + a_1. \tag{3.4}$$

Let us solve the simplest case, when $a_1 = 0$. The same results can be obtained by changing the argument of the Lambert function and then using (3.4) to get wanted m.

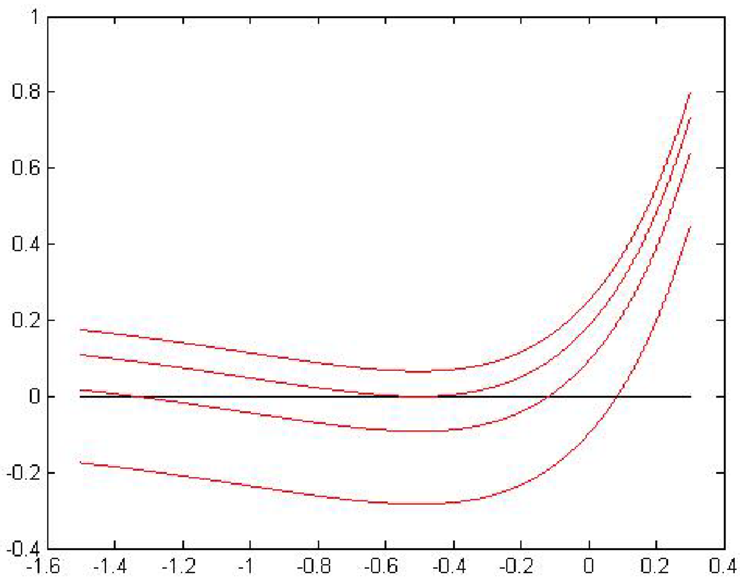

Figure 3.1: Graph of $f(m) = me^{2m} - a_2$ for $a_2 = \{-\frac{1}{4}, -\frac{1}{2e}, -\frac{1}{4e}, \frac{1}{10}\}$.

So, the characteristic equation becomes $me^{md} - a_2 = 0$.

Define a function

$$f(m) = me^{md} - a_2,$$

where a_2 is a parameter. For some values of a_2 it has real solutions, but it turns out, that in all cases for $f(m)$ has infinitely many complex solutions. With $d > 0$, Figure 3.1 shows that there are 4 cases for the number of roots of the function $f(m)$ (for $a_2 \neq 0$):

- When $a_2 < -\frac{1}{de} < 0$, there are no real roots of $f(m)$.
- If $a_2 = -\frac{1}{de}$, then there is exactly one real solution of $f(m) = 0$.
- When $-\frac{1}{de} < a_2 < 0$, then $f(m)$ has two real roots, both negative.
- If $a_2 > 0$, then it has one root, which is positive.

Case 1 For $a_2 < -\frac{1}{de} < 0$ there are only complex solutions of $f(m)$, so we seek for the complex numbers $m = a + bi,\ a, b \in \mathbb{R}$. The characteristic equation becomes

$$(a + bi)e^{(a+bi)d} = a_2$$

$$(a + bi)e^{ad}(\cos(bd) + i\sin(bd)) = a_2$$

After multiplying out we obtain the following equations for the real and complex parts

$$a\cos(bd) - b\sin(bd) = a_2 e^{ad} \tag{3.5}$$

$$a\sin(bd) + b\cos(bd) = 0, \tag{3.6}$$

from where

$$a = -b\frac{\cos(bd)}{\sin(bd)} = -b\cot(bd), b \neq 0. \tag{3.7}$$

Substituting a from (3.7) into (3.5) leads to

$$-b\Big[\frac{\cos^2(bd)}{\sin(bd)} + \sin(bd)\Big] = a_2 e^{bd\cot(bd)}$$

From where we get the equation for b

$$b = -a_2\sin(bd)e^{bd\cot(bd)} \tag{3.8}$$

For $a_2 < -\frac{1}{de}$ it is possible to find the solutions as the intersections of line $y = x$, with the one-parameter family of curves

$$y = -a_2\sin(dx)e^{dx\cot(dx)},$$

where d is fixed and a_2 is the parameter.

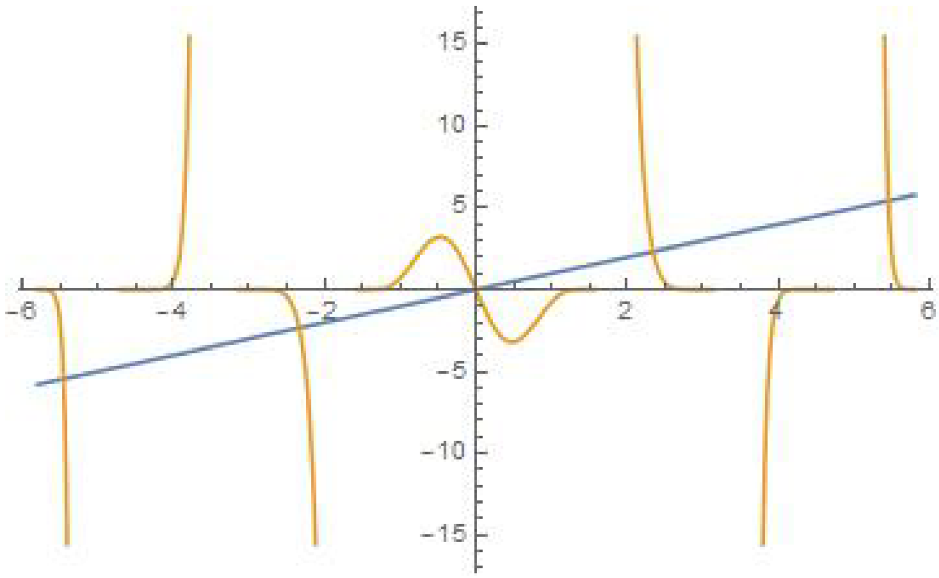

Figure 3.2: Graphs of the functions $y = x$ and $y = -2\sin(2x)e^{2x\cot(2x)}$.

Figure 3.2 shows that equation (3.8) has infinitely many solutions; denote them by $b_k, k = 1, 2, \ldots$. From (3.7) we obtain a_k. Since $m = a_k + ib_k, k = 1, 2, \ldots$ $(b_k \neq 0)$ the formal solution to the DDE is

$$y(t) = \sum_{k=1}^{\infty} e^{a_k t}(C_{1k}\cos(b_k t) + C_{2k}\sin(b_k t)),$$

where C_{1k} and C_{2k} are arbitrary constants.

Case 2 $a_2 = -\frac{1}{de}$

The single root can be found using (3.7)

$$\lim_{b \longrightarrow 0} -b\cot(bd) = -\frac{1}{d}$$

which produces another characteristic solution $e^{-\frac{1}{d}t}$ and the solution becomes

$$y(t) = C_0 e^{-\frac{1}{d}t} + \sum_{k=1}^{\infty} e^{a_k t}(C_{1k}\cos(b_k t) + C_{2k}\sin(b_k t)),$$

where C_0, C_{1k} and C_{2k} are arbitrary constants.

Case 3 When $-\frac{1}{de} < a_2 < 0$, then $f(m)$ has two real roots, both negative. One of them is bigger than $-\frac{1}{d}$, the second one smaller. Both can be found using Newton's Method. Starting with $\rho_0 = -\frac{1}{d}$ (respectively $\rho_0 = -\frac{2}{d}$) and for $k \in \mathbb{N}$ define $\rho_{k+1} = \rho_k - \frac{f(\rho_k)}{f'(\rho_k)}$. Then $m_0 = lim_{k \longrightarrow \infty}\rho_k$ (resp. $m_1 = lim_{k \longrightarrow \infty}\rho_k$). Which gives the solution

$$y(t) = C_1 e^{m_0 t} + C_2 e^{m_1 t} + \sum_{k=1}^{\infty} e^{a_k t}(C_{1k}\cos(b_k t) + C_{2k}\sin(b_k t)).$$

Case 4 $a_2 > 0$

The only root m_3 can be again found by Newton's Method with starting point $\rho_0 = 1$. The solution is

$$y(t) = C_3 e^{m_3 t} + \sum_{k=1}^{\infty} e^{a_k t}(C_{1k}\cos(b_k t) + C_{2k}\sin(b_k t)).$$

The equation $\dot{y} = a_2 y(t-d)$ has a solution

$$y(t) = C_0 e^{-\frac{1}{d}t} + C_1 e^{m_0 t} + C_2 e^{m_1 t} + C_3 e^{m_3 t} + \sum_{k=1}^{\infty} e^{a_k t}(C_{1k}\cos(b_k t) + C_{2k}\sin(b_k t))$$

where a_k and b_k satisfy equations (3.5) and (3.6). Provided that

1. $C_0 = C_1 = C_2 = C_3 = 0$ when $a_2 < -\frac{1}{de}$.
2. $C_1 = C_2 = C_3 = 0$ and C_0 is arbitrary when $a_2 = -\frac{1}{de}$.
3. $C_0 = C_3 = 0$, and C_1 and C_2 are arbitrary. m_0 and m_1 are real roots of $me^{md} - a_2 = 0$ when $-\frac{1}{de} < a_2 < 0$.
4. $C_0 = C_1 = C_2 = 0$ and C_3 is arbitrary and m_3 is the real root of $me^{md} - a_2 = 0$ when $a_2 > 0$.

The Method of Steps

The Method of steps converts DDE into ODE over some specific interval, using the known history function for that interval. The resulting equation is solved and the process is repeated for next interval, using the newly found solution as a history function for next interval. We will apply this process to the problem (3.1), (3.2).

Step 1 We know, that on the interval $[-d, 0]$, the solution is $y(t) = g(t)$. When $t \in [0, d]$, $t - d \in [-d, 0]$, so $y(t-d)$ becomes $g(t-d)$. Using this fact, on $[0, d]$ the equation (3.1) is *not* DDE but an ODE, with initial condition $y(0) = g(0)$. Thus after solving the ODE we obtain the solution $y_1(t)$, $t \in [0, d]$ (the existence of the solution is guaranteed, due to expecting class C^1 functions in the problem).

Step 2 If $t \in [d, 2d]$, then $y(t-d)$ is the solution from Step 1, namely $y_1(t-d)$. And for this interval the equation (3.1) is again an ODE with initial condition $y(d) = y_1(d)$. Solution on the interval $[d, 2d]$ is $y_2(t)$.

These steps may be continued for subsequent intervals. Continuing with the process we can obtain the solution for any interval $[0, a]$, $a > 0, a \in \mathbb{R}$.

The existence of the solution of DDE on the specific interval depends on the existence of the solution of ODE on the interval.

Example 3.1.

$$\dot{y}(t) = 4y(t-2)$$
$$y(t) = 1, \quad t \leq 0.$$

To application of the Method of Steps it is necessary to know the length of subintervals, in this case the delay is constant so the intervals are of the length $d = 2$.

1. For $t \in [0,2]$: $\dot{y} = 4$
   Integrating we obtain: $y_1(t) = 4t + c_1$
   The initial value $y(0) = 1$ gives $c_1 = 1$
   So the solution is $y(t) = 4t + 1, \ t \in [0,2]$. To continue we have to know the value in 2, which is the initial value for the next interval: $y_1(2) = 9$.

2. For $t \in [2,4]$: $\dot{y} = 4(4t+1) = 16t + 4$
   The solution is: $y_2(t) = 8t^2 + 4t + c_2$
   From the initial value $y_1(2) = y_2(2)$ which gives $c_2 = -31$.
   Then the solution is $y_2(t) = 8t^2 + 4t - 31$,
   and the initial value for the next interval is $y(4) = 119$

3. For $t \in [4,6]$: $\dot{y} = 4(8t^2 + 4t - 31)$
   The solution is: $y_3(t) = \frac{32}{3}t^3 + 8t^2 - 124t + c_3$
   The initial value gives $c_3 = -\frac{587}{3}$, then the solution is $y_3(t) = \frac{32}{3}t^3 + 8t^2 - 124t - \frac{587}{3}$.

   This can be continued to obtain whole solution for $t > 0$.

   It is shown in the next example, where the derivative does not exist at 1.

Example 3.2.

$$\dot{y} = \frac{1}{t-1}y(t) + \frac{1}{t-1} \ t \in [0,2],$$
$$y(t) = 1, \ t \in [-2,0].$$

Generally the solution $y(t)$ of

$$\dot{y}(t) = f(t, y(t), y(t-\tau)), \quad y(t) = g(t), t \in [-\tau, 0]$$

has a jump discontinuity of $\dot{y}(t)$ at 0 (that is $lim_{t\to 0^+}\dot{y}(t) \neq lim_{t\to 0^-}\dot{y}(t)$). This initial discontinuity propagates. The second derivative has a jump discontinuity at $t = \tau$, third derivative has a jump at $t = 2\tau$, etc.

Discontinuities increase in order for *retarded* DDEs, but generally they do not, if the equation involves delayed terms with derivatives, *neutral* DDEs.

3.2 Gantry crane

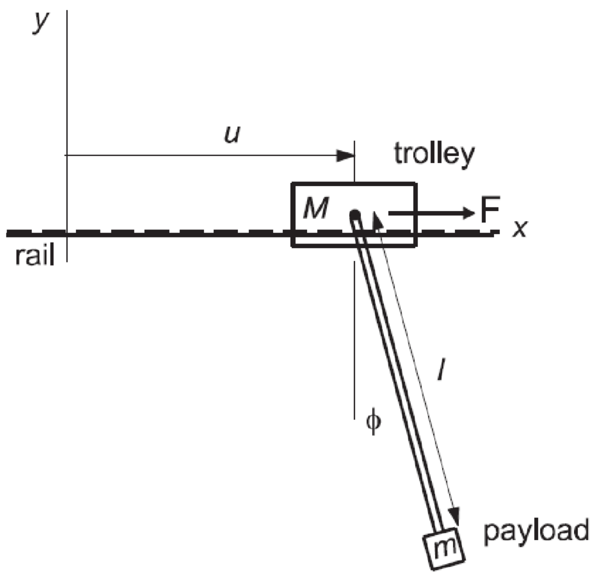

Figure 3.3: Simple pendulum model of a gantry crane (can be found in [6]).

Gantry cranes are used for moving objects within shipyards, railyards, factories. The cranes can lift several hundred tons depending how big the crane is. It is important to move the payloads rapidly and smoothly. Here arises problem that if the payload moves too fast it may start to swing and the operator may loose control of it. Different strategies for controlling the swinging not including the operator were examined. Time-delayed feedback controller was developed to add damping. The next section focuses on the mathematical formulation of the model and finding conditions for this type of controller.

Let us consider the simpler pendulum model of container crane in the Figure 3.3. Suppose that the cable is inextensible or its length is slowly varying compared to the time of oscillations.

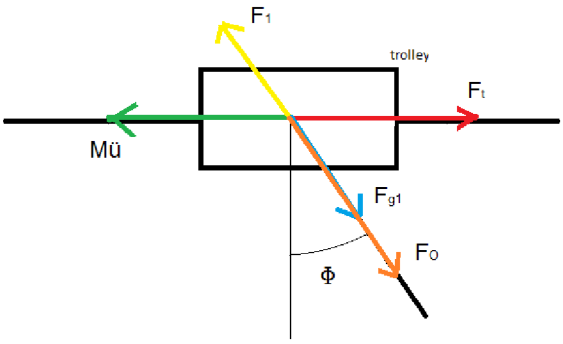

Figure 3.4: Forces acting on the trolley.

The forces acting on the trolley can be seen in the Figure 3.4. Using second Newton's law on the trolley (neglecting the friction and air draft) we obtain the following equation for the forces:

$$M\ddot{u} + F_{11} - F_{g11} - F_{O1} - F = 0$$

All the forces in equation are the components of the forces in the Figure 3.4 in u direction, because the trolley does not move in the other directions. This equation can be interpreted in terms with u and ϕ as

$$M\ddot{u} + m\ddot{u}\sin^2(\phi) - mg\sin(\phi)\cos(\phi) - ml(\dot{\phi})^2\sin(\phi) - F = 0 \tag{3.9}$$

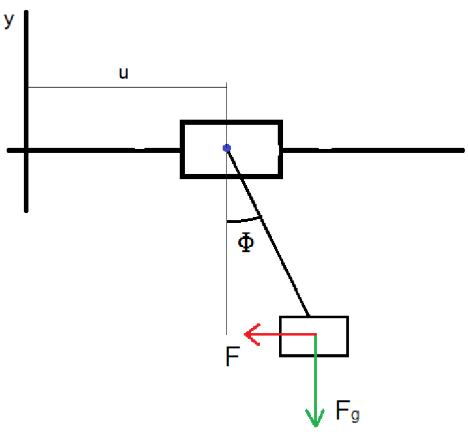

Figure 3.5: Forces influencing the trolley with momentum

The second equation can be obtained using third Newton's law. The forces influencing the momentum of the trolley are in the Figure 3.5.

$$ml^2\ddot{\phi} = -l(F_{g2} + F_2) = -l(mg\sin(\phi) + m\ddot{u}\cos(\phi))$$

After division by the nonzero terms we obtain:

$$l\ddot{\phi} + g\sin(\phi) + \ddot{u}\cos(\phi) = 0. \tag{3.10}$$

Let us rewrite the equations (3.9) and (3.10) into the different form. Multiplying (3.10) by $m\cos(\phi)$ we obtain

$$ml\ddot{\phi}\cos(\phi) + mg\sin(\phi)\cos(\phi) = -m\ddot{u}(1 - \sin^2(\phi)).$$

Hence,

$$m\ddot{u}\sin^2(\phi) = ml\ddot{\phi}\cos(\phi) + mg\sin(\phi)\cos(\phi) + m\ddot{u}. \tag{3.11}$$

Substituting (3.11) into (3.9) leads to

$$M\ddot{u} + ml\ddot{\phi}\cos(\phi) + mg\sin(\phi)\cos(\phi) + m\ddot{u} - mg\sin(\phi)\cos(\phi) - ml(\dot{\phi})^2\sin(\phi) - F = 0$$

Then the equations of the gantry crane are

$$\begin{aligned}(M+m)\ddot{u} + ml\ddot{\phi}\cos(\phi) - ml(\dot{\phi})^2\sin(\phi) - F &= 0 \\ l\ddot{\phi} + g\sin(\phi) + \ddot{u}\cos(\phi) &= 0.\end{aligned} \tag{3.12}$$

For moderate oscillations, when the angle $\phi < \frac{\pi}{2}$ holds

$$\ddot{u} = -\frac{l\ddot{\phi}}{\cos(\phi)} - g\tan(\phi) \tag{3.13}$$

Applying (3.13) into the equation (3.12) we get equation for ϕ only

$$(M+m)\left(-\frac{l\ddot{\phi}}{\cos(\phi)} - g\tan(\phi)\right) + ml\ddot{\phi}\cos(\phi) - ml(\dot{\phi})^2\sin(\phi) - F = 0 \tag{3.14}$$

Introduce the dimensionless time into the equation $s = \omega t$, where $\omega \equiv \sqrt{\frac{M+m}{Ml}g}$ is the frequency of the linearized trolley-payload system $Ml\ddot{\phi} + (M+m)g\phi = 0$. Equation (3.14) becomes

$$\frac{1+\rho\sin^2(\phi)}{\cos(\phi)}\ddot{\phi} + \tan(\phi) + \rho(\dot{\phi})^2\sin(\phi) + h = 0 \tag{3.15}$$

Differentiation is now with respect to the nondimensional time s and $\rho = \frac{m}{M}$ is the ratio of the weights of trolley and payload. The function h represents the nondimensional control force.

A simple delayed feedback

$$h(s) = k(\phi(s-\tau) - \phi(s))$$

was used by Pyragas in [15] to control the chaotic behaviour of the system. The only equilibrium point is $\phi = 0$.

Using the controller developed by Pyragas, the equation (3.15) can be rewritten into the second order delay differential equation

$$\ddot{\phi(s)} + \frac{\cos(\phi(s))}{1 + \rho\sin(\phi(s))}[\tan(\phi(s)) + \rho(\dot{\phi(s)})^2 \sin(\phi(s)) + k(\phi(s-\tau) - \phi(s))] = 0$$

3.2.1 Linearized system

The linearized equation of gantry crane without friction, with the Pyragas' controller is then

$$\ddot{\phi(s)} + \phi(s) + k(\phi(s-\tau) - \phi(s)) = 0. \tag{3.16}$$

The values for k, when the equation is stable can be found by introducing $\phi(s) = e^{i\sigma s}$ into the equation.

$$-\sigma^2 + 1 + k(e^{-i\sigma\tau} - 1) = 0$$

Using $e^{iz} = \cos z + i \sin z$, $z \in \mathbb{R}$ and the evenness of cosine, resp. oddness of sine, we get the following equations for real and imaginary parts:

$$\begin{aligned} -\sigma^2 + 1 + k(\cos(\sigma\tau) - 1) &= 0 \\ -k\sin(\sigma\tau) &= 0 \end{aligned}$$

From these equations we get three cases

1. $$k_0 = 0 \quad \text{and} \quad \sigma_0 = 1,$$

2. $$\tau_1 = 2\pi n \quad \text{and} \quad \sigma_1 = 1,$$

3. $$k_3 = \frac{1}{2}\left[1 - \left(\frac{\pi n}{\tau}\right)\right] \quad \text{and} \quad \sigma_3 = \frac{(2n+1)\pi}{\tau},$$

where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Software Matlab has a function *dde23* for solving the delayed differential equations. The graphs were generated using these conditions:

- the constant delay $\tau = 0.3$,
- constant used in the Pyragas' controller in the interval of stability $k = \frac{1}{2}$,
- the weight of the trolley $M = 10$,
- the weight of the payload $m = 100$,
- the rounded value of gravity $g = 9.81$,
- the force provided by an ideal motor as a constant $F(t) = \frac{7}{(M+m)g}$,
- the initial position $y_0 = 0.08$,
- the initial speed $\dot{y_0} = 0$.

Figure 3.8 contains four curves, each one is the solution of the equation of gantry crane using different controller.

1. The red curve represents the solution curve for the nonlinear equation of gantry crane with Pyragas' controller 3.6.

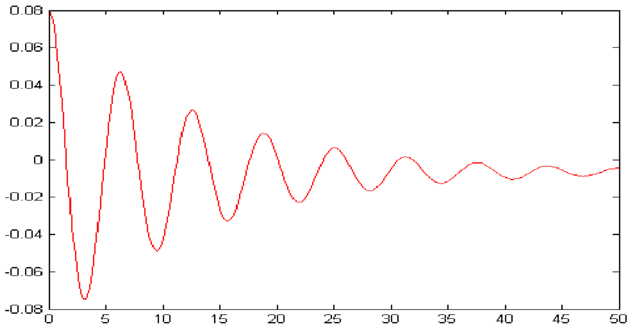

Figure 3.6: Graph of the solution for the nonlinear equation with Pyragas' controller.

2. The solution of the linearized equation with Pyragas' controller is represented by black curve.

3. The blue curve is the solution of the nonlinear equation without the controller 3.7.

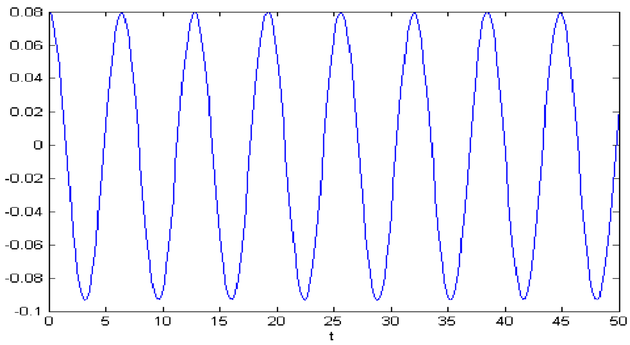

Figure 3.7: Graph of the nonlinear system without controller.

4. The green one is the solution of linearized equation without Pyragas' controller.

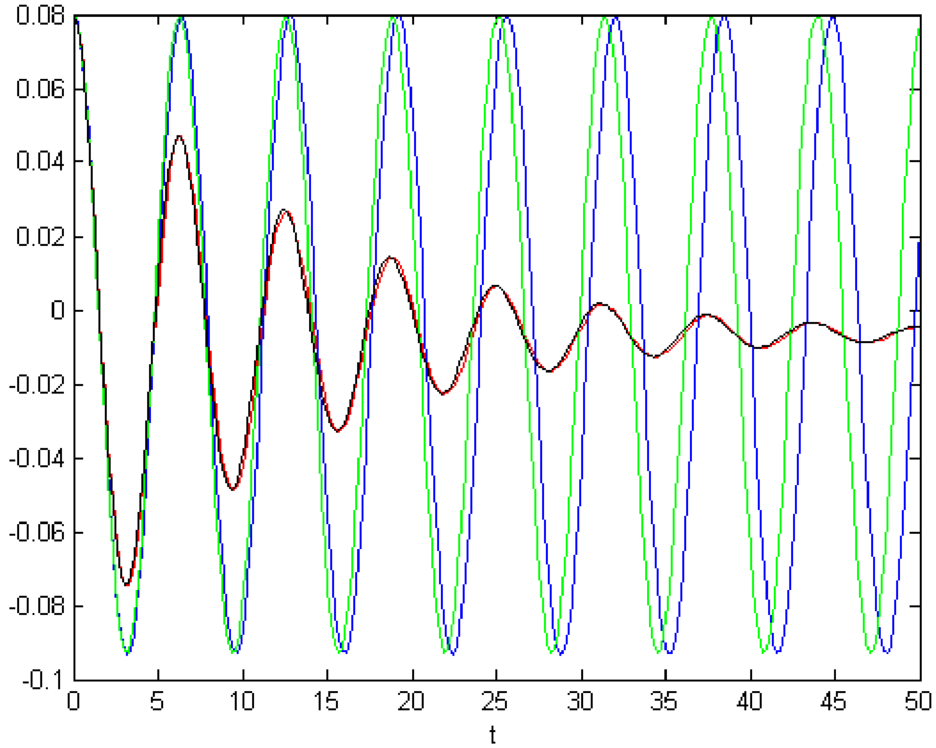

Figure 3.8: Graphs of the gantry crane.

The graphs in the Figure 3.8 show that the Pyragas' controller adds damping to the system with the used conditions. It also can be seen that the difference between the solution of linearized and the nonlinear system in small time interval is negligible, but it increases taking bigger time interval.

3.2.2 Real gantry crane

If we do not neglect friction the equations slightly change and we obtain for angle ϕ and position u:

$$(m+M)(\ddot{u}+c\dot{u})+ml\ddot{\phi}\cos(\phi)-ml(\dot{\phi})^2\sin(\phi)+cml\dot{\phi}\cos(\phi)=F(t)$$
$$l(\ddot{\phi}+c\dot{\phi})+g\sin(\phi)+\cos(\phi)(\ddot{u}+c\dot{u})=0$$

where c is the friction coefficient, l is the length of the inextensible cable, M, m are the weights of trolley, respectively payload (model was introduced in [6]).

Using the dimensionless variables and Pyragas' controller, the final equation for ϕ is

$$\ddot{\phi(s)}+2\mu\dot{\phi(s)}+\frac{\cos(\phi(s))}{1+\rho\sin(\phi(s))}[\tan(\phi(s))+\rho(\dot{\phi(s)})^2\sin(\phi(s))+k(\phi(s-\tau)-\phi(s))]=0.$$

Linearizing leads to

$$\ddot{\phi(s)}+2\mu\dot{\phi(s)}+\phi(s)+k(\phi(s-\tau)-\phi(s))=0.$$

Again introducing solution $\phi=e^{-i\sigma t}$ into the previous equation we obtain the system of equations for real and imaginary parts:

$$-\sigma^2+1+k(\cos(\sigma\tau)-1)=0$$
$$2\mu\sigma-k\sin(\sigma\tau)=0$$

The cases from the previous section are bounds of the interval of stability. For specific value of the delay τ, there is an interval for k when the solution is stable.

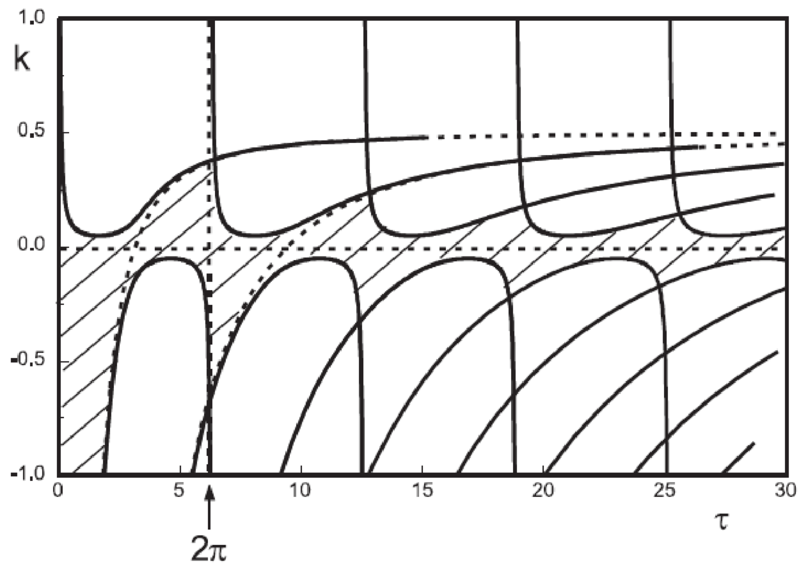

Figure 3.9: Dependence of k on the delay τ, generated with $\mu = 0.025$ (see [6]).

The broken lines in the Figure 3.9 correspond to the case without friction. The crosshatched domain corresponds to a stable case.

4 Oscillation

4.1 Oscillation of linear differential equation with non-constant delay

In this section we will deal with the oscilatory properties of the linear differential equation with non-constant delay on the interval $\langle 0, +\infty)$

$$\ddot{u}(t) + p_1(t)u(\tau_1(t)) + p_2(t)u(\tau_2(t)) = 0, \tag{4.1}$$

where $p_{1,2} : \mathbb{R}^+ \longrightarrow \mathbb{R}_+$ are continuous functions and $\tau_{1,2} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are also continuous functions satisfying

$$\begin{gathered} \tau_1(t) \leq t, \quad \tau_2(t) \leq t \quad \text{for } t \geq 0 \\ \lim_{t \longrightarrow +\infty} \tau_1(t) = \lim_{t \longrightarrow +\infty} \tau_2(t) = +\infty. \end{gathered} \tag{4.2}$$

Let us note that the linearized equation of the gantry crane (3.16) is a special case of the equation (4.1) with $p_1(t) = (1-k), \tau_1(t) = t, p_2(t) = k, \tau_2(t) = t - \tau$.
Let us denote the function $\tau(t)$ as

$$\tau(t) = \min_{t \geq 0}\{\tau_1(t), \tau_2(t)\}. \tag{4.3}$$

Definition 4.1. Consider $t_0 \in \mathbb{R}$ and $a_0 = \min\{\tau(t) : t \geq t_0\}$. The continuous function $u : \langle a_0, +\infty) \longrightarrow \mathbb{R}$ is said to be the solution of the equation (4.1) on the interval $\langle t_0, +\infty)$ if the function u is twice differentiable on the interval $\langle t_0, +\infty)$ and it satisfies the equation (4.1) on the interval $\langle t_0, +\infty)$.

Definition 4.2. The solution u of the equation (4.1) is said to be oscillatory if it has a sequence of zeros tending to infinity.

Let us state sufficient conditions under which every solution of the equation (4.1) is oscillatory. The main theorem generalizes the theorem stated in [13].
Suppose that the functions $p_1(t)$ and $p_2(t)$ satisfy

$$\int_0^{+\infty} s(p_1(s) + p_2(s))ds = +\infty. \tag{4.4}$$

Before formulation of the main theorem, let us state and prove two auxiliary lemmas.

Lemma 4.3. *Let u be a solution to the equation (4.1) on the interval $\langle t_u, +\infty)$ which satisfies*

$$u(t) > 0 \quad \text{for } t \geq t_u. \tag{4.5}$$

Then, there exist $t_0 \in \mathbb{R}, t_0 \geq t_u$ such that

$$\frac{T_1}{T_2} \leq \frac{u(T_1)}{u(T_2)} \quad \text{for } T_2 \geq T_1 \geq t_0.$$

Proof. Using $p_1, p_2 \geq 0$, (4.2) and (4.5) it can be proved that $\dot{u}(t) \geq 0$ for t large enough. Moreover, the equation (4.1) is homogeneous, which implies that without loss of generality we can assume $u(t) \geq 1$ for large t. (4.2) implies that there exist $t_1 \geq t_u$ such that

$$\dot{u}(t) \geq 0, \quad u(\tau(t)) \geq 1 \quad \text{for } t \geq t_1. \tag{4.6}$$

From the equation (4.1) we get

$$\frac{d}{dt}(t\dot{u} - u) = -t(p_1(t)u(\tau_1(t)) + p_2(t)u(\tau_2(t))) \quad \text{for } t \geq t_u.$$

Integrating the last equation from t_1 to t gives

$$t\dot{u}(t) - u(t) = \delta(t_1) - \int_{t_1}^{t} s(p_1(s)u(\tau_1(s)) + p_2(s)u(\tau_2(s))ds,$$

where $\delta(t_1) = t_1\dot{u}(t_1) - u(t_1)$ is a constant. Using the last equality, (4.2) and (4.4), it is possible to find number $t_2 \geq t_1$ such that

$$t\dot{u}(t) - u(t) \leq -\int_{t_2}^{t} s(p_1(s)u(\tau_1(s)) + p_2(s)u(\tau_2(s))ds \leq 0 \quad \text{for} \quad t \geq t_2.$$

Last inequalities give

$$\frac{d}{dt}\left(\frac{u(t)}{t}\right) = \frac{1}{t^2}(t\dot{u}(t) - u(t)) \leq 0 \quad \text{for } t \geq t_2. \tag{4.7}$$

Hence, for $T_2 \geq T_1 \geq t_2$ we get

$$\frac{u(T_2)}{T_2} \leq \frac{u(T_1)}{T_1},$$

which is the desired inequality. □

Remark 4.4. Notice that the Lemma 4.3 implies that there exists $t_0 \geq t_u$ such that

$$\left(\frac{\tau(t)}{t}\right) \leq \frac{u(\tau(t))}{u(t)} \quad \text{for } t \geq t_0, \tag{4.8}$$

where $\tau(t)$ is defined in (4.3).

Lemma 4.5. *Let u be a solution to the equation (4.1) satisfying the inequality (4.5) on the interval $\langle t_u, +\infty)$. Then, there exists a finite limit*

$$c_0 = \lim c(t, u),$$

where the function $c(t, u)$ is defined as follows

$$c(t, u) := \int_{t_u}^{t} \frac{1}{u(s)}(p_1(s)u(\tau_1(s)) + p_2(s)u(\tau_2(s))ds. \tag{4.9}$$

Proof. Let us use Riccati's transformation

$$\rho(t) := \frac{\dot{u}(t)}{u(t)} \quad \text{for } t \geq t_u,$$

then

$$\dot{\rho}(t) = \frac{\ddot{u}(t)u(t) - (\dot{u}(t))^2}{u^2(t)} \quad \text{for } t \geq t_u.$$

Substituting $\ddot{u}$ from (4.1) into the previous equation leads to

$$\dot{\rho}(t) = -p_1(t)\frac{u(\tau_1(t))}{u(t)} - p_2(t)\frac{u(\tau_2(t))}{u(t)} - \rho^2(t) \quad \text{for } t \geq t_u.$$

Integrating from t_u to t gives

$$\rho(t) - \rho(t_u) = -\int_{t_u}^{t} \frac{1}{u(s)}(p_1(s)u(\tau_1(s)) + p_2(s)u(\tau_2(s))ds - \int_{t_u}^{t} \rho^2(s)ds \quad \text{for } t \geq t_u. \tag{4.10}$$

If $\int_{t_u}^{+\infty} \rho^2(s)ds = +\infty$, then in view of (4.5), (4.6) and (4.10) we obtain a contradiction

$$0 \leq \lim_{t \longrightarrow +\infty} \rho(t) \leq -\infty.$$

Consequently,

$$\int_{t_u}^{+\infty} \rho^2(s)ds < +\infty. \tag{4.11}$$

On the other hand from (4.1) and (4.7) for t large enough, we obtain

$$\frac{u(t)}{t}\left(\frac{\dot{u}(t)}{u(t)} - \frac{1}{t}\right) \leq 0.$$

Hence,

$$0 \leq \rho(t) \leq \frac{1}{t},$$

which gives

$$\lim_{t \longrightarrow +\infty} \rho(t) = 0. \tag{4.12}$$

Let us rewrite the equation (4.10) as follows

$$\rho(t) = \rho(t_u) - \int_{t_u}^{+\infty} \rho^2(s)ds - c(t,u) + \int_{t}^{+\infty} \rho^2(s)ds,$$

Taking the limit $t \longrightarrow +\infty$, and using (4.11) and (4.12) we obtain

$$\lim_{t \longrightarrow +\infty} c(t,u) = \rho(t_u) - \int_{t_u}^{+\infty} \rho^2(s)ds = c_0 \in \mathbb{R}.$$

□

Theorem 4.6. *Let the following conditions hold*

$$\int_0^{+\infty} s(p_1(s)+p_2(s))ds = +\infty,$$

and

$$\int_0^{+\infty} \frac{\tau(s)}{s}(p_1(s)+p_2(s))ds = +\infty, \tag{4.13}$$

where the function $\tau(s)$ is defined in (4.3). Then, every solution of the equation (4.1) is oscillatory.

Proof. Assume by contradiction that the equation (4.1) has a solution which is not oscillatory, but satisfies (4.5). Lemma 4.3 implies that there exist $t_0 \geq t_u$ such that (4.8) holds. Clearly

$$\int_0^t \frac{\tau(s)}{s}(p_1(s)+p_2(s))ds = \int_0^{t_0} \frac{\tau(s)}{s}(p_1(s)+p_2(s))ds + \int_{t_u}^t \frac{\tau(s)}{s}(p_1(s)+p_2(s))ds - \\ - \int_{t_u}^{t_0} \frac{\tau(s)}{s}(p_1(s)+p_2(s))ds.$$

Hence, in view of (4.8), we get

$$\int_0^t \frac{\tau(s)}{s}(p_1(s)+p_2(s))ds \leq \int_0^{t_0} \frac{\tau(s)}{s}(p_1(s)+p_2(s))ds + \int_{t_u}^t \frac{u(\tau(s))}{u(s)}(p_1(s)+p_2(s))ds - \\ - \int_{t_u}^{t_0} \frac{\tau(s)}{s}(p_1(s)+p_2(s))ds \quad \text{for } t \geq t_0.$$

Using the conditions (4.3) and (4.6), from the last inequality we obtain

$$\int_0^t \frac{\tau(s)}{s}(p_1(s)+p_2(s))ds \leq \int_0^{t_0} \frac{\tau(s)}{s}(p_1(s)+p_2(s))ds + \int_{t_u}^t \frac{1}{u(s)}(p_1(s)\tau_1(s)+p_2(s)\tau_2(s))ds - \\ - \int_{t_u}^{t_0} \frac{\tau(s)}{s}(p_1(s)+p_2(s))ds, \qquad t \geq t_0.$$

One can see that the second term on the right side of the inequality is the function $c(t,u)$ defined in (4.9). Consequently, according to Lemma 4.5 we have

$$\int_0^{+\infty} \frac{\tau(s)}{s}(p_1(s)+p_2(s))ds < +\infty.$$

The last relation is a contradiction with (4.13), which implies that the Theorem 4.6 holds. □

5 Conclusion

In the thesis there are presented three main chapters: Nonlinear systems, Delayed differential equations, Oscillation of the linear differential equation with non-constant delay.

In the first we deal with the systems of ODE, which have numerous applications in many disciplines (engineering, physics, mechanics,...). If the equations are nonlinear it is difficult to find an exact solution, which is the reason to use the linearization and study the behaviour around the stationary solution of the linearized system. In the chapter there are stated cases when it is possible to use the linearization in small neighbourhood of the stationary solution, or use of the approximation with Taylor polynomial of higher order to simplify the system and get the idea of the behaviour of the system.

We showed the application of the theorems on the equation of the pendulum. It can be seen that the behaviour in small neighbourhood of the stationary solution is almost the same as in the nonlinear equation. To get more precise results we used also the approximation with the Taylor polynomial of 3rd and 9th order. The figures show that the approximation holds only in small neighbourhood the stationary solutions of the simplified systems slightly differ from the nonlinear system, and the solutions around them is also almost the same.

In the second chapter are investigated the delayed differential equations. DDEs also have numerous applications, they are mostly used in biological models, mechanics, medicine, engineering processes. There are cases when the usage of DDE is even better to describe the problem, then the usage of ODE. Finding the exact solution of DDE is possible only in special cases. We introduced two methods of solving DDEs.

We investigated the stability of the simplified equation of the gantry crane with the delayed feedback controller (using Pyragas' controller). For linearized equation we derived the conditions for the controller such that the solution is stable. Using the software MATLAB we showed that the behaviour of the linearized system in small time interval is almost the same as the nonlinear one. The figures show that the controller adds damping to the system. The Pyragas' controller is purely linear, which implies the question if it is possible to design a nonlinear controller for better stability.

The last chapter is dedicated to study of the oscillation of the equation with non-constant delay. The linearized equation of the gantry crane is the specific type of the of the linear differential equation with delay. We derived the conditions which guarantee us that every solution is oscillatory. Presented results generalize (in certain sense) some results stated in [13].

There are several directions of research, where the results of the thesis can be developed. Designing the nonlinear controller for better stability of the gantry crane. It can be useful to find another conditions for the oscillation of the equation. From the application point of view, e.g. vibrations during machining, heredity in physics, it can be useful to study the equation with a term with the first derivative

$$\ddot{u}(t) + q(t)\dot{u}(t) + p(t)u(\tau(t)) = 0.$$

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6 List of abbreviations and symbols

$\mathbb{N}$	set of natural numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	set of positive real numbers
$\mathbb{R}^n$	n-dimensional real coordinate space
$\mathbb{Z}$	set of integers
C^k	continuous functions with continuous derivatives till the order $k \in \mathbb{N} \cup \{0\}$
$[a, b]$	closed interval of real numbers
$\mathbf{x}$	the n-dimensional vector $(x_1, ..., x_n)^T$
$\dot{x}(t)$, $\frac{dx(t)}{dt}$	first derivative of $x(t)$ with respect to t
$\ddot{x}(t)$	second derivative of $x(t)$ with respect to t
DDE	delay differential equations
FE	functional equations
ODE	ordinary differential equation