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Mathematical and physical models of fluids – properties of solutions

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Abstrakt: Zakladní rovnice, které reprezentují matematický popis mechaniky kontinua, mají často tři prostorové dimenze a jednu časovou. Jejich nevýhodou je, že jejich analytické řešení je často nedosažitelné a jeho numerická aproximace výpočetně velmi náročná. Z těchto důvodů jsou takovéto modely často různými způsoby zjednodušovány. Jednou z možností, jak model zjednodušit, je snížení počtu prostorových dimenzí. Otázkou ovšem zůstává, jak dimenzionální redukci provést matematicky korektně. Zabývali jsme se nestacionárními Navier-Stokesovými rovnicemi pro stlačitelné nelineárně viskózní tekutiny v trojrozměrné oblasti. Nejprve jsme studovali dynamiku stlačitelných tekutin v oblastech, kde dominuje pouze jedna prostorová dimenze. Představili jsme odvození jednorozměrného modelu z trojrozměrných Navier-Stokesových rovnic. Následně jsme rozšířili současný rámec poznání tím, že jsme aplikovali dimenzionální redukci na nestacionární Navier-Stokesovy rovnice pro stlačitelné nelineárně viskózní tekutiny v deformované trojrozměrné oblasti se dvěma dominantními prostorovými dimenzemi. Zjistili jsme, že deformace oblasti netriviálne ovlivňuje výsledné limitní rovnice.

Klíčová slova: Navier-Stokesovy rovnice, Stlačitelné tekutiny, Nelineární viskozita, Asymptotická analýza, Redukce dimenze, Deformovaná oblast, Křivočaré souřadnice

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Key words: Navier-Stokes equations, Compressible fluids, Nonlinear viscosity, Asymptotic analysis, Dimension reduction, Curved domain, Curvilinear coordinates

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I hereby declare that this dissertation thesis has been completed independently under the supervision of RNDr. Rostislav Vodák, Ph.D. All the materials and resources are cited with regard to the scientific ethics, copyrights and laws protecting intellectual property. This thesis or its parts were not submitted to obtain any other or the same academic title.

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signature

Contents

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Chapter 1

Introduction

Governing equations representing mathematical description of continuum mechanics have often three spatial dimensions and one temporal dimension. However, their analytical solution is usually unattainable, and numerical approximation of the solution unduly complicated and computationally demanding. Therefore, these models are frequently simplified in various ways. One option of a simplification is a reduction of the number of spatial dimensions. We focused on nonsteady Navier-Stokes equations for compressible nonlinearly viscous fluids in a three-dimensional domain. These equations need a simplification, when possible, to be effectively solved.

The existence of weak solutions for three-dimensional models of fluid dynamics has already been studied. For instance, Pierre-Louis Lions proved the global solvability of Navier-Stokes equations for compressible linearly viscous fluids [17]. Further, Eduard Feireisl extensively studied global existence theory for the full Navier-Stokes-Fourier system [11]. A comprehensive overview on results achieved in the case of Newtonian compressible fluids is given in [22]. Concerning non-Newtonian fluids, Mamontov [18, 19] proved the existence of a global weak solution for compressible Navier-Stokes equations. This knowledge allows us to step forward in finding the solution (or at least its approximation). One possibility to achieve that is by performing a dimension reduction of the equations. Without the proven existence of a weak solution, it would be pointless to study the asymptotic behavior of the equations.

An asymptotic analysis was performed in linear elasticity for rods and beams [13, 14, 24], and for plates and shells [4, 6, 7], at first. Subsequently, rigorous derivation of lower-dimensional models was done also for fluids. An asymptotic analysis of three-dimensional steady Navier-Stokes equations based on the asymptotic expansion was presented in [21]. For comparison, the same result was achieved directly in [28] without the need to apply any asymptotic expansion. Regarding nonsteady Navier-Stokes equations for incompressible fluids, they were simplified into a lower-dimensional model in [12]. Further, a three-dimensional system for barotropic Navier-Stokes equations was asymptotically analyzed and the resulting one-dimensional and two-dimensional models were presented in [27] and [20], respectively. It was also shown that weak solutions of both three-dimensional Navier-Stokes equations for barotropic flows and three-dimensional full Navier-Stokes-Fourier equations tend to strong solutions of the respective one-dimensional system as the three-dimensional model tends to the one-dimensional model [3, 5]. Recently, Ducomet et al. [8] presented a rigorous derivation of a two-dimensional model from the three-dimensional compressible barotropic Navier-Stokes-Poisson system with radiation.

New difficulties arise by considering non-Newtonian fluids (i. e. fluids having nonlinear viscous stress tensor). This problem was tackled for the first time in [26], where a two-dimensional model was derived by a suitable scaling from nonsteady Navier-Stokes equations for compressible fluids. Our aim is to extend the current framework by dealing with nonsteady Navier-Stokes equations for compressible nonlinearly viscous fluids.

We study the dynamics of a compressible fluid in a thin pipe $\Omega_\varepsilon \subset \mathbb{R}^3$ (see Chapter 3) and in a curved three-dimensional domain $\tilde{\Omega}_\varepsilon$ with two dominant dimensions (see Chapter 4). The motion of a compressible fluid is described by its velocity $\mathbf{u}$ and density ρ. The time evolution of $\mathbf{u}$ and ρ is governed by the continuity and momentum equations

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t (\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \operatorname{div} \mathbb{S} + \rho \mathbf{f} \quad \text{in } \Omega \times (0,T), \tag{1.2}$$

where $T > 0$, p is the pressure, $\mathbb{S}$ stands for the viscous stress tensor and $\mathbf{f}$ represents the external forces [18].

Let us suppose that the fluid is isothermal and non-Newtonian. It means that (without the loss of generality)

$$\mathbb{S} = P(|D\mathbf{u}|)D\mathbf{u}, \quad p = \rho.$$

Similarly as in [26], we assume that the function P satisfies, for any U, V belonging to Orlicz class $[\tilde{L}_M(\Omega)]^9$ (see Definition 2.9, in section 2.3), the following five conditions

$$\int_\Omega P(|U|)|U|^2 \, \mathrm{d}x \geq \int_\Omega M(|U|) \, \mathrm{d}x, \tag{1.3}$$

$$\int_\Omega (P(|U|)U - P(|V|)V) : (U - V) \, \mathrm{d}x \geq 0, \tag{1.4}$$

$$P(z)|z|^2 \text{ is a convex function for } z \geq 0, \tag{1.5}$$

$$\int_\Omega N(P(|U|)|U|) \, \mathrm{d}x \leq C \left(1 + \int_\Omega M(|U|) \, \mathrm{d}x\right), \tag{1.6}$$

$$P(|U - \lambda V|)(U - \lambda V) \xrightarrow{M} P(|U|)U, \text{ for } \lambda \to 0. \tag{1.7}$$

For example, function

$$P(z) = \begin{cases} \frac{M(z)}{z}, & \text{for } z \neq 0, \\ 0, & \text{for } z = 0 \end{cases}$$

satisfies all conditions (1.3)-(1.7).

First, we introduce Orlicz spaces and Young functions with a logarithmic and an exponential growth (see Chapter 2), because this knowledge is necessary to prove our main results. Additionally, Chapter 2 summarizes the basic notation used throughout the thesis. Afterwards, we study the dynamics of a compressible fluid in thin domains with only one dominant dimension. In Chapter 3, a rigorous derivation of a one-dimensional model from the three-dimensional Navier-Stokes equations is presented. Our first main result, concerning the one-dimensional model, is summarized in Theorem 3.4 (section 3.4). Subsequently, we deal with nonsteady Navier-Stokes equations for compressible nonlinearly viscous fluids in a deformed three-dimensional domain. Chapter 4 focuses on a rigorous derivation of a two-dimensional model. Our second main result, concerning the two-dimensional model in a curved domain, is stated in Theorem 4.3 (section 4.5).

Chapter 2

Preliminaries

The basic notation is summarized in this section. Afterwards, we pay our attention to Young functions and their properties. Subsequently, we give a brief introduction to Orlicz spaces. More information and details about the Orlicz spaces can be found in [16]. In addition, we focus on a special class of Young functions with an exponential growth and their complementary functions, because the theory concerning these Young functions and respective Orlicz spaces is needed in the subsequent sections.

2.1 Basic notation

We adopt the notation "$\cdot$" and "$:$" for the scalar product of vectors and tensors, respectively, and "$\otimes$" for the tensor product. The Cartesian product of two sets is denoted by "$\times$" as well as the cross product of two vectors without danger of confusion. Symbol $|\cdot|$ stands for either the Lebesgue measure of a measurable set or the Euclidean norm defined as $|Z| = \sqrt{Z_{ij}Z_{ij}}$, where $Z \in \mathbb{R}^{m,n}$, m, $n \in \mathbb{N}$. We use Einstein summation convention for notational brevity. Symbols C and C_n, $n \in \mathbb{N}$, stand for unspecified positive constants.

We emphasize the connection of a function to Ω_ε and $\tilde{\Omega}_\varepsilon$ by subscript ε, and symbols "$^-$" and "$\tilde{}$", respectively. On the other hand, objects without symbol "$^-$" or "$\tilde{}$" are connected to the referential domain Ω (see sections 3.1 and 4.1). Since ε is always positive, we write only $\varepsilon \to 0$ instead of $\varepsilon \to 0^+$ for simplicity. Symbols $\bar{D}$, $\tilde{D}$ and D represent a symmetric part of the gradient, i. e. $\bar{D}_{ij}\bar{\mathbf{u}}_\varepsilon = \frac{1}{2}(\bar{\partial}_i \bar{u}_{\varepsilon,j} + \bar{\partial}_j \bar{u}_{\varepsilon,i})$, $\tilde{D}_{ij}\bar{\mathbf{u}}_\varepsilon = \frac{1}{2}(\tilde{\partial}_i \tilde{u}_{\varepsilon,j} + \tilde{\partial}_j \tilde{u}_{\varepsilon,i})$ and $D_{ij}\mathbf{u} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$.

Let $Q \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a bounded domain. We denote by ∂Q the boundary of Q. Bounded domain Q is called a Lipschitz domain if its boundary can be expressed by Lipschitz continuous functions (see [16] for the precise definition). We write $\partial Q \in \mathcal{C}^{0,1}$. The following three options of writing a matrice are used:

$$A = \left(\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3\right) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

where $\mathbf{a}^j = (a_{1j}, a_{2j}, a_{3j})^{\mathrm{T}}$. All vectors $\mathbf{x} \in \mathbb{R}^3$ in the text are column vectors.

We employ the standard notation of the following function spaces and their norms:

Spaces of continuously differentiable functions up to order m	–	$\mathcal{C}^m(\bar{Q})$, $\|\cdot\|_{\mathcal{C}^m(\bar{Q})}$
Lebesgue spaces	–	$L^p(Q)$, $\|\cdot\|_p$
Sobolev spaces	–	$W^{1,p}(Q)$, $\|\cdot\|_{1,p}$
Duals of $W^{1,p}(Q)$	–	$[W^{1,p}(Q)]^*$, $\|\cdot\|_{[W^{1,p}(Q)]^*}$
Orlicz spaces	–	$L_\Phi(Q)$, $\|\cdot\|_{L_\Phi(Q)}$
Sobolev-Orlicz spaces	–	$W^1L_\Phi(Q)$, $\|\cdot\|_{W^1L_\Phi(Q)}$
Duals of $W^1L_\Phi(Q)$	–	$[W^1L_\Phi(Q)]^*$, $\|\cdot\|_{[W^1L_\Phi(Q)]^*}$
Bochner spaces	–	$L^p(0,T;X)$, $\|\cdot\|_{L^p(0,T;X)}$, $\mathcal{C}^m(\langle 0,T\rangle;X)$, $\|\cdot\|_{\mathcal{C}^m(\langle 0,T\rangle;X)}$

where $Q \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is a bounded domain, $p \in \langle 1, +\infty) \cup \{+\infty\}$, $m \in \mathbb{N} \cup \{0\}$ and X is a Banach space. In addition, $\mathcal{C}_0^m(\bar{Q})$ denotes spaces of continuously differentiable functions up to order m, $m \in \mathbb{N} \cup \{0\}$, with compact support. Naturally, $\mathcal{C}^0(\bar{Q}) = \mathcal{C}(\bar{Q})$ is the space of continuous functions. Next, we denote the space of smooth and compactly supported functions endowed with the inductive limit topology by $\mathcal{D}(Q)$. Its dual space is denoted by $\mathcal{D}^*(Q)$.

2.2 Young functions and their properties

A generalization of Lebesgue spaces was the motivation for the concept of Young functions and Orlicz spaces. A function u defined on $Q \subset \mathbb{R}^n$ belongs to $L^p(Q)$ if

$$\int_Q \Phi(|u(x)|)\ \mathrm{d}x = \int_Q |u(x)|^p\ \mathrm{d}x < +\infty,$$

where $\Phi(z) = z^p$. It is possible to substitute the function Φ with a more general function called a Young function.

Definition 2.1. Φ is a **Young function**, if there exists a function φ such that

$$\Phi(z) = \int_0^z \varphi(s)\ \mathrm{d}s, \quad z \geq 0$$

and the following conditions hold:

(i) $\varphi(0) = 0$,

(ii) $\varphi(s) > 0$ for $s > 0$,

(iii) φ is right continuous,

(iv) φ is non-decreasing,

(v) $\lim_{s \to +\infty} \varphi(s) = +\infty$.

Definition 2.2. Let φ be the first derivative of a Young function Φ, which means that

$$\Phi(z) = \int_0^z \varphi(s)\ \mathrm{d}s.$$

The function Ψ is called the **complementary function** to the Young function Φ if

$$\Psi(z) = \int_0^z \psi(s)\ \mathrm{d}s,$$

where $\psi(z) = \sup\{s, \varphi(s) \leq z\}$, $z \geq 0$. If there exists an inversion of φ, then $\psi = \varphi^{-1}$.

Remark 2.3. If function Ψ is a complementary function to Φ, then also Φ is complementary to Ψ. In addition, Ψ is a Young function. Therefore, we can call Φ, Ψ as a **pair of complementary Young functions**.

There is a special class of Young functions which plays an important role in the theory of Orlicz spaces.

Definition 2.4. A Young function Φ satisfies the **$\boldsymbol{\Delta_2}$-condition**, if there exist $C > 0$ and $z_0 \geq 0$ such that

$$\Phi(2z) \leq C\Phi(z), \quad \forall z \geq z_0.$$

If $z_0 = 0$, we say that Φ satisfies the **global $\boldsymbol{\Delta_2}$-condition** (we write $\Phi \in \Delta_2$).

Sometimes, the explicit formula for a Young function is unknown and only its complementary function can be used to decide, whether a Young function belongs to Δ_2.

Theorem 2.5. *A Young function Φ satisfies the Δ_2-condition if and only if there exist $C > 0$ and $z_0 > 0$ such that*

$$\Psi(z) \leq \frac{1}{2C}\Psi(Cz), \quad \forall z \geq z_0,$$

where Ψ is the complementary function to Φ.

Proof: see [16], page 139.

Two special types of ordering can be introduced for Young functions. The first ordering concerns the equivalence property of Young functions.

Definition 2.6. Let Φ_1 and Φ_2 be two Young functions. If there exist $C > 0$ and $z_0 > 0$ such that

$$\Phi_1(z) \leq \Phi_2(Cz), \quad \forall z \geq z_0,$$

then we write

$$\Phi_1 \prec \Phi_2.$$

If $\Phi_1 \prec \Phi_2$ and also $\Phi_2 \prec \Phi_1$, we say that Φ_1 and Φ_2 are **equivalent**.

The second ordering of Young functions is useful for the embedding theorem of Orlicz spaces (see Theorem 2.20).

Definition 2.7. Let Φ_1 and Φ_2 be two Young functions. If

$$\lim_{z \to +\infty} \frac{\Phi_1(z)}{\Phi_2(\lambda z)} = 0,$$

for any $\lambda > 0$, then we write

$$\Phi_1 \prec\prec \Phi_2.$$

Lemma 2.8. *Let Φ_1 and Φ_2 be two Young functions, Ψ_1 and Ψ_2 be the respective complementary functions. If $\Phi_1 \prec\prec \Phi_2$, then $\Psi_1 \succ\succ \Psi_2$.*

Proof: see [15], page 114.

2.3 Orlicz spaces

Orlicz spaces generalize the concept of Lebesgue spaces. Prior to the definition of Orlicz spaces, we define Orlicz classes.

Definition 2.9. Let Φ be a Young function and $Q \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is an open subset. We say that $u \in \tilde{L}_\Phi(Q)$, if

$$\int_Q \Phi(|u(x)|)\mathrm{d}x < +\infty.$$

The set $\tilde{L}_\Phi(Q)$ is called an **Orlicz class**.

We remark that the equality of elements in $\tilde{L}_\Phi(Q)$ is the equality almost everywhere (similarly as in Lebesgue spaces) and the elements of $\tilde{L}_\Phi(Q)$ are still called "functions" without a confusion. The following two theorems are useful to obtain a better notion about Orlicz classes.

Theorem 2.10. *Let $|Q| < +\infty$ and $u \in L^1(Q)$. Then there exists a Young function Φ such that $u \in \tilde{L}_\Phi(Q)$.*

Proof: see [16], page 131.

It follows from Theorem 2.10 that $L^1(Q)$ can be viewed as the union of all Orlicz classes. The hierarchy of Orlicz classes is given by their respective Young functions.

Theorem 2.11. *Let us assume that $|Q| < +\infty$ and Φ_1, Φ_2 are two Young functions. It holds that*

$$\tilde{L}_{\Phi_2}(Q) \subset \tilde{L}_{\Phi_1}(Q)$$

if and only if

$$\Phi_1(z) \leq C\Phi_2(z), \quad \forall z \geq z_0,$$

for some $C > 0$ and $z_0 > 0$.

Proof: see [16], page 140.

An Orlicz class is only a convex subset of $L^1(Q)$ (see [16], page 130), in general. Therefore, we define Orlicz spaces.

Definition 2.12. Let $u : Q \to \mathbb{R}$, $Q \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a measurable function and let Φ, Ψ be a pair of complementary Young functions. The set $L_\Phi(Q)$ of all u such that $\|u\|_{L_\Phi(Q)} < +\infty$ is called the **Orlicz space**. The positive number $\|u\|_{L_\Phi(Q)}$ is defined as

$$\|u\|_{L_\Phi(Q)} = \sup_v \int_Q |u(x)v(x)|\mathrm{d}x,$$

where the supremum is taken over all functions $v \in \tilde{L}_\Psi(Q)$ satisfying condition $\int_Q \Psi(|v(x)|)\mathrm{d}x \leq 1$.

Theorem 2.13. *The Orlicz space $L_\Phi(Q)$ is a Banach space and $\|\cdot\|_{L_\Phi(Q)}$ is the norm on $L_\Phi(Q)$.*

Proof: see [16], pages 145 and 156.

Orlicz spaces can be alternatively defined as follows.

Definition 2.14. Let Φ be a Young function. The space $E_\Phi(Q)$ is defined as the closure of the set of all bounded measurable functions defined on Q with respect to the norm $\|\cdot\|_{L_\Phi(Q)}$.

In general, Definitions 2.12 and 2.14 are not equivalent. They coincide if and only if the Δ_2-condition holds (see Definition 2.4).

Theorem 2.15. *Let Φ be a Young function. It holds that*

$$E_\Phi(Q) \subseteq \tilde{L}_\Phi(Q) \subseteq L_\Phi(Q).$$

In addition, Φ satisfies the Δ_2-condition if and only if

$$E_\Phi(Q) = \tilde{L}_\Phi(Q) = L_\Phi(Q).$$

Proof: see [16], page 164.

Theorem 2.16. *Let Φ and Ψ be a pair of Young functions. In general, it holds that $L_\Psi(Q) = [E_\Phi(Q)]^*$.*

Proof: see [16], pages 169 and 171.

The hierarchy of Orlicz spaces is clarified in the following statements. It depends on the respective Young functions.

Theorem 2.17. *Let us suppose that Φ_1 and Φ_2 are Young functions. Then $L_{\Phi_1}(Q) \hookrightarrow L_{\Phi_2}(Q)$ if and only if $\Phi_1 \succ \Phi_2$.*

Proof: see [16], pages 185 and 187.

Remark 2.18. We note that the inclusion $L_{\Phi_1}(Q) \subset L_{\Phi_2}(Q)$ is equivalent to the embedding $L_{\Phi_1}(Q) \hookrightarrow L_{\Phi_2}(Q)$ in case of Orlicz spaces (see [16], page 187).

Corollary 2.19. Young functions Φ_1 and Φ_2 are equivalent if and only if

$$L_{\Phi_1}(Q) = L_{\Phi_2}(Q).$$

Proof: The assertion is a consequence of Definition 2.6, Theorem 2.17 and Remark 2.18.

Theorem 2.20. *Let Φ_1 and Φ_2 be Young functions. If $\Phi_1 \succ\succ \Phi_2$, then*

$$L_{\Phi_1}(Q) \hookrightarrow E_{\Phi_2}(Q).$$

Proof: see [16], page 189.

Besides the strong convergence in the Orlicz space $L_\Phi(Q)$ given in terms of the norm $\|\cdot\|_{L_\Phi(Q)}$, we can also define the E_Ψ-weak convergence.

Definition 2.21. A sequence $\{u_n\}_{n=1}^{+\infty} \subset L_\Phi(Q)$ converges $\mathbf{E_\Psi}$**-weakly** to $u \in L_\Phi(Q)$, if

$$\lim_{n\to+\infty} \int_Q (u_n(x) - u(x))v(x)\mathrm{d}x = 0, \quad \forall v \in E_\Psi(Q).$$

We write $u_n \stackrel{\Psi}{\rightharpoonup} u$.

Remark 2.22. It follows from Theorem 2.16 that the weak-* convergence in $L_\Phi(Q)$ is equivalent to the E_Ψ-weak convergence. Therefore, the boundedness of sequence $\{u_n\}_{n=1}^{+\infty}$ in $L_\Phi(Q)$ implies the existence of E_Ψ-weakly convergent subsequence of $\{u_n\}_{n=1}^{+\infty}$.

Frequently, we are not interested in functions only but we are concerned also with their derivatives. Therefore, we define Sobolev-Orlicz spaces. The definition of the Sobolev-Orlicz spaces is similar to the definition of Sobolev spaces, which were constructed from Lebesgue spaces.

Definition 2.23. The **Sobolev-Orlicz space** $W^1L_\Phi(Q)$ is the set of all functions u such that

$$\|u\|_{W^1L_\Phi(Q)} = \sqrt{\sum_{\alpha,|\alpha|\leq 1} \|D^\alpha u\|^2_{L_\Phi(Q)}} < +\infty,$$

where D^α denotes distributional derivatives, and $\|\cdot\|_{W^1 L_\Phi(Q)}$ is the norm of $W^1 L_\Phi(Q)$. Further, $W^1 E_\Phi(Q)$ and $W^1_0 L_\Phi(Q)$ are the closures of $C^\infty(\bar{Q})$ and $C^\infty_0(\bar{Q})$, respectively, with respect to $\|\cdot\|_{W^1 L_\Phi(Q)}$.

Finally, we present inequalities which are necessary for deriving estimates in the subsequent sections.

Theorem 2.24. *(Hölder's inequality)*
Let $u \in L_\Phi(Q)$ and $v \in L_\Psi(Q)$, where Φ, Ψ is a pair of complementary Young functions. Then $uv \in L^1(Q)$ and

$$\int_Q |u(x)v(x)|\mathrm{d}x \leq \|u\|_{L_\Phi(Q)} \|v\|_{L_\Psi(Q)}. \tag{2.1}$$

Proof: see [16], page 152.

Theorem 2.25. *(Young's inequality)*
Let a, $b \in \langle 0, +\infty)$ and Φ, Ψ be a complementary Young functions. It holds that

$$ab \leq \Phi(a) + \Psi(b). \tag{2.2}$$

Proof: see [16], page 65.

Corollary 2.26. Assume that Φ, Ψ is a pair of complemetary Young functions. Further, we suppose that $u \in \tilde{L}_\Phi(Q)$ and $v \in \tilde{L}_\Psi(Q)$. Then

$$\int_Q |u(x)v(x)| \ \mathrm{d}x \leq \int_Q \Phi(|u(x)|) \ \mathrm{d}x + \int_Q \Psi(|v(x)|) \ \mathrm{d}x \tag{2.3}$$

and therefore $uv \in L^1(Q)$.

Proof: see [16], page 136.

Corollary 2.27. Let $u \in L_\Phi(Q)$. It holds that

$$\|u\|_{L_\Phi(Q)} \leq \int_Q \Phi(|u(x)|)\mathrm{d}x + 1. \tag{2.4}$$

Hence, $\tilde{L}_\Phi(Q) \subset L_\Phi(Q)$.

Proof: see [16], page 145.

Theorem 2.28. *(Jensen's inequality)*
Let us assume that $\Phi : \mathbb{R} \to \mathbb{R}$ is a convex function and $\alpha(x)$ is positive almost everywhere in $Q \subset \mathbb{R}^n$, $n \in \mathbb{N}$. Then

$$\Phi\left(\frac{\int_Q \alpha(x)u(x)\mathrm{d}x}{\int_Q \alpha(x)\mathrm{d}x}\right) \leq \frac{\int_Q \alpha(x)\Phi(u(x))\mathrm{d}x}{\int_Q \alpha(x)\mathrm{d}x} \tag{2.5}$$

for any non-negative function $u : Q \to \mathbb{R}$ supposing that all the integrals in (2.5) are meaningful.

Proof: see [16], page 133.

2.4 Special Young functions

We focus on Young functions with a logarithmic or an exponential growth. These Young functions are used in the following sections to analyze the asymptotic behavior of solutions to the equations (1.1) and (1.2).

Definition 2.29. Let us define Young functions $\Phi_\gamma(z) = (1+z)\ln^\gamma(1+z)$, $\gamma > 1$, and $\Phi_1(z) = z\ln(z+1)$. Functions Ψ_γ, $\gamma \geq 1$, denote the complementary functions to Φ_γ, $\gamma \geq 1$. Subsequently, we define $M(z) = \mathrm{e}^z - z - 1$ and its complementary function $N(z) = (1+z)\ln(1+z) - z$. Further, we denote $\Phi_{1/\alpha}(z)$, $\alpha \in (1, +\infty)$, the Young functions with growth $z\ln^{1/\alpha} z$, $z \geq z_0 > 0$, and their complementary functions $\Psi_{1/\alpha}(z)$.

It is apparent that $\Phi_\gamma(z) = O(z \ln^\gamma z)$, $\gamma > 0$, and $M(z) = O(\mathrm{e}^z)$. Furthermore, $\Psi_\gamma(z) = O(\mathrm{e}^{z^{1/\gamma}})$, $\gamma > 0$, and $N(z) = O(z \ln z)$. Hence, pairs $\Phi_1(z)$ and $N(z)$, and $\Psi_1(z)$ and $M(z)$ are equivalent.

Lemma 2.30. *Functions $\Phi_1(z)$ and $N(z)$ are equivalent Young functions. Similarly, $\Psi_1(z)$ and $M(z)$ are also equivalent Young functions, because they are complementary functions to $\Phi_1(z)$ and $N(z)$, respectively. Therefore, $L_{\Phi_1}(Q) = L_N(Q)$ and also $L_{\Psi_1}(Q) = L_M(Q)$.*

Proof: It is sufficient to prove that $\Phi_1(z)$ and $N(z)$ are equivalent. The rest of the statement is a direct consequence of Lemma 2.8 and Corollary 2.19. Since $z \geq \ln(1+z)$, for all $z \geq 0$, we get

$$N(z) = \Phi_1(z) + \ln(1+z) - z \leq \Phi_1(z), \quad \forall z \geq 0.$$

On the other hand,

$$\begin{aligned} \Phi_1(z) &\leq \Phi_1(z) + \ln(1+2z) + z(\ln(1+2z) - 2) \leq \\ &\leq 2z \ln(1+2z) + \ln(1+2z) - 2z = N(2z), \quad \forall z \geq 0.5(\mathrm{e}^2 - 1), \end{aligned}$$

because $\ln(1+2z) \geq 2$, for all $z \geq 0.5(\mathrm{e}^2 - 1)$. $\square$

Remark 2.31. Since $O(z) < O(z \ln^\gamma z) < O(z^p) < O(\mathrm{e}^{z^{1/\gamma}})$, for any $\gamma > 0$ and $p \geq 2$, it stems from Theorem 2.17 that

$$L^\infty(Q) \hookrightarrow L_{\Psi_\gamma}(Q) \hookrightarrow L^p(Q) \hookrightarrow L_{\Phi_\gamma}(Q) \hookrightarrow L^1(Q).$$

By applying a similar approach as in [25], we prove the following two properties of Young functions with a logarithmic growth.

Lemma 2.32. *Young functions Φ_γ, $\gamma \geq 1$, satisfy the global Δ_2-condition.*

Proof: Directly, from the properties of logarithmic functions, we have

$$\begin{aligned} \Phi_\gamma(2z) &= (1+2z) \ln^\gamma(1+2z) \leq 2(1+z) \ln^\gamma(1+z)^2 \leq \\ &\leq 2^{\gamma+1}(1+z) \ln^\gamma(1+z) = 2^{\gamma+1}\Phi_\gamma(z), \quad \forall \gamma > 1. \end{aligned}$$

Similarly for Φ_1, we get

$$\begin{aligned}\Phi_1(2z) = 2z\ln(1+2z) &\leq 2z\ln(1+z)^2 \leq \\ &\leq 4z\ln(1+z) = 4\Phi_1(z).\end{aligned}$$

$\square$

Lemma 2.33. *Let us suppose that $\gamma_2 > \gamma_1 \geq 1$, then $\Phi_{\gamma_2} \succ\succ \Phi_{\gamma_1}$ and hence also $\Psi_{\gamma_2} \prec\prec \Psi_{\gamma_1}$.*

Proof: We define $C \in \mathbb{R}$ such that $C = 0$, if $\gamma_1 = 1$, and $C = 1$, if $\gamma_1 > 1$. Let us calculate the limit from Definition 2.7:

$$\begin{aligned}\lim_{z\to+\infty} \frac{\Phi_{\gamma_1}(z)}{\Phi_{\gamma_2}(\lambda z)} &= \lim_{z\to+\infty} \frac{(C+z)\ln^{\gamma_1}(1+z)}{(1+\lambda z)\ln^{\gamma_2}(1+\lambda z)} \\ &\leq \frac{1}{\lambda}\lim_{z\to+\infty} \frac{\ln^{\gamma_1}(1+z)}{(\ln\lambda + \ln(1+z))^{\gamma_2}} = 0,\end{aligned}$$

for $\lambda \in (0,1)$ and

$$\begin{aligned}\lim_{z\to+\infty} \frac{\Phi_{\gamma_1}(z)}{\Phi_{\gamma_2}(\lambda z)} &= \lim_{z\to+\infty} \frac{(C+z)\ln^{\gamma_1}(1+z)}{(1+\lambda z)\ln^{\gamma_2}(1+\lambda z)} \\ &\leq \lim_{z\to+\infty} \frac{\ln^{\gamma_1}(1+z)}{\ln^{\gamma_2}(1+z)} = 0,\end{aligned}$$

for $\lambda \geq 1$. $\square$

Remark 2.34. It follows from Theorem 2.20 and Lemmas 2.32 and 2.33 that:

- If $u \in L_{\Phi_\gamma}(Q)$, $\gamma \geq 1$, then $\int_Q \Phi_\gamma(|u(x)|)\ \mathrm{d}x < +\infty$, because the Δ_2-condition holds and thus $L_{\Phi_\gamma}(Q) = \tilde{L}_{\Phi_\gamma}(Q)$.
- If $u \in L_{\Psi_\gamma}(Q)$, $\gamma \geq 1$, then $\int_Q \Psi_{\gamma'}(|u(x)|)\ \mathrm{d}x < +\infty$, for all $\gamma' > \gamma$, because $\Psi_\gamma \succ\succ \Psi_{\gamma'}$ and therefore $L_{\Psi_\gamma}(Q) \hookrightarrow E_{\Psi_{\gamma'}}(Q) \subset \tilde{L}_{\Psi_{\gamma'}}(Q)$.

Chapter 3

Derivation of a 1D model

We focus on derivation of a one-dimensional model from equations (1.1)-(1.2) under Navier boundary conditions [1]. The problem in question is described in detail in section 3.1. Subsequently, the transformation of governing equations and energy equality is performed in section 3.2. Finally, section 3.3 contains the proof of our main result, which is stated in section 3.4.

3.1 Statement of the problem

We study the motion of a compressible fluid in a thin pipe. The dynamics of a compressible fluid is governed by equations (1.1)-(1.2). We employ notation $\bar{\mathbf{u}}_\varepsilon$ and $\bar{\rho}_\varepsilon$ for the velocity and the density, respectively, in equations (1.1)-(1.2) to highlight the connection to Ω_ε. Similar notation is applied also for other functions connected to Ω_ε.

Domain $\Omega_\varepsilon \subset \mathbb{R}^3$ is defined by the use of a referential domain $\Omega = (0,1) \times S$ with $S \subset \mathbb{R}^2$, $|S| = 1$ and $\partial S \in C^{0,1}$, and mapping $\mathbf{R}_\varepsilon : \Omega \to \Omega_\varepsilon$ so that

$$\mathbf{R}_\varepsilon : (x_1, x_2, x_3) \longmapsto (x_1, \varepsilon x_2, \varepsilon x_3).$$

It means that $\Omega_\varepsilon = (0,1) \times \varepsilon S$. As well as in [22], section 4.17.2.4, we suppose that Ω is not axially symmetric. Axial symmetry would mean that the appearance of Ω remains unchanged if rotated around an axis along the first spatial dimension.

Symbols $\mathbf{n}$ and $\bar{\mathbf{n}}_\varepsilon$ stand for unit outward normals to Ω and Ω_ε, respectively. Similarly, $\mathbf{t}$ and $\bar{\mathbf{t}}_\varepsilon$ are vectors from the corresponding tangent planes. We employ the following notation for the borders of domains Ω and Ω_ε:

$$\Gamma_1 = (0,1) \times \partial S,\ \Gamma_2 = \{0,1\} \times S,$$
$$\Gamma_{1,\varepsilon} = \mathbf{R}_\varepsilon\left(\Gamma_1\right),\ \Gamma_{2,\varepsilon} = \mathbf{R}_\varepsilon\left(\Gamma_2\right).$$

To ensure the well-posedness of our problem [26], we prescribe Navier boundary conditions

$$\bar{\mathbf{t}}_\varepsilon \cdot \left(P(|\bar{D}\bar{\mathbf{u}}_\varepsilon|)\bar{D}\bar{\mathbf{u}}_\varepsilon\bar{\mathbf{n}}_\varepsilon\right) + h(\varepsilon)\bar{\mathbf{u}}_\varepsilon \cdot \bar{\mathbf{t}}_\varepsilon = 0 \quad \text{on } \Gamma_{1,\varepsilon} \times (0,T), \tag{3.1}$$

$$\bar{\mathbf{t}}_\varepsilon \cdot \left(P(|\bar{D}\bar{\mathbf{u}}_\varepsilon|)\bar{D}\bar{\mathbf{u}}_\varepsilon\bar{\mathbf{n}}_\varepsilon\right) + q\bar{\mathbf{u}}_\varepsilon \cdot \bar{\mathbf{t}}_\varepsilon = 0 \quad \text{on } \Gamma_{2,\varepsilon} \times (0,T), \tag{3.2}$$

$$\bar{\mathbf{u}}_\varepsilon \cdot \bar{\mathbf{n}}_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon \times (0,T). \tag{3.3}$$

We suppose that $h(\varepsilon) > 0$ behaves like $O(\varepsilon)$ and $q > 0$. The asymptotic behavior of $h(\varepsilon)$ will be discussed during derivation of weak convergences of density and velocity field (section 3.3.3).

We consider the initial conditions for the density and the momentum

$$\bar{\rho}_\varepsilon(\bar{x}, 0) = \bar{\rho}_{0,\varepsilon}(\bar{x}) \geq 0, \quad \forall \bar{x} \in \Omega_\varepsilon$$
$$\left(\bar{\rho}_\varepsilon\bar{\mathbf{u}}_\varepsilon\right)(\bar{x}, 0) = \left(\bar{\rho}_\varepsilon\bar{\mathbf{u}}_\varepsilon\right)_0(\bar{x}, 0), \quad \forall \bar{x} \in \Omega_\varepsilon.$$

The variational formulation of our problem is

$$\int_0^T \int_{\Omega_\varepsilon} \left(\bar{\rho}_\varepsilon \partial_t \bar{\varphi} + \bar{\rho}_\varepsilon \bar{\mathbf{u}}_\varepsilon \cdot \bar{\nabla}\bar{\varphi}\right) \mathrm{d}\bar{x}\mathrm{d}t = 0, \tag{3.4}$$

$$\begin{aligned}
&\int_0^T \int_{\Omega_\varepsilon} \left(\bar{\rho}_\varepsilon \bar{\mathbf{u}}_\varepsilon \cdot \partial_t \bar{\psi} + \bar{\rho}_\varepsilon \bar{\mathbf{u}}_\varepsilon \otimes \bar{\mathbf{u}}_\varepsilon : \bar{D}\bar{\psi} + \bar{\rho}_\varepsilon \bar{\mathrm{div}}\bar{\psi}\right) \mathrm{d}\bar{x}\mathrm{d}t \\
&\quad = \int_0^T \int_{\Omega_\varepsilon} \left(P(|\bar{D}\bar{\mathbf{u}}_\varepsilon|)\bar{D}\bar{\mathbf{u}}_\varepsilon : \bar{D}\bar{\psi} - \bar{\rho}_\varepsilon \bar{\mathbf{f}}_\varepsilon \cdot \bar{\psi}\right) \mathrm{d}x\mathrm{d}t \\
&+h(\varepsilon) \int_0^T \int_{\Gamma_{1,\varepsilon}} \bar{\mathbf{u}}_\varepsilon \cdot \bar{\psi}\ \mathrm{d}\bar{\Gamma}\mathrm{d}t + q \int_0^T \int_{\Gamma_{2,\varepsilon}} \bar{\mathbf{u}}_\varepsilon \cdot \bar{\psi}\ \mathrm{d}\bar{\Gamma}\mathrm{d}t,
\end{aligned} \tag{3.5}$$

for any $\bar{\varphi} \in \mathcal{D}\left(\mathbb{R}^3 \times (0,T)\right)$ and $\bar{\psi} \in C_0^\infty(0,T; C^\infty(\bar{\Omega}_\varepsilon)^3)$ satisfying condition $\bar{\psi} \cdot \bar{\mathbf{n}}_\varepsilon|_{\partial\Omega_\varepsilon \times (0,T)} = 0$.

3.2 Transformation and related results

We transform the governing equations and the energy equality to the referential domain. First, we denote

$$
\begin{aligned}
\mathbf{u}_\varepsilon &: \Omega \times \langle 0, T\rangle \to \mathbb{R}^3,\\
\rho_\varepsilon &: \Omega \times \langle 0, T\rangle \to \mathbb{R},
\end{aligned}
$$

where $\mathbf{u}_\varepsilon(x,t) = \bar{\mathbf{u}}_\varepsilon(\mathbf{R}_\varepsilon(x),t)$ and $\rho_\varepsilon(x,t) = \bar{\rho}_\varepsilon(\mathbf{R}_\varepsilon(x),t)$, for all $x \in \Omega$. Since $\bar{x} = \mathbf{R}_\varepsilon(x)$, $\bar{x} \in \Omega_\varepsilon$, we can write $\mathbf{u}_\varepsilon(x,t) = \bar{\mathbf{u}}_\varepsilon(\bar{x},t)$ and $\rho_\varepsilon(x,t) = \bar{\rho}_\varepsilon(\bar{x},t)$.

We express the spatial gradient of a scalar function $\bar{\varphi}$ according to the chain rule as

$$
\bar{\nabla}\bar{\varphi}(\bar{x},t) = \bar{\nabla}\bar{\varphi}(\mathbf{R}_\varepsilon^{-1}(x),t) = \nabla_\varepsilon \varphi,
$$

where gradient $\nabla_\varepsilon = (\partial_1, \varepsilon^{-1}\partial_2, \varepsilon^{-1}\partial_3)$. Additionally, divergence div_ε is defined as $\mathrm{div}_\varepsilon \varphi = \partial_1\varphi + \varepsilon^{-1}\partial_2\varphi + \varepsilon^{-1}\partial_3\varphi$.

Similarly, we transform the symmetric part of the gradient of a vector function $\bar{\mathbf{u}}_\varepsilon$ and arrive at $\bar{D}\bar{\mathbf{u}}_\varepsilon(\bar{x},t) = \bar{D}\bar{\mathbf{u}}_\varepsilon(\mathbf{R}_\varepsilon^{-1}(x),t) = \omega_\varepsilon(\mathbf{u}_\varepsilon(x,t))$, where

$$
\omega_\varepsilon(\mathbf{u}_\varepsilon) = \begin{pmatrix} \partial_1 u_{1,\varepsilon} & \frac{1}{2}(\partial_1 u_{2,\varepsilon} + \varepsilon^{-1}\partial_2 u_{1,\varepsilon}) & \frac{1}{2}(\partial_1 u_{3,\varepsilon} + \varepsilon^{-1}\partial_3 u_{1,\varepsilon}) \\ \cdot & \varepsilon^{-1}\partial_2 u_{2,\varepsilon} & \frac{1}{2}\varepsilon^{-1}(\partial_2 u_{3,\varepsilon} + \partial_3 u_{2,\varepsilon}) \\ \mathrm{sym} & \cdot & \varepsilon^{-1}\partial_3 u_{3,\varepsilon} \end{pmatrix}. \tag{3.6}
$$

3.2.1 Transformation of the governing equations

According to [7], we use the following equalities

$$
\begin{aligned}
\mathrm{d}\bar{x} &= \varepsilon^2 \ \mathrm{d}x,\\
\mathrm{d}\bar{\Gamma} &= \varepsilon \ \mathrm{d}\Gamma \quad \text{on } \Gamma_1,\\
\mathrm{d}\bar{\Gamma} &= \varepsilon^2 \ \mathrm{d}\Gamma \quad \text{on } \Gamma_2
\end{aligned}
$$

to arrive at the transformed equations of the variational formulation (3.4)–(3.5).

Now, we can divide both equations by ε^2 and arrive at transformed governing equations

$$
\int_0^T \int_\Omega (\rho_\varepsilon \partial_t \varphi + \rho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_\varepsilon \varphi) \ \mathrm{d}x\mathrm{d}t = 0, \tag{3.7}
$$

$$\int_0^T \int_\Omega \left[\rho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \psi + \rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \omega_\varepsilon(\psi) + \rho_\varepsilon \mathrm{div}_\varepsilon \psi\right] \,\mathrm{d}x\mathrm{d}t$$

$$= \int_0^T \int_\Omega \left[P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right) \omega_\varepsilon(\mathbf{u}_\varepsilon) : \omega_\varepsilon(\psi) - \rho_\varepsilon \mathbf{f}_\varepsilon \cdot \psi\right] \,\mathrm{d}x\mathrm{d}t$$

$$+ \frac{h(\varepsilon)}{\varepsilon} \int_0^T \int_{\Gamma_1} \mathbf{u}_\varepsilon \cdot \psi \,\mathrm{d}\Gamma\mathrm{d}t + q \int_0^T \int_{\Gamma_2} \mathbf{u}_\varepsilon \cdot \psi \,\mathrm{d}\Gamma\mathrm{d}t, \tag{3.8}$$

for any $\varphi \in \mathcal{D}\left(\mathbb{R}^3 \times (0,T)\right)$ and $\psi \in C_0^\infty(0,T;[C^\infty(\bar{\Omega})]^3)$, $\psi \cdot \mathbf{n}|_{\partial\Omega \times (0,T)} = 0$.

Imposing the same transformation also to the renormalized continuity equation (see [17] or [19] for its original form) leads to

$$\int_0^T \int_\Omega b(\rho_\varepsilon)\partial_t \varphi + b(\rho_\varepsilon)\mathbf{u}_\varepsilon \cdot \nabla_\varepsilon \varphi + \left[\left(b(\rho_\varepsilon) - \rho_\varepsilon b'(\rho_\varepsilon)\right) \mathrm{div}_\varepsilon \mathbf{u}_\varepsilon\right] \varphi \,\mathrm{d}x\mathrm{d}t = 0, \tag{3.9}$$

for any $\varphi \in \mathcal{D}\left(\mathbb{R}^3 \times (0,T)\right)$.

3.2.2 Energy equality and its transformation

For any $t \in \langle 0, T\rangle$, we have the energy equality expressed by the following formula [19]

$$\int_{\Omega_\varepsilon} \left(\bar{\rho}_\varepsilon(t) \frac{|\bar{\mathbf{u}}_\varepsilon(t)|^2}{2} + \bar{\rho}_\varepsilon(t) \ln(\bar{\rho}_\varepsilon(t))\right) \,\mathrm{d}\bar{x} +$$

$$+ \int_0^t \int_{\Omega_\varepsilon} P(|\bar{D}\bar{\mathbf{u}}_\varepsilon|)\bar{D}\bar{\mathbf{u}}_\varepsilon : \bar{D}\bar{\mathbf{u}}_\varepsilon \,\mathrm{d}\bar{x}\,\mathrm{d}s + h(\varepsilon) \int_0^t \int_{\Gamma_{1,\varepsilon}} |\bar{\mathbf{u}}_\varepsilon|^2 \,\mathrm{d}\bar{\Gamma}\,\mathrm{d}s +$$

$$+ q \int_0^t \int_{\Gamma_{2,\varepsilon}} |\bar{\mathbf{u}}_\varepsilon|^2 \,\mathrm{d}\bar{\Gamma}\mathrm{d}s = \int_0^t \int_{\Omega_\varepsilon} \bar{\rho}_\varepsilon \bar{\mathbf{f}}_\varepsilon \cdot \bar{\mathbf{u}}_\varepsilon \,\mathrm{d}\bar{x}\mathrm{d}s +$$

$$+ \int_{\Omega_\varepsilon} \left(\frac{|(\bar{\rho}_\varepsilon \bar{\mathbf{u}}_\varepsilon)_0|^2}{2\bar{\rho}_{0,\varepsilon}} + \bar{\rho}_{0,\varepsilon} \ln(\bar{\rho}_{0,\varepsilon})\right) \,\mathrm{d}\bar{x}. \tag{3.10}$$

By transforming (3.10), we obtain

$$
\begin{aligned}
&\int_\Omega \left(\rho_\varepsilon(t)\frac{|\mathbf{u}_\varepsilon(t)|^2}{2} + \rho_\varepsilon(t)\ln(\rho_\varepsilon(t))\right)\ \mathrm{d}x \\
&+\int_0^t\int_\Omega P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right)|\omega_\varepsilon(\mathbf{u}_\varepsilon)|^2\ \mathrm{d}x\mathrm{d}s + \\
&+\frac{h(\varepsilon)}{\varepsilon}\int_0^t\int_{\Gamma_1}|\mathbf{u}_\varepsilon|^2\ \mathrm{d}\Gamma\mathrm{d}s + q\int_0^t\int_{\Gamma_2}|\mathbf{u}_\varepsilon|^2\ \mathrm{d}\Gamma\mathrm{d}s = \\
&=\int_0^t\int_\Omega \rho_\varepsilon\mathbf{g}_\varepsilon\cdot\mathbf{v}_\varepsilon\ \mathrm{d}x\mathrm{d}s + \int_\Omega\left(\frac{|(\rho_\varepsilon\mathbf{u}_\varepsilon)_0|^2}{2\rho_{0,\varepsilon}} + \rho_{0,\varepsilon}\ln(\rho_{0,\varepsilon})\right)\ \mathrm{d}x. \qquad (3.11)
\end{aligned}
$$

for any $t \in \langle 0, T\rangle$, where

$$
\begin{aligned}
\mathbf{g}_\varepsilon &= (f_{1,\varepsilon}, \varepsilon^{-1}f_{2,\varepsilon}, \varepsilon^{-1}f_{3,\varepsilon}),\\
\mathbf{v}_\varepsilon &= (u_{1,\varepsilon}, \varepsilon u_{2,\varepsilon}, \varepsilon u_{3,\varepsilon}),
\end{aligned}
$$

It is obvious that $\mathbf{g}_\varepsilon\cdot\mathbf{v}_\varepsilon = \mathbf{f}_\varepsilon\cdot\mathbf{u}_\varepsilon$, but we need to use this notation for making a priori estimates (see inequality 3.17), because a variant of Korn's inequality holds for $\mathbf{v}_\varepsilon$ (see Lemma 3.2).

3.2.3 Related results

It is necessary to mention that equations (3.7)-(3.8) with non-slip boundary conditions have a weak solution in a sufficiently regular domain for any $\varepsilon \in (0,1)$. Moreover, any weak solution satisfies the energy equality (3.11) and it can be constructed as a limit of Rothe approximations (see [19], Theorem 3.5). The non-slip boundary conditions mean that surface integrals in (3.8) and (3.11) disappears. We remark that $\gamma > 7/2$ in [19], while our result was achieved for a slightly more general $\gamma > 3$.

According to [26], we can treat the case of slip boundary conditions similarly as the barotropic case [22]. In our case, we use the Navier boundary conditions (3.1)-(3.3), because the slip boundary conditions are their special case ($h(\varepsilon) = q = 0$) and the generalization poses no additional technical problems to the existence proof. The case of non-slip boundary conditions would lead to the zero velocity in the limit. Thus, it was not an interesting choice of boundary conditions for us.

Since we are dealing with a domain which has a shape similar to a cylinder, the assumption on the regularity of the boundary of Ω can be relaxed by simplifying and slightly modifying the approach presented in [10].

3.3 Derivation of the limiting 1D equations

The first step of the proof concerns a variant of the first Korn's inequality (see section 3.3.1). We need this inequality to perform a priori estimates in section 3.3.2 and afterwards show the boundedness of $\{\rho_\varepsilon\}_{\varepsilon\in(0,1)}$ and $\{\mathbf{u}_\varepsilon\}_{\varepsilon\in(0,1)}$, and perform weak limits. Subsequently, we pass to limits in equations (3.7)-(3.8) in section 3.3.3. Finally in section 3.3.4, the limit passage is performed also for the energy equality (3.11).

3.3.1 Korn's inequality

From [9], we know that for any $\mathbf{w} \in [W^{1,p}(\Omega)]^3$, $p \geq 2$, there exists constant $C > 0$ such that the following estimate holds

$$\|\mathbf{w}\|_{1,p} \leq C \left(\|D\mathbf{w}\|_p + \|\mathbf{w}\|_p \right). \tag{3.12}$$

Lemma 3.1. *Let (3.12) hold for any $\mathbf{w} \in [W^{1,p}(\Omega)]^3$, $p \geq 2$. Then, there exists constant $\bar{C}(\Omega, p) > 0$ such that*

$$\|\mathbf{w}\|_{1,p} \leq \bar{C}(\Omega, p) \left(\|D\mathbf{w}\|_p + \|\mathbf{w}\|_{2,\Gamma} \right).$$

Proof: Let us suppose the contrary: without loss of generality, there exist a sequence $\{\mathbf{w}_n\}_{n=1}^{+\infty} \subset [W^{1,p}(\Omega)]^3$ such that $\|\mathbf{w}_n\|_{1,p} = 1$ and

$$\frac{1}{n} \geq \left(\|D\mathbf{w}_n\|_p + \|\mathbf{w}_n\|_{2,\Gamma} \right).$$

Then (passing to a subsequence if necessary), we get

$$\begin{aligned}
\mathbf{w}_n &\to \mathbf{w} \quad \text{in } [L^p(\Omega)]^3, \\
\nabla \mathbf{w}_n &\rightharpoonup \nabla \mathbf{w} \quad \text{in } [L^p(\Omega)]^9, \\
D\mathbf{w}_n &\to 0 \quad \text{in } [L^p(\Omega)]^9, \\
\mathbf{w}_n &\to 0 \quad \text{in } [L^2(\Gamma)]^3.
\end{aligned}$$

From (3.12), $\nabla \mathbf{w}_n \to \nabla \mathbf{w}$ in $[L^p(\Omega)]^9$ and thus $\|\mathbf{w}\|_{1,p} = 1$. However, $D\mathbf{w} = 0$ and $\mathbf{w}|_\Gamma = 0$. It means that $\mathbf{w} = 0$ (see [7], Theorem 1.7-3) and we arrive at contradiction with $\|\mathbf{w}\|_{1,p} = 1$. □

Without the loss of generality, we denote $\mathbf{u}_\varepsilon = \mathbf{u}_\varepsilon(t)$ in the following theorem. Variable $t \in \langle 0, T\rangle$ is arbitrary but fixed.

Theorem 3.2. *Let $\mathbf{u}_\varepsilon \in [W^{1,p}(\Omega)]^3$, $p > 3$, be such that $\mathbf{u}_\varepsilon \cdot \mathbf{n} = 0$ on $\Gamma = \{0\} \times S$. We define $\mathbf{v}_\varepsilon = (u_{1,\varepsilon}, \varepsilon u_{2,\varepsilon}, \varepsilon u_{3,\varepsilon}) \in [W^{1,p}(\Omega)]^3$. Then, there exists $C = C(\Omega, p) > 0$, such that*

$$\|\mathbf{v}_\varepsilon\|_{1,p} \leq C \left(\|\omega_\varepsilon(\mathbf{u}_\varepsilon)\|_p + \|\mathbf{u}_\varepsilon\|_{2,\Gamma} \right), \quad \forall \varepsilon > 0, \tag{3.13}$$

where $\omega_\varepsilon(\mathbf{u}_\varepsilon)$ is defined by (3.6).

Proof: Let us assume the contrary. Without the loss of generality, there exists a sequence $\{\mathbf{v}_{\varepsilon_n}\}_{n=1}^{+\infty}$ defined by $\{\mathbf{u}_{\varepsilon_n}\}_{n=1}^{+\infty}$, where $\varepsilon_n \to 0$ as n tends to infinity, such that $\|\mathbf{v}_{\varepsilon_n}\|_{1,p} = 1$ and

$$\frac{1}{n} \geq \|\omega_{\varepsilon_n}(\mathbf{u}_{\varepsilon_n})\|_p + \|\mathbf{u}_{\varepsilon_n}\|_{2,\Gamma}.$$

Hence,

$$\mathbf{u}_{\varepsilon_n} \to 0 \text{ in } [L^2(\Gamma)]^3, \quad \omega_{\varepsilon_n}(\mathbf{u}_{\varepsilon_n}) \to 0 \text{ in } [L^p(\Omega)]^9. \tag{3.14}$$

From boundedness of sequence $\{\mathbf{v}_{\varepsilon_n}\}_{n=1}^{+\infty}$ and embedding $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathcal{C}(\bar{\Omega})$, we deduce the following convergences (passing to a subsequence if necessary)

$$\mathbf{v}_{\varepsilon_n} \rightharpoonup \mathbf{v} \quad \text{in } [W^{1,p}(\Omega)]^3, \tag{3.15}$$

$$\mathbf{v}_{\varepsilon_n} \to \mathbf{v} \quad \text{in } [\mathcal{C}(\bar{\Omega})]^3. \tag{3.16}$$

According to the definition of $\mathbf{v}_{\varepsilon_n}$, it holds that

$$D\mathbf{v}_{\varepsilon_n} = \begin{pmatrix} \partial_1 u_{1,\varepsilon_n} & \frac{\varepsilon_n}{2}(\partial_1 u_{2,\varepsilon_n} + \varepsilon_n^{-1}\partial_2 u_{1,\varepsilon_n}) & \frac{\varepsilon_n}{2}(\partial_1 u_{3,\varepsilon_n} + \varepsilon_n^{-1}\partial_3 u_{1,\varepsilon_n}) \\ \cdot & \varepsilon_n \partial_2 u_{2,\varepsilon_n} & \frac{\varepsilon_n}{2}(\partial_2 u_{3,\varepsilon_n} + \partial_3 u_{2,\varepsilon_n}) \\ \mathrm{sym} & \cdot & \varepsilon_n \partial_3 u_{3,\varepsilon_n} \end{pmatrix}.$$

Due to the second convergence in (3.14) and definition of $\omega_{\varepsilon_n}(\mathbf{u}_{\varepsilon_\mathbf{n}})$ given by relation (3.6), we arrive at $D\mathbf{v}_{\varepsilon_n} \to 0$ in $[L^p(\Omega)]^9$.

Finally, we prove that $\|\mathbf{v}\|_{1,p} = 1$ and simultaneously $\mathbf{v} = 0$ to arrive at a contradiction. We apply the Korn's inequality (see Lemma 3.1) as follows

$$\|\mathbf{v}_{\varepsilon_n}\|_{1,p} \le \bar{C}(\Omega, p)\left(\|D\mathbf{v}_{\varepsilon_n}\|_p + \|\mathbf{v}_{\varepsilon_n}\|_{2,\Gamma}\right).$$

Since $\mathbf{u}_{\varepsilon_n} \to 0$ in $[L^2(\Gamma)]^3$, also $\mathbf{v}_{\varepsilon_n} \to 0$ in $[L^2(\Gamma)]^3$. Furthermore, $D\mathbf{v}_{\varepsilon_n} \to 0$ in $[L^p(\Omega)]^9$. Thus, it implies convergence $\mathbf{v}_{\varepsilon_n} \to 0$ in $[W^{1,p}(\Omega)]^3$, which together with (3.15) give us $\mathbf{v}_{\varepsilon_n} \to \mathbf{v} = 0$ in $[W^{1,p}(\Omega)]^3$. This strong convergence and $\|\mathbf{v}_{\varepsilon_n}\|_{1,p} = 1$ mean that also $\|\mathbf{v}\|_{1,p} = 1$. To sum it up, $\mathbf{v} = 0$ in Ω and $\|\mathbf{v}\|_{1,p} = 1$, which is a contradiction. $\square$

3.3.2 Boundedness and weak limits

Now, we make a priori estimates. Equation (3.7) implies the conservation of mass which can be expressed as

$$\int_\Omega \rho_\varepsilon(t)\ \mathrm{d}x = \int_\Omega \rho_{0,\varepsilon}\ \mathrm{d}x, \quad \forall t \in (0,T).$$

Therefore, the first integral on the right-hand side of the energy equality (3.11) can be estimated as follows

$$\begin{aligned} \left|\int_0^t \int_\Omega \rho_\varepsilon \mathbf{g}_\varepsilon \cdot \mathbf{v}_\varepsilon\ \mathrm{d}x\mathrm{d}s\right| &\le \int_0^t \|\mathbf{v}_\varepsilon(s)\|_\infty \|\mathbf{g}_\varepsilon(s)\|_\infty \int_\Omega \rho_\varepsilon(s)\ \mathrm{d}x\mathrm{d}s \\ &\le C(\rho_{0,\varepsilon}, \mathbf{g}_\varepsilon) \int_0^t \|\mathbf{v}_\varepsilon(s)\|_{1,p}\ \mathrm{d}s, \quad p > 3. \end{aligned}$$

In the view of inequalities (1.3) and (3.13), and the Young's inequality, we arrive at

$$\left|\int_0^t \int_\Omega \rho_\varepsilon \mathbf{g}_\varepsilon \cdot \mathbf{v}_\varepsilon \, \mathrm{d}x\mathrm{d}s\right| \leq C\left(C_1 \int_0^t \int_\Omega P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right) |\omega_\varepsilon(\mathbf{u}_\varepsilon)|^2 \mathrm{d}x\mathrm{d}s \right.$$
$$\left. + C_1 \int_0^t \int_{\{0\}\times S} |\mathbf{u}_\varepsilon|^2 \mathrm{d}S\mathrm{d}s + C_2(C_1)\right), \qquad (3.17)$$

where $C_1 > 0$ can be made arbitrarily small.

Due to (1.3) and (3.17), we obtain from (3.11) boundedness

$$\{\sqrt{\rho_\varepsilon}\,|\mathbf{u}_\varepsilon|\}_{\varepsilon\in(0,1)} \text{ in } L^\infty\left(0,T;L^2(\Omega)\right), \qquad (3.18)$$

$$\{\rho_\varepsilon\}_{\varepsilon\in(0,1)} \text{ in } L^\infty\left(0,T;L_{\Phi_1}(\Omega)\right), \qquad (3.19)$$

$$\{\omega_\varepsilon(\mathbf{u}_\varepsilon)\}_{\varepsilon\in(0,1)} \text{ in } [\tilde{L}_M\left(\Omega\times(0,T)\right)]^9, \qquad (3.20)$$

$$\{\mathbf{v}_\varepsilon\}_{\varepsilon\in(0,1)} \text{ in } L^p(0,T;[W^{1,p}(\Omega)]^3)\cap L^2(0,T;[L^2(\partial\Omega)]^3) \qquad (3.21)$$

for any $p > 3$. From (3.21), we get immediately the boundedness

$$\{u_{1,\varepsilon}\}_{\varepsilon\in(0,1)} \text{ in } L^p\left(0,T;W^{1,p}(\Omega)\right)\cap L^2\left(0,T;L^2(\partial\Omega)\right). \qquad (3.22)$$

Boundedness (3.20) gives us the following convergences

$$\partial_2 u_{3,\varepsilon} + \partial_3 u_{2,\varepsilon} \to 0, \quad \partial_\alpha u_{\alpha,\varepsilon} \to 0, \quad \text{in } L_M(\Omega\times(0,T)),\ \alpha = 2,3.$$

Now, we can prove even the boundedness of $\{\varepsilon^{-1}u_{\alpha,\varepsilon}\}_{\varepsilon\in(0,1)}$, $\alpha = 2,3$, in $L_M(\Omega\times(0,T))$. Let us denote $\mathbf{w}_\varepsilon = (\varepsilon^{-1}u_{2,\varepsilon}, \varepsilon^{-1}u_{3,\varepsilon})$. We begin with the Korn's inequality in a two-dimensional space [9]:

$$\|\mathbf{w}_\varepsilon\|_{W^{1,p}(S)} \leq C_1\left(\|D\mathbf{w}_\varepsilon\|_{L^p(S)} + \|\mathbf{w}_\varepsilon\|_{L^p(S)}\right), \quad p > 2, \qquad (3.23)$$

where $x_1 \in (0,1)$ and $t \in (0,T)$ are arbitrary but fixed. From (3.23) and axial non-symmetry of Ω, via the standard compactness argument (as in [22] for proving inequality (4.17.19)), we deduce

$$\|\mathbf{w}_\varepsilon\|_{L^p(S)} \leq C_2\,\|D\mathbf{w}_\varepsilon\|_{L^p(S)}. \qquad (3.24)$$

Due to compact embedding of $W^{1,p}(S)$ in $L^\infty(S)$ and inequality (3.24), we can arrive from (3.23) to the following inequality

$$\|C\mathbf{w}_\varepsilon\|^p_{L^\infty(S)} \leq \|D\mathbf{w}_\varepsilon\|^p_{L^p(S)}\,, \quad p > 2,$$

where $C = C(S,p) > 0$. Applying Young function Ψ_p and Jensen's inequality gives us

$$\int_S \Psi_p(|C\mathbf{w}_\varepsilon|^p)\mathrm{d}x_2\mathrm{d}x_3 \leq C_3 \int_S \Psi_p(|D\mathbf{w}_\varepsilon|^p)\mathrm{d}x_2\mathrm{d}x_3, \quad p > 2.$$

Since $\Psi_p(z^p)$ behaves like $M(z)$, we arrive at

$$\int_S M(|C\mathbf{w}_\varepsilon|)\mathrm{d}x_2\mathrm{d}x_3 \leq C_3 \left(\int_S M(|D\mathbf{w}_\varepsilon|)\mathrm{d}x_2\mathrm{d}x_3 + 1 \right).$$

After integrating over $x_1 \in (0,1)$ and $t \in (0,T)$, we get

$$\int_0^T \int_\Omega M(|C\mathbf{w}_\varepsilon|)\mathrm{d}x\mathrm{d}t \leq C_3 \left(\int_0^T \int_\Omega M(|D\mathbf{w}_\varepsilon|)\mathrm{d}x\mathrm{d}t + T \right). \tag{3.25}$$

Let us remark that (see inequality (2.4))

$$C\left\|\mathbf{w}_\varepsilon\right\|_{L_M(\Omega\times(0,T))} \leq \int_0^T \int_\Omega M(|C\mathbf{w}_\varepsilon|)\mathrm{d}x\mathrm{d}t + 1. \tag{3.26}$$

Inequalities (3.25) and (3.26) give us

$$C\left\|\mathbf{w}_\varepsilon\right\|_{L_M(\Omega\times(0,T))} \leq C_3 \left(\int_0^T \int_\Omega M(|D\mathbf{w}_\varepsilon|)\mathrm{d}x\mathrm{d}t + T \right) + 1. \tag{3.27}$$

The right-hand side of inequality (3.27) is bounded for any $\varepsilon \in (0,1)$ due to (3.20). Thus, it ensures the boundedness

$$\{\varepsilon^{-1}u_{\alpha,\varepsilon}\}_{\varepsilon\in(0,1)} \text{ in } L_M(\Omega\times(0,T)),\ \alpha = 2,3. \tag{3.28}$$

Boundedness of $\{\rho_\varepsilon\}_{\varepsilon\in(0,1)}$ in $L^\infty(0,T;L_{\Phi_1}(\Omega))$ can be extended to the space $L^\infty(0,T;L_{\Phi_\gamma}(\Omega))$. We remind that $\gamma > 3$ (see Theorem 3.4 in section 3.4).

We proceed in the following way. First, we test the equation (3.9) by function $\varphi = \varphi(t) \in \mathcal{C}_0^\infty(0,T)$ with $b(z) = \Phi_\gamma(z)$. We get

$$\int\limits_0^T \int_\Omega \Phi_\gamma(\rho_\varepsilon)\varphi'(t) + \left[\left(\Phi_\gamma(\rho_\varepsilon) - \rho_\varepsilon \Phi'_\gamma(\rho_\varepsilon)\right) \operatorname{div}_\varepsilon \mathbf{u}_\varepsilon\right] \varphi(t) \, \mathrm{d}x\mathrm{d}t = 0, \tag{3.29}$$

Function $\Phi_\gamma(z) - z\Phi'_\gamma(z)$ behaves asymptotically like $\Phi_{\gamma-1}(z)$. Furthermore, there exists $C_1 > 0$ such that $\Phi_1(\Phi_{\gamma-1}(z)) \leq C_1(\Phi_\gamma(z)+1)$ for $z \geq 0$ [26]. Due to equivalence of Young functions M and Ψ_1, relations (1.3), (3.20) and the Young's inequality, we deduce the estimate

$$\begin{aligned} &\left|\int_0^T \int_\Omega \left(\Phi_\gamma(\rho_\varepsilon) - \rho_\varepsilon \Phi'_\gamma(\rho_\varepsilon)\right) \operatorname{div}_\varepsilon \mathbf{u}_\varepsilon \, \mathrm{d}x\mathrm{d}t\right| \\ &\leq C(T) \left(\int_0^T \int\limits_\Omega \left(\Phi_\gamma(\rho_\varepsilon) + P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right) |\omega_\varepsilon(\mathbf{u}_\varepsilon)|^2\right) \, \mathrm{d}x\mathrm{d}t + 1\right), \end{aligned} \tag{3.30}$$

where $C(T) > 0$. With respect to (3.29), (3.30), (3.66) and the Gronwall's lemma, we obtain the boundedness of

$$\{\rho_\varepsilon\}_{\varepsilon\in(0,1)} \text{ in } L^\infty(0,T; L_{\Phi_\gamma}(\Omega)). \tag{3.31}$$

In the following step, we focus on boundedness of $\{\partial_t \rho_\varepsilon\}_{\varepsilon\in(0,1)}$. Let us test equation (3.7) by function $\varphi(x,t) = \varphi_1(t)\psi(x)$, where $\varphi_1 \in L^{p'}(0,T)$, $1/p + 1/p' = 1$, $p > 3$, and $\psi \in [W^1 L_{\Psi_{\gamma-1}}(\Omega)]^3$, $\gamma > 3$. We can write

$$\begin{aligned} &\left|\int_0^T \varphi'_1 \int_\Omega \rho_\varepsilon \psi \, \mathrm{d}x\mathrm{d}t\right| = \left|\int_0^T \varphi_1 \int_\Omega \rho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_\varepsilon \psi \, \mathrm{d}x\mathrm{d}t\right| = \\ &= \left|\int_0^T \varphi_1 \int_\Omega \rho_\varepsilon (u_{1,\varepsilon}\partial_1\psi + \varepsilon^{-1}u_{2,\varepsilon}\partial_2\psi + \varepsilon^{-1}u_{3,\varepsilon}\partial_3\psi) \, \mathrm{d}x\mathrm{d}t\right|. \end{aligned} \tag{3.32}$$

From (3.28) and (3.32), we get the boundedness of

$$\{\partial_t \rho_\varepsilon\}_{\varepsilon\in(0,1)} \text{ in } L^p\left(0,T; [W^1 L_{\Psi_{\gamma-1}}(\Omega)]^*\right). \tag{3.33}$$

For instance, boundedness of the last term on the right-hand side of equation (3.32) can be demonstrated as follows

$$\left|\int_0^T \int_\Omega \rho_\varepsilon \varepsilon^{-1} u_{3,\varepsilon} \varphi_1 \partial_3 \psi \ \mathrm{d}x\mathrm{d}t\right| \leq \left\|\varepsilon^{-1} u_{3,\varepsilon}\right\|_{L_M(\Omega\times(0,T))} \left\|\rho_\varepsilon \varphi_1 \partial_3 \psi\right\|_{L_N(\Omega\times(0,T))},$$

where the boundedness of the first norm on the right-hand has been already proved – see (3.28). The second norm is less or equal than (see inequality (2.4))

$$\begin{aligned}
&\int_0^T \int_\Omega N(\rho_\varepsilon |\varphi_1||\partial_3\psi|) \ \mathrm{d}x\mathrm{d}t + 1 \leq \\
&\leq \int_0^T \int_\Omega |\varphi_1||\partial_3\psi| N(\rho_\varepsilon) + \rho_\varepsilon |\partial_3\psi| N(|\varphi_1|) + \rho_\varepsilon |\varphi_1| N(|\partial_3\psi|) \ \mathrm{d}x\mathrm{d}t + C,
\end{aligned}$$

where "the worst term" can be finally estimated as

$$\begin{aligned}
&\int_0^T |\varphi_1| \int_\Omega |\partial_3\psi| N(\rho_\varepsilon) \ \mathrm{d}x\mathrm{d}t \leq \int_0^T |\varphi_1| \int_\Omega \Psi_{\gamma-1}(|\partial_3\psi|) + \Phi_\gamma(\rho_\varepsilon) \ \mathrm{d}x\mathrm{d}t \leq \\
&\leq \|\varphi_1\|_{L^1(0,T)} \left(\int_\Omega \Psi_{\gamma-1}(|\partial_3\psi|) \ \mathrm{d}x + \left\| \int_\Omega \Phi_\gamma(\rho_\varepsilon) \ \mathrm{d}x \right\|_{L^\infty(0,T)} \right).
\end{aligned}$$

By the use of (3.19)-(3.21), (3.31), (3.33), compact embedding of $W^{1,p}(\Omega)$ in $E_{\Psi_1}(\Omega)$, isometric isomorphism of $[E_{\Psi_1}(0,1)]^*$ and $L_{\Phi_1}(0,1)$, continuous embedding of $[W^{1,p}(\Omega)]^*$ in $[W^1 L_{\Psi_\gamma}(\Omega)]^*$ and theorem on compact embedding [23], we get (passing to subsequences if necessary)

$$\rho_\varepsilon \overset{*}{\rightharpoonup} \rho \ \text{ in } L^\infty\left(0,T; L_{\Phi_\gamma}(\Omega)\right), \tag{3.34}$$

$$\rho_\varepsilon \to \rho \ \text{ in } \mathcal{C}(\langle 0,T\rangle; [W^{1,p}(\Omega)]^*), \tag{3.35}$$

$$\omega_\varepsilon(\mathbf{u}_\varepsilon) \overset{N}{\rightharpoonup} \zeta \tag{3.36}$$

$$u_{1,\varepsilon} \rightharpoonup u_1 \ \text{ in } L^p\left(0,T; W^{1,p}(\Omega)\right) \cap L^2\left(0,T; L^2(\partial\Omega)\right). \tag{3.37}$$

Let us remind that we already have (from (3.28))

$$u_{\alpha,\varepsilon} \to 0 = u_\alpha \quad \text{in } L_M(\Omega \times (0,T)), \ \alpha = 2,3, \tag{3.38}$$

which means that $u_2 = u_3 = 0$ almost everywhere in $\Omega \times (0,T)$.

We prove that the limiting function $\mathbf{u}$ does not depend on the second and the third spatial variables. Boundedness (3.20) implies the following convergences

$$\varepsilon \partial_1 u_{\alpha,\varepsilon} + \partial_\alpha u_{1,\varepsilon} \to 0, \quad \text{in } L_M(\Omega \times (0,T)),\ \alpha = 2,3.$$

With respect to (3.38), we arrive at $\partial_\alpha u_1 = 0$ almost everywhere in $\Omega \times (0,T)$, $\alpha = 2,3$. Hence, we get $u_1 = u_1(x_1,t) \in L^p(0,T;W^{1,p}(0,1))$ with $u_1(0,t) = u_1(1,t) = 0$, $t \in (0,T)$.

Let us pay our attention to convergences of nonlinear terms in equation (3.8). Convergences (passing to subsequences if necessary)

$$\rho_\varepsilon u_{1,\varepsilon} \rightharpoonup \rho u_1 \quad \text{in } L^p(0,T;L_{\Phi_\gamma}(\Omega)) \tag{3.39}$$

$$\rho_\varepsilon u_{\alpha,\varepsilon} \to 0 \quad \text{in } L_{\Phi_{\gamma-1}}(\Omega \times (0,T)),\ \alpha = 2,3, \tag{3.40}$$

where $\gamma > 3$ (see Theorem 3.4), are immediate consequences of (3.35), (3.37), (3.38) and theorem concerning compact embedding [23]. For instance, we prove convergences (3.40). According to the Hölder's inequality, it holds that

$$\begin{aligned} \|\rho_\varepsilon u_{\alpha,\varepsilon}\|_{L_{\Phi_{\gamma-1}}(\Omega\times(0,T))} &= \sup_\varphi \int_0^T \int_\Omega |\rho_\varepsilon u_{\alpha,\varepsilon}\varphi| \,\mathrm{d}x\mathrm{d}t \leq \\ &\leq C\, \|u_{\alpha,\varepsilon}\|_{L_M(\Omega\times(0,T))} \sup_\varphi \|\rho_\varepsilon \varphi\|_{L_N(\Omega\times(0,T))} \end{aligned}$$

where the supremum is taken over all functions $\varphi \in \tilde{L}_{\Psi_{\gamma-1}}(\Omega \times (0,T))$ such that

$$\int_0^T \int_\Omega \Psi_{\gamma-1}(|\varphi|) \,\mathrm{d}x\mathrm{d}t \leq 1.$$

From (3.38), we already know that $\|u_{\alpha,\varepsilon}\|_{L_M(\Omega\times(0,T))} \to 0$. Therefore, it is sufficient to show the boundedness of $\|\rho_\varepsilon \varphi\|_{L_N(\Omega\times(0,T))}$ for proving (3.40). The equivalence of Orlicz spaces L_N and L_{Φ_1}, and the Young's inequality give us

$$\|\rho_\varepsilon \varphi\|_{L_N(\Omega\times(0,T))} \leq \int_0^T \int_\Omega \Phi_1(\rho_\varepsilon|\varphi|)\ \mathrm{d}x\mathrm{d}t + C \leq$$

$$\leq \int_0^T \int_\Omega \rho_\varepsilon \Phi_1(|\varphi|)\ \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega |\varphi|\Phi_1(\rho_\varepsilon)\ \mathrm{d}x\mathrm{d}t + C. \tag{3.41}$$

The second integral on the right-hand side of (3.41) is "the worst" and it is less or equal than

$$\int_0^T \int_\Omega \Psi_{\gamma-1}(|\varphi|)\ \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \Phi_{\gamma-1}(\Phi_1(\rho_\varepsilon))\ \mathrm{d}x\mathrm{d}t \leq$$

$$\leq \int_0^T \int_\Omega \Psi_{\gamma-1}(|\varphi|)\ \mathrm{d}x\mathrm{d}t + C\int_0^T \int_\Omega \Phi_\gamma(\rho_\varepsilon)\ \mathrm{d}x\mathrm{d}t.$$

Hence, we conclude that convergences (3.40) hold true.

To overcome the second term on the left-hand side in equation (3.8), we consider "the worst integrals" in (3.8) and prove their boundedness. First, we show that (3.22), (3.31) and (3.38) lead to boundedness of

$$\int_0^T \int_\Omega \rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \omega_\varepsilon(\psi)\ \mathrm{d}x\mathrm{d}t \tag{3.42}$$

for any $\varepsilon \in (0,1)$ and test function ψ such that $\psi \cdot \mathbf{n}|_{\partial\Omega\times(0,T)} = 0$. There are three types of terms in (3.42), but we analyze in detail only "the worst one": $\rho_\varepsilon u_{\alpha,\varepsilon} u_{\beta,\varepsilon}[\omega_\varepsilon(\psi)]_{\alpha\beta}$, $\alpha,\beta = 2,3$. Let us apply notation $[\omega_\varepsilon(\psi(x,t))]_{\alpha\beta} = \varepsilon^{-1}\varphi(t)\bar{\psi}(x)$ with $\varphi \in L^q(0,T)$ and $\bar{\psi} \in E_{\Psi_\gamma}(\Omega)$, $2/p + 1/q = 1$ and $\gamma > 3$ (see Theorem 3.4 in section 3.4).

By the use of Hölder's inequality, we get

$$\left|\int_0^T \int_\Omega \rho_\varepsilon u_{\alpha,\varepsilon} u_{\beta,\varepsilon} \varepsilon^{-1}\varphi\bar{\psi}\ \mathrm{d}x\mathrm{d}t\right| \leq$$

$$\leq \left\|\varepsilon^{-1} u_{\alpha,\varepsilon} u_{\beta,\varepsilon}\right\|_{L_{\Psi_2}(\Omega\times(0,T))} \left\|\rho_\varepsilon \varphi\bar{\psi}\right\|_{L_{\Phi_2}(\Omega\times(0,T))} \tag{3.43}$$

Both norms on the right-hand side of inequality (3.43) are bounded. Regarding $\|\varepsilon^{-1}u_{\alpha,\varepsilon}u_{\beta,\varepsilon}\|_{L_{\Psi_2}(\Omega\times(0,T))}$, it holds that

$$\left\|\varepsilon^{-1}u_{\alpha,\varepsilon}u_{\beta,\varepsilon}\right\|_{L_{\Psi_2}(\Omega\times(0,T))} = \sup_{\varphi_1}\int_0^T\int_\Omega |\varepsilon^{-1}u_{\alpha,\varepsilon}u_{\beta,\varepsilon}\varphi_1|\ \mathrm{d}x\mathrm{d}t \leq$$

$$\leq \|u_{\alpha,\varepsilon}\|_{L_M(\Omega\times(0,T))}\sup_{\varphi_1}\left\|\varepsilon^{-1}u_{\beta,\varepsilon}\varphi_1\right\|_{L_N(\Omega\times(0,T))},$$

where $\varphi_1 \in \tilde{L}_{\Phi_2}(\Omega\times(0,T))$ such that $\int_0^T\int_\Omega \Phi_2(|\varphi_1|)\ \mathrm{d}x\mathrm{d}t \leq 1$. From (3.38), we know that $\|u_{\alpha,\varepsilon}\|_{L_M(\Omega\times(0,T))} \to 0$. Further, we can write that

$$\left\|\varepsilon^{-1}u_{\beta,\varepsilon}\varphi_1\right\|_{L_N(\Omega\times(0,T))} = \sup_{\varphi_2}\int_0^T\int_\Omega |\varepsilon^{-1}u_{\beta,\varepsilon}\varphi_1\varphi_2|\ \mathrm{d}x\mathrm{d}t \leq$$

$$\leq \left\|\varepsilon^{-1}u_{\beta,\varepsilon}\right\|_{L_M(\Omega\times(0,T))}\sup_{\varphi_2}\|\varphi_1\varphi_2\|_{L_N(\Omega\times(0,T))},$$

where the first norm is bounded (see 3.28) and $\varphi_2 \in \tilde{L}_M(\Omega\times(0,T))$ such that $\int_0^T\int_\Omega M(|\varphi_2|)\ \mathrm{d}x\mathrm{d}t \leq 1$. Finally,

$$\|\varphi_1\varphi_2\|_{L_N(\Omega\times(0,T))} \leq C_1\int_0^T\int_\Omega \Phi_1(|\varphi_1\varphi_2|)\mathrm{d}x\mathrm{d}t + 1,$$

where the integral on the right-hand side is lower or equal than

$$C_2\left(\int_0^T\int_\Omega |\varphi_2|\Phi_1(|\varphi_1|) + |\varphi_1|\Phi_1(|\varphi_2|)\mathrm{d}x\mathrm{d}t\right) \leq$$

$$\leq C_3\left(\int_0^T\int_\Omega M(|\varphi_2|)\mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega \Phi_2(|\varphi_1|)\mathrm{d}x\mathrm{d}t\right) \leq 2C_3$$

Hence, $\|\varepsilon^{-1}u_{\alpha,\varepsilon}u_{\beta,\varepsilon}\|_{L_{\Psi_2}(\Omega\times(0,T))} \to 0$ as $\varepsilon \to 0$.

Concerning the second norm on the right-hand side of inequality (3.43), we justify its boundedness in the following way

$$\|\rho_\varepsilon\varphi\bar{\psi}\|_{L_{\Phi_2}(\Omega\times(0,T))} \leq \int_0^T\int_\Omega \Phi_2(\rho_\varepsilon|\varphi||\bar{\psi}|)\ \mathrm{d}x\mathrm{d}t + C_1 \leq$$

$$\leq \int_0^T\int_\Omega |\varphi||\bar{\psi}|\Phi_2(\rho_\varepsilon)\ \mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega \rho_\varepsilon\Phi_2(|\varphi||\bar{\psi}|)\ \mathrm{d}x\mathrm{d}t +$$

$$+2\int_0^T\int_\Omega \Phi_1(\rho_\varepsilon)\Phi_1(|\varphi||\bar{\psi}|)\ \mathrm{d}x\mathrm{d}t + C_2,$$

where "the worst term" can be estimated as

$$\int_0^T \int_\Omega |\varphi||\bar{\psi}|\Phi_2(\rho_\varepsilon)\ \mathrm{d}x\mathrm{d}t \leq$$

$$\leq C\|\varphi\|_{L^1(0,T)} \left(\left\| \int_\Omega \Phi_\gamma(\rho_\varepsilon)\ \mathrm{d}x \right\|_{L^\infty(0,T)} + \int_\Omega \Psi_{\gamma-2}(|\bar{\psi}|)\ \mathrm{d}x \right).$$

In summary, the first norm on the right hand side of inequality (3.43) converges to zero and the second norm is bounded. It implies that

$$\left| \int_0^T \int_\Omega \rho_\varepsilon u_{\alpha,\varepsilon} u_{\beta,\varepsilon} \varepsilon^{-1} \varphi\bar{\psi}\ \mathrm{d}x\mathrm{d}t \right| \to 0 \quad \text{for } \varepsilon \to 0.$$

We conclude that integral (3.42) is bounded for any $\varepsilon \in (0,1)$ and test function ψ such that $\psi \cdot \mathbf{n}|_{\partial\Omega\times(0,T)} = 0$, $\varphi \in L^q(0,T)$ and $\bar{\psi} \in [E_{\Psi_\gamma}(\Omega)]^9$. Subsequently, we show that also

$$\int_0^T \int_\Omega P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right) \omega_\varepsilon(\mathbf{u}_\varepsilon) : \omega_\varepsilon(\psi)\ \mathrm{d}x\mathrm{d}t \tag{3.44}$$

is bounded for any $\varepsilon \in (0,1)$. For the sake of simplicity, we employ the same notation in the decomposition of ψ into its spatial and temporal part as in the analysis of integral (3.42), i. e. $\psi(x,t) = \varphi(t)\bar{\psi}(x)$, where $\varphi \in E_{\Psi_{1/\alpha}}(0,T)$, $\alpha > 2$, and $\bar{\psi} \in [W^1E_{\Psi_{1/2}}(\Omega)]^3$, $\partial_2\bar{\psi} = \partial_3\bar{\psi} = 0$.

We remark that

$$\omega_\varepsilon(\bar{\psi}) = \omega(\bar{\psi}) = \begin{pmatrix} \partial_1\bar{\psi}_1 & \frac{1}{2}\partial_1\bar{\psi}_2 & \frac{1}{2}\partial_1\bar{\psi}_3 \\ \cdot & 0 & 0 \\ \text{sym} & \cdot & 0 \end{pmatrix},$$

which is not longer dependent on ε. Due to Young's inequality, it holds that

$$\left| \int_0^T \int_\Omega P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right) \omega_\varepsilon(\mathbf{u}_\varepsilon) : \omega(\bar{\psi})\varphi\ \mathrm{d}x\mathrm{d}t \right| \leq \left(|\Omega| \int_0^T \Psi_{1/\alpha}(|\varphi|)\ \mathrm{d}t + \right.$$

$$\left. + \int_0^T \int_\Omega \Phi_{1/\alpha}\left(P(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|)|\omega_\varepsilon(\mathbf{u}_\varepsilon)||\omega(\bar{\psi})| \right) \mathrm{d}x\mathrm{d}t \right), \tag{3.45}$$

where $\alpha > 2$. For brevity, let us denote $w_\varepsilon = P(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|)|\omega_\varepsilon(\mathbf{u}_\varepsilon)|$. Since $w_\varepsilon \in L_{\Phi_1}(\Omega\times(0,T))$ implies $w_\varepsilon \in L_{\Phi_{(\alpha-1)/\alpha}}(0,T; L_{\Phi_{1/\alpha}}(\Omega))$, which follows from Jensen's inequality and estimate

$$\Phi_{(\alpha-1)/\alpha}(\Phi_{1/\alpha}(z)) \leq 2\Phi_1(z) + C, \quad z \geq 0, \tag{3.46}$$

the second term on the right-hand side of (3.45) is less or equal than

$$\begin{aligned}
&\int_0^T \int_\Omega |\omega(\bar{\psi})|\Phi_{1/\alpha}(w_\varepsilon) + w_\varepsilon \Phi_{1/\alpha}(|\omega(\bar{\psi})|) \, \mathrm{d}x\mathrm{d}t \leq \\
&\leq \int_0^T \int_\Omega \Phi_{(\alpha-1)/\alpha}(\Phi_{1/\alpha}(w_\varepsilon)) + \Psi_{(\alpha-1)/\alpha}(|\omega(\bar{\psi})|) \, \mathrm{d}x\mathrm{d}t + \\
&+ \int_0^T \int_\Omega \Phi_1(w_\varepsilon) + \Psi_1(\Phi_{1/\alpha}(|\omega(\bar{\psi})|)) \, \mathrm{d}x\mathrm{d}t \leq \\
&\leq 3\int_0^T \int_\Omega \Phi_1\left(P(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|)|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right) \mathrm{d}x\mathrm{d}t + \\
&+ \int_0^T \int_\Omega \Psi_{(\alpha-1)/\alpha}\left(|\omega(\bar{\psi})|\right) \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \Psi_{1/2}\left(|\omega(\bar{\psi})|\right) \mathrm{d}x\mathrm{d}t + C,
\end{aligned}$$

where $\alpha > 2$. Due to property (1.6), we conclude that integral (3.44) is bounded.

Terms (3.42) and (3.44) represent "the worst integrals" in (3.8). Thus, we omit the estimates of the others and take $\psi(x,t) = \varphi(t)\bar{\psi}(x)$, where $\varphi \in E_{\Psi_{1/\alpha}}(0,T)$ with $\alpha > 2$, and $\bar{\psi} \in [W^1E_{\Psi_{1/2}}(\Omega)]^3$ such that $\bar{\psi} = (\bar{\psi}_1(x_1), 0, 0)$ and complies with $\bar{\psi}_1(0) = \bar{\psi}_1(1) = 0$, as a test function. By the use of estimates (3.42) and (3.44), we demonstrate how to perform a limit passage in the second term on the left-hand side of equation (3.8). Let us test the equation (3.8) by function $\psi(x,t) = \varphi(t)\bar{\psi}(x)$, where $\varphi \in \mathcal{C}_0^\infty(0,T)$ and $\bar{\psi} \in [W^1E_{\Psi_{1/2}}(\Omega)]^3$, $\bar{\psi} = (\bar{\psi}_1(x_1), 0, 0)$ and $\bar{\psi}_1(0) = \bar{\psi}_1(1) = 0$. We get

$$\begin{aligned}
&\left|\int_0^T \varphi' \int_\Omega \rho_\varepsilon \mathbf{u}_\varepsilon \cdot \bar{\psi} \, \mathrm{d}x\mathrm{d}t\right| \leq \int_0^T |\varphi| \int_\Omega \left(|\rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \omega(\bar{\psi})| + |\rho_\varepsilon \partial_1 \bar{\psi}| + \right. \\
&\left. + |P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right)\omega_\varepsilon(\mathbf{u}_\varepsilon) : \omega(\bar{\psi})| + |\rho_\varepsilon \mathbf{f}_\varepsilon \cdot \bar{\psi}|\right) \, \mathrm{d}x\mathrm{d}t + \\
&+ \frac{h(\varepsilon)}{\varepsilon} \int_0^T |\varphi| \int_{\Gamma_1} |\mathbf{u}_\varepsilon \cdot \bar{\psi}| \, \mathrm{d}\Gamma\mathrm{d}t.
\end{aligned} \tag{3.47}$$

Let us remark that

$$q\int_0^T |\varphi| \int_{\Gamma_2} |\mathbf{u}_\varepsilon \cdot \bar{\psi}| \, \mathrm{d}\Gamma \mathrm{d}t = 0$$

due to the choice of test function ($\bar{\psi} = 0$ on Γ_2).

Considering the density of $\mathcal{C}_0^\infty(0,T)$ in $E_{\Psi_{1/2}}(0,T)$, embedding $L_{\Psi_{1/\alpha}}(0,T) \hookrightarrow E_{\Psi_{1/2}}(0,T) \subset \tilde{L}_{\Psi_{1/2}}(0,T)$, $\alpha > 2$, and boundedness of all terms on the right-hand side of the inequality (3.47) – see (3.42) and (3.44), we deduce boundedness

$$\{\partial_t \int_S \rho_\varepsilon \mathbf{u}_\varepsilon \, \mathrm{d}x_2 x_3\}_{\varepsilon\in(0,1)} \text{ in } L_{\Phi_{1/\alpha}}(0,T;([W^1 L_{\Psi_{1/2}}(0,1)]^*)^3). \tag{3.48}$$

By the use of (3.39), (3.48), compact embedding of $W^{1,p}(0,1)$ in $E_{\Psi_1}(0,1)$, isometric isomorphism of $[E_{\Psi_1}(0,1)]^*$ and $L_{\Phi_1}(0,1)$, continuous embedding of $[W^{1,p}(0,1)]^*$ in $[W^1 L_{\Psi_1}(0,1)]^*$ and theorem concerning compact embedding [23], we get (passing to subsequences if necessary)

$$\begin{aligned} &\int_S \rho_\varepsilon u_{1,\varepsilon} \, \mathrm{d}x_2 \mathrm{d}x_3 \to \int_S \rho u_1 \, \mathrm{d}x_2 \mathrm{d}x_3 \\ &\text{in } \mathcal{C}(\langle 0,T\rangle\,; [W^{1,p}(0,1)]^*). \end{aligned} \tag{3.49}$$

In order to perform a limit passage in the second term on the left-hand side of equation (3.8), we need the following lemma which can be proven in a similar way as Proposition 3.2 in [27] and Lemma 6.2 in [2].

Lemma 3.3. *Assume that $\{\mathbf{u}_\varepsilon\}_{\varepsilon\in(0,1)}$ satisfies condition (3.20) and $\{\mathbf{v}_\varepsilon\}_{\varepsilon\in(0,1)}$, where $\mathbf{v}_\varepsilon = (u_{1,\varepsilon}, \varepsilon u_{2,\varepsilon}, \varepsilon u_{3,\varepsilon})$, satisfies condition (3.21). Then for any $p > 3$ (passing to a subsequence if necessary), it holds that*

$$\|u_{1,\varepsilon} - \frac{1}{|S|}\int_S u_{1,\varepsilon} \, \mathrm{d}x_2 \mathrm{d}x_3\|_{L^p(0,T;L^\infty(\Omega))} \to 0, \ \textit{for } \varepsilon \to 0. \tag{3.50}$$

Lemma 3.3 can be applied in the following way. It holds that

$$
\begin{aligned}
&\int_0^T \int_0^1 \int_S \rho_\varepsilon u_{1,\varepsilon} u_{1,\varepsilon} \psi \, \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_1 \mathrm{d}t = \\
&= \int_0^T \int_0^1 \int_S \rho_\varepsilon u_{1,\varepsilon} (u_{1,\varepsilon} - \frac{1}{|S|} \int_S u_{1,\varepsilon} \mathrm{d}x_2 \mathrm{d}x_3) \psi \, \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_1 \mathrm{d}t + \\
&+ \int_0^T \int_0^1 \left(\int_S \rho_\varepsilon u_{1,\varepsilon} \, \mathrm{d}x_2 \mathrm{d}x_3 \right) \left(\frac{1}{|S|} \int_S u_{1,\varepsilon} \mathrm{d}x_2 \mathrm{d}x_3 \right) \psi \, \mathrm{d}x_1 \mathrm{d}t
\end{aligned}
$$

where $\psi \in \mathcal{C}_0^\infty(0,T;\mathcal{C}^\infty(\bar{\Omega}))$, $\partial_2 \psi = \partial_3 \psi = 0$. The first integral on the right-hand side tends to zero for $\varepsilon \to 0$ due to convergence (3.39) and Lemma 3.3. Concerning the second integral, we apply strong convergence (3.49) and weak convergence

$$
\int_S u_{1,\varepsilon} \, \mathrm{d}x_2 \mathrm{d}x_3 \rightharpoonup \int_S u_1 \, \mathrm{d}x_2 \mathrm{d}x_3 \text{ in } L^p(0,T;W^{1,p}(0,1)),
$$

which follows from (3.37). In addition, it holds that

$$
\int_S \rho u_1 \, \mathrm{d}x_2 \mathrm{d}x_3 = \hat{\rho} u_1
$$

$$
\int_S u_1 \, \mathrm{d}x_2 \mathrm{d}x_3 = u_1,
$$

where $\hat{\rho} = \int_S \rho \, \mathrm{d}x_2 \mathrm{d}x_3$, because $\mathbf{u}$ is independent of x_2 and x_3. Hence, convergence

$$
\int_0^T \int_\Omega \rho_\varepsilon u_{1,\varepsilon} u_{1,\varepsilon} \psi \, \mathrm{d}x \mathrm{d}t \to \int_0^T \int_0^1 \hat{\rho} u_1 u_1 \psi \, \mathrm{d}x_1 \mathrm{d}t \tag{3.51}
$$

is an immediate consequence of (3.37), (3.49) and (3.50). Convergence (3.51) is applied in the next section to overcome the nonlinearity in the second term on the left-hand side of (3.8)

3.3.3 Limit of the governing equations

Now, we can perform limit passages in (3.7) and (3.8). Throughout this section, we denote an integral of a function in the second and third spatial variable over set S by symbol "$\hat{}$" over the function. Obviously, these integrals depend only on x_1. For example, we write $\hat{\rho} = \int_S \rho \; \mathrm{d}x_2 \mathrm{d}x_3$.

We remark that prescribed behavior (3.66) enables us to use the Gronwall's lemma in the proof of boundedness (3.31). Further, we assume that $h(\varepsilon) > 0$ in (3.8) satisfies the condition $h(\varepsilon) \sim O(\varepsilon)$ to ensure the convergence of $\frac{h(\varepsilon)}{\varepsilon}$ to a real positive number.

First, we test the equation (3.7) by function $\varphi \in \mathcal{D}(\mathbb{R} \times \langle 0, T \rangle)$. We arrive at

$$\int_0^T \int_\Omega \rho_\varepsilon \partial_t \varphi + \rho_\varepsilon u_{1,\varepsilon} \partial_1 \varphi \; \mathrm{d}x \mathrm{d}t = 0.$$

Subsequently, we perform the limit passage for $\varepsilon \to 0$, apply convergences (3.35) and (3.39), and get

$$\int_0^T \int_0^1 \hat{\rho} \partial_t \varphi + \hat{\rho} u_1 \partial_1 \varphi \; \mathrm{d}x_1 \mathrm{d}t = 0, \tag{3.52}$$

for any $\varphi \in \mathcal{D}\left(\mathbb{R} \times \langle 0, T \rangle\right)$.

Second, we test the equation (3.8) by function $\psi = (\psi_1(x_1, t), 0, 0)$, where $\psi_1 \in \mathcal{C}_0^\infty(0, T; \mathcal{C}^\infty(\langle 0, 1 \rangle))$ complies with $\psi_1(0, t) = \psi_1(1, t) = 0$, for all $t \in (0, T)$. We will show the limit passage for each term in (3.8) separately.

(a) $\rho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \psi$

Since convergence (3.39) holds, we get

$$\int_0^T \int_\Omega \rho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \psi \; \mathrm{d}x \mathrm{d}t \to \int_0^T \int_0^1 \hat{\rho} u_1 \partial_t \psi_1 \; \mathrm{d}x_1 \mathrm{d}t,$$

for $\varepsilon \to 0$.

(b) $\rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \omega_\varepsilon(\psi)$

From the definition of the test function ψ, we know that

$$\omega_\varepsilon(\psi) = \omega(\psi) = \begin{pmatrix} \partial_1 \psi_1 & 0 & 0 \\ \cdot & 0 & 0 \\ \mathrm{sym} & \cdot & 0 \end{pmatrix}. \tag{3.53}$$

After applying convergence (3.51), we conclude that

$$\int_0^T \int_\Omega \rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \omega_\varepsilon(\psi)\ \mathrm{d}x\mathrm{d}t \to \int_0^T \int_0^1 \hat{\rho} u_1^2 \partial_1 \psi_1\ \mathrm{d}x_1\mathrm{d}t,$$

for $\varepsilon \to 0$.

(c) $\rho_\varepsilon \mathrm{div}_\varepsilon \psi$

Since $\mathrm{div}_\varepsilon \psi = \partial_1 \psi_1$, we have (see convergence (3.34))

$$\int_0^T \int_\Omega \rho_\varepsilon \mathrm{div}_\varepsilon \psi\ \mathrm{d}x\mathrm{d}t \to \int_0^T \int_0^1 \hat{\rho} \partial_1 \psi_1\ \mathrm{d}x_1\mathrm{d}t,$$

for $\varepsilon \to 0$.

(d) $P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right)\omega_\varepsilon(\mathbf{u}_\varepsilon) : \omega_\varepsilon(\psi)$

It holds that (see convergence (3.36))

$$\int_0^T \int_\Omega P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right)\omega_\varepsilon(\mathbf{u}_\varepsilon) : \omega_\varepsilon(\psi)\ \mathrm{d}x\mathrm{d}t \to \int_0^T \int_\Omega \overline{P(|\zeta|)\zeta} : \omega(\psi)\ \mathrm{d}x\mathrm{d}t,$$

for $\varepsilon \to 0$, where $\omega(\psi)$ is defined by (3.53) and

$$\zeta = \begin{pmatrix} \partial_1 u_1 & \zeta_{12} & \zeta_{13} \\ \cdot & \zeta_{22} & \zeta_{23} \\ \mathrm{sym} & \cdot & \zeta_{33} \end{pmatrix}. \tag{3.54}$$

Later, we will show that

$$\int_0^t \int_\Omega \overline{P(|\zeta|)\zeta} : \omega(\psi)\ \mathrm{d}x\mathrm{d}s = |S| \int_0^t \int_0^1 P(|\partial_1 u_1|)\partial_1 u_1 \partial_1 \psi_1\ \mathrm{d}x_1\mathrm{d}s,$$

for any $t \in (0, T)$.

(e) $\rho_\varepsilon \mathbf{f}_\varepsilon \cdot \psi$

Since $\mathbf{f}_\varepsilon \cdot \psi = f_{1,\varepsilon}\psi_1$, convergence (3.34) holds and $\mathbf{f}_\varepsilon \to \mathbf{f}$ in $[L^\infty(\Omega \times (0,T))]^3$ (see assumptions of Theorem 3.4 in section 3.4), we obtain

$$\int_0^T \int_\Omega \rho_\varepsilon \mathbf{f}_\varepsilon \cdot \psi \ \mathrm{d}x\mathrm{d}t \to \int_0^T \int_0^1 \widehat{\rho f_1}\psi_1 \ \mathrm{d}x_1\mathrm{d}t,$$

for $\varepsilon \to 0$, where f_1 denotes the limit of $f_{1,\varepsilon}$.

(f) $\frac{h(\varepsilon)}{\varepsilon}\mathbf{u}_\varepsilon \cdot \psi$

According to the supposed behavior of $h(\varepsilon)$, i. e. $h(\varepsilon) \sim O(\varepsilon)$, we can use convergence (3.37) to derive

$$\varepsilon^{-1} \int_0^T \int_{\Gamma_1} h(\varepsilon)\mathbf{u}_\varepsilon \cdot \psi \ \mathrm{d}\Gamma\mathrm{d}t \to |\partial S| h \int_0^T \int_0^1 u_1\psi_1 \ \mathrm{d}x_1\mathrm{d}t$$

for $\varepsilon \to 0$, where h is a positive constant.

(g) $\mathbf{u}_\varepsilon \cdot \psi$

Due to the choice of test function ψ, we have

$$\int_0^T \int_{\Gamma_2} \mathbf{u}_\varepsilon \cdot \psi \ \mathrm{d}\Gamma\mathrm{d}t = \int_0^T \int_{\Gamma_2} u_{1,\varepsilon}\psi_1 \ \mathrm{d}\Gamma\mathrm{d}t = 0,$$

for all $\varepsilon \in (0,1)$. Thus, this integral vanishes in the limit of the governing equations.

Finally, we arrive at

$$\begin{aligned}
&\int_0^T \int_0^1 \hat{\rho} u_1 \partial_t \psi_1 + \hat{\rho} u_1^2 \partial_1 \psi_1 + \hat{\rho} \partial_1 \psi_1 \mathrm{d}x_1\mathrm{d}t = \\
&= \int_0^T \int_\Omega \overline{P(|\zeta|)\zeta} : \omega(\psi) \ \mathrm{d}x\mathrm{d}t - \int_0^T \int_0^1 \widehat{\rho f_1}\psi_1 \mathrm{d}x_1\mathrm{d}t + \\
&+ |\partial S| h \int_0^T \int_0^1 u_1\psi_1 \mathrm{d}x_1\mathrm{d}t.
\end{aligned} \tag{3.55}$$

3.3.4 Limit of the energy equality

Applying similar approach as in section 3.3.3, convexity and Jensen's inequality, we perform the limit passage for $\varepsilon \to 0$ also in the energy equality (3.11). We arrive at the following inequality

$$\begin{aligned} &\int_0^1 \hat{\rho}\frac{|u_1|^2}{2} + \hat{\rho}\ln(\hat{\rho})\ \mathrm{d}x_1 + \int_0^t \int_\Omega \overline{P\left(|\zeta|\right)|\zeta|^2}\ \mathrm{d}x\mathrm{d}s + \\ &+|\partial S| h \int_0^t \int_0^1 |u_1|^2\ \mathrm{d}x_1\mathrm{d}s \leq \int_0^t \int_0^1 \widehat{\rho f_1} u_1\ \mathrm{d}x_1\mathrm{d}s + \\ &+\int_0^1 \frac{|(\rho\mathbf{u})_0|^2}{2\rho_0}\ \mathrm{d}x_1 + \int_0^1 \rho_0\ln(\rho_0)\ \mathrm{d}x_1. \end{aligned} \tag{3.56}$$

By the use of a similar procedure as in [19], Lemmas 3.2 and 3.3, based on the renormalized continuity equation and the Steklov function, we derive the energy equality

$$\begin{aligned} &\int_0^1 \hat{\rho}\frac{|u_1|^2}{2} + \hat{\rho}\ln(\hat{\rho})\ \mathrm{d}x_1 + \int_0^t \int_\Omega \overline{P\left(|\zeta|\right)\zeta} : \omega(\mathbf{u})\ \mathrm{d}x_1\mathrm{d}s + \\ &+|\partial S| h \int_0^t \int_0^1 |u_1|^2\ \mathrm{d}x_1\mathrm{d}s = \int_0^t \int_0^1 \widehat{\rho f_1} u_1\ \mathrm{d}x_1\mathrm{d}s + \\ &+\int_0^1 \frac{|(\rho\mathbf{u})_0|^2}{2\rho_0}\ \mathrm{d}x_1 + \int_0^1 \rho_0\ln(\rho_0)\ \mathrm{d}x_1. \end{aligned} \tag{3.57}$$

from (3.52) and (3.55), where $\omega(\mathbf{u})$ is defined in the same way as $\omega(\psi)$ in relation (3.53). It means that its only nonzero term is $[\omega(\mathbf{u})]_{11} = \partial_1 u_1$.

Since function $P(|z|)z$ is monotone, we get

$$\begin{aligned} 0 \leq &\lim_{\varepsilon\to 0} \int_0^t \int_\Omega \left(P(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|)\omega_\varepsilon(\mathbf{u}_\varepsilon) - P(|T|)T\right) : \left(\omega_\varepsilon(\mathbf{u}_\varepsilon) - T\right)\ \mathrm{d}x\mathrm{d}s = \\ &= \lim_{\varepsilon\to 0} \int_0^t \int_\Omega P(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|)|\omega_\varepsilon(\mathbf{u}_\varepsilon)|^2 \mathrm{d}x\mathrm{d}s - \\ &- \int_0^t \int_\Omega \overline{P(|\zeta|)\zeta} : T + P(|T|)T : \zeta + P(|T|)|T|^2\ \mathrm{d}x\mathrm{d}s \end{aligned} \tag{3.58}$$

for any symmetric $T \in [\tilde{L}_M(\Omega \times (0,T))]^9$. As a consequence of (3.11), (3.57), convexity and Jensen's inequality, we arrive at

$$
\begin{aligned}
&\lim_{\varepsilon \to 0} \int_0^t \int_\Omega P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right) |\omega_\varepsilon(\mathbf{u}_\varepsilon)|^2 \mathrm{d}x\mathrm{d}s = \\
&= \lim_{\varepsilon \to 0} \Bigg(- \int_\Omega \rho_\varepsilon \frac{|\mathbf{u}_\varepsilon|^2}{2} + \rho_\varepsilon \ln(\rho_\varepsilon)\ \mathrm{d}x - \\
&- \frac{h(\varepsilon)}{\varepsilon} \int_0^t \int_{\Gamma_1} |\mathbf{u}_\varepsilon|^2\ \mathrm{d}\Gamma\mathrm{d}s - q \int_0^t \int_{\Gamma_2} |\mathbf{u}_\varepsilon|^2\ \mathrm{d}\Gamma\mathrm{d}s + \\
&+ \int_0^t \int_\Omega \rho_\varepsilon \mathbf{f}_\varepsilon \cdot \mathbf{u}_\varepsilon\ \mathrm{d}x\mathrm{d}s + \int_\Omega \frac{|(\rho_\varepsilon \mathbf{u}_\varepsilon)_0|^2}{2\rho_{0,\varepsilon}}\ \mathrm{d}x + \int_\Omega \rho_{0,\varepsilon} \ln(\rho_{0,\varepsilon})\ \mathrm{d}x \Bigg) \leq \\
&\leq - \int_0^1 \hat{\rho} \frac{|u_1|^2}{2} + \hat{\rho} \ln(\hat{\rho})\ \mathrm{d}\hat{x} - |\partial S| h \int_0^t \int_0^1 |u_1|^2\ \mathrm{d}x_1\mathrm{d}s + \\
&+ \int_0^t \int_0^1 \widehat{\rho f_1} u_1\ \mathrm{d}x_1\mathrm{d}s + \int_0^1 \frac{|(\rho \mathbf{u})_0|^2}{2\rho_0}\ \mathrm{d}x_1 + \int_0^1 \rho_0 \ln(\rho_0)\ \mathrm{d}x_1 = \\
&= \int_0^t \int_\Omega \overline{P\left(|\zeta|\right) \zeta} : \omega(\mathbf{u})\ \mathrm{d}x\mathrm{d}s
\end{aligned} \tag{3.59}
$$

Hence and from (3.58), we get

$$
0 \leq \int_0^t \int_\Omega \left(\overline{P(|\zeta|)\zeta} - P(|T|)T \right) : (\omega(\mathbf{u}) - T)\, \mathrm{d}x\mathrm{d}s.
$$

Taking $T = \omega(\mathbf{u}) + \lambda\omega(\psi)$ and $T = \omega(\mathbf{u}) - \lambda\omega(\psi)$, for $\lambda > 0$, $\psi = (\psi_1, 0, 0)$, where $\psi_1 \in \mathcal{C}_0^\infty(0,T; \mathcal{C}^\infty(\bar{\Omega}))$ is such that $\partial_2\psi_1 = \partial_3\psi_1 = 0$ and $\psi_1(0,t) = \psi_1(1,t) = 0$, for all $t \in (0,T)$, we conclude that

$$
\begin{aligned}
&\int_0^t \int_\Omega \overline{P(|\zeta|)\zeta} : \omega(\psi)\mathrm{d}x\mathrm{d}s = |S| \int_0^t \int_0^1 P(|\omega(\mathbf{u})|)\omega(\mathbf{u}) : \omega(\psi)\ \mathrm{d}x_1\mathrm{d}s = \\
&= |S| \int_0^t \int_0^1 P(|\partial_1 u_1|)\partial_1 u_1 \partial_1 \psi_1\ \mathrm{d}x_1\mathrm{d}s.
\end{aligned} \tag{3.60}
$$

3.4 Main theorem for the 1D model

To sum it up, the limit equations together with the energy equality are given by the following formulas

$$\int_0^T \int_0^1 \hat{\rho}\partial_t\varphi + \hat{\rho}u_1\partial_1\varphi \; \mathrm{d}x_1\mathrm{d}t = 0, \tag{3.61}$$

for any $\varphi \in \mathcal{D}(\mathbb{R} \times \langle 0, T\rangle)$,

$$\begin{aligned}
&\int_0^T \int_0^1 \hat{\rho}u_1\partial_t\psi_1 + \hat{\rho}u_1^2\partial_1\psi_1 + \hat{\rho}\partial_1\psi_1\mathrm{d}x_1\mathrm{d}t = \\
&= |S|\int_0^T \int_0^1 P(|\partial_1 u_1|)\partial_1 u_1\partial_1\psi_1 \; \mathrm{d}x_1\mathrm{d}t - \int_0^T \int_0^1 \widehat{\rho f_1}\psi_1\mathrm{d}x_1\mathrm{d}t + \\
&+|\partial S|h\int_0^T \int_0^1 u_1\psi_1\mathrm{d}x_1\mathrm{d}t, \qquad (3.62)
\end{aligned}$$

for any $\psi = (\psi_1(x_1), 0, 0)$, where $\psi_1 \in \mathcal{C}_0^\infty(0, T; \mathcal{C}^\infty(\langle 0, 1\rangle))$ complies with condition $\psi_1(0, t) = \psi_1(1, t) = 0$, for all $t \in (0, T)$,

$$\begin{aligned}
&\int_0^1 \hat{\rho}\frac{|u_1|^2}{2} + \hat{\rho}\ln(\hat{\rho}) \; \mathrm{d}x_1 + |S|\int_0^t \int_\Omega P\left(|\partial_1 u_1|\right)|\partial_1 u_1|^2 \; \mathrm{d}x_1\mathrm{d}s + \\
&+|\partial S|h\int_0^t \int_0^1 |u_1|^2 \; \mathrm{d}x_1\mathrm{d}s = \int_0^t \int_0^1 \widehat{\rho f_1}u_1 \; \mathrm{d}x_1\mathrm{d}s + \qquad (3.63)\\
&+\int_0^1 \frac{|(\rho\mathbf{u})_0|^2}{2\rho_0} \; \mathrm{d}x_1 + \int_0^1 \rho_0\ln(\rho_0) \; \mathrm{d}x_1. \qquad (3.64)
\end{aligned}$$

Finally, we summarize our main result in the following theorem.

Theorem 3.4. *Let us assume that couples $(\rho_\varepsilon, \mathbf{u}_\varepsilon)$, $\varepsilon \in (0, 1)$, satisfying*

$$\begin{aligned}
&\rho_\varepsilon \in L^\infty(0, T; L_{\Phi_\gamma}(\Omega)), \\
&\mathbf{v}_\varepsilon \in L^p(0, T; [W^{1,p}(\Omega)]^3) \cap L^2(0, T; [L^2(\partial\Omega)]^3),
\end{aligned}$$

with $\mathbf{v}_\varepsilon = (u_{1,\varepsilon}, \varepsilon u_{2,\varepsilon}, \varepsilon u_{3,\varepsilon})$ and Ω being not axially symmetric, $\partial\Omega \in \mathcal{C}^{0,1}$, are weak solutions to the equations (3.7)-(3.8) and (3.11) with initial states $\rho_{0,\varepsilon} \in$

$L_{\Phi_\gamma}(\Omega)$ and $\frac{|(\rho_\varepsilon \mathbf{u}_\varepsilon)_0|^2}{2\rho_{0,\varepsilon}} \in L^1(\Omega)$ satisfying

$$\int_S \rho_{0,\varepsilon} \ln(\rho_{0,\varepsilon})\ \mathrm{d}x_2 \mathrm{d}x_3 \to \rho_0 \ln(\rho_0) \quad \text{in } L^1(0,1), \tag{3.65}$$

$$\int_S \Phi_\gamma(\rho_{0,\varepsilon})\ \mathrm{d}x_2 \mathrm{d}x_3 \to \Phi_\gamma(\rho_0) \quad \text{in } L^1(0,1), \tag{3.66}$$

$$\int_S \frac{|(\rho_\varepsilon \mathbf{u}_\varepsilon)_0|^2}{2\rho_{0,\varepsilon}}\ \mathrm{d}x_2 \mathrm{d}x_3 \to \frac{|(\rho \mathbf{u})_0|^2}{2\rho_0} \quad \text{in } L^1(0,1), \tag{3.67}$$

for arbitrary but fixed $\gamma > 3$ and $p > 3$. In addition, we assume that Navier boundary conditions (3.1)-(3.3) hold and $\omega_\varepsilon(\mathbf{u}_\varepsilon) \in [\tilde{L}_M(\Omega \times (0,T))]^9$.

Further, we suppose that function P complies with conditions (1.3)-(1.7), $\mathbf{f}_\varepsilon \to \mathbf{f}$ in $[L^\infty(\Omega \times (0,T))]^3$, $h(\varepsilon) > 0$ behaves like $O(\varepsilon)$, see (3.1), and $q > 0$, see (3.2). Then (passing to subsequences if necessary)

$$\begin{aligned}
\rho_\varepsilon &\overset{*}{\rightharpoonup} \rho \quad \text{in } L^\infty\left(0,T; L_{\Phi_\gamma}(\Omega)\right), \\
\rho_\varepsilon &\to \rho \quad \text{in } \mathcal{C}\left(\langle 0,T\rangle; [W^{1,p}(\Omega)]^*\right), \\
\omega_\varepsilon(\mathbf{u}_\varepsilon) &\overset{N}{\rightharpoonup} \omega(\mathbf{u}) \\
u_{1,\varepsilon} &\rightharpoonup u_1 \quad \text{in } L^p\left(0,T; W^{1,p}(\Omega)\right) \cap L^2\left(0,T; L^2(\partial\Omega)\right), \\
u_{\alpha,\varepsilon} &\to 0 \quad \text{in } L_M(\Omega \times (0,T)),\ \alpha = 2,3.
\end{aligned}$$

In addition, couple $(\hat{\rho}, u_1)$, where $u_1 = u_1(x_1)$ and $\hat{\rho} = \int_S \rho\ \mathrm{d}x_2 \mathrm{d}x_3$, is a weak solution to the equations (3.61)-(3.62) and complies with the energy equality (3.63).

Chapter 4

Derivation of a 2D model in a curved domain

We focus on a rigorous derivation of a two-dimensional model from equations (1.1)-(1.2) over a curved domain under Navier boundary conditions [2]. First, we describe the problem in section 4.1. In section 4.2, the deformation of the domain in question is expressed in the curvilinear coordinates. The transformation of the both governing equations and energy equality is performed in section 4.3. Finally, section 4.4 contains the proof of our main result, which is formulated in section 4.5.

4.1 Statement of the problem

We are interested in the motion of a compressible fluid in a thin domain. The dynamics of a compressible fluid are described by the continuity and momentum equations (1.1)-(1.2). We denote the velocity and the density as $\tilde{\mathbf{u}}_\varepsilon$ and $\tilde{\rho}_\varepsilon$, respectively, in equations (1.1)-(1.2) to highlight the connection to $\tilde{\Omega}_\varepsilon$. We employ similar notation also for other functions connected to $\tilde{\Omega}_\varepsilon$.

The domain $\tilde{\Omega}_\varepsilon \subset \mathbb{R}^3$ is defined by the use of a reference domain $\Omega = S \times (0,1)$, $S \subset \mathbb{R}^2$, $\partial S \in C^{0,1}$, and the mapping $\Theta_\varepsilon : \Omega \to \tilde{\Omega}_\varepsilon$ so that

$$\Theta_\varepsilon : (x_1, x_2, x_3) \longmapsto \theta(x_1, x_2) + \varepsilon x_3 \mathbf{a}_3(x_1, x_2),$$

where $\theta : S \to \mathbb{R}^3$ and

$$\begin{aligned}
\mathbf{a}_1 &= (\partial_1\theta_1, \partial_1\theta_2, \partial_1\theta_3)^{\mathrm{T}}, \\
\mathbf{a}_2 &= (\partial_2\theta_1, \partial_2\theta_2, \partial_2\theta_3)^{\mathrm{T}}, \\
\mathbf{a}_3 &= \frac{\mathbf{a}_1 \times \mathbf{a}_2}{\|\mathbf{a}_1 \times \mathbf{a}_2\|}.
\end{aligned}$$

We suppose that $\mathbf{a}_j$, $\partial_\alpha \mathbf{a}_j$ and $\partial^2_{\alpha\beta}\mathbf{a}_3 \in [L^\infty(\Omega)]^3$, where α, $\beta = 1, 2$ and $j = 1, 2, 3$.

Symbols $\mathbf{n}$ and $\tilde{\mathbf{n}}_\varepsilon$ stand for unit outward normals to Ω and $\tilde{\Omega}_\varepsilon$, respectively. Similarly, $\mathbf{t}$ (resp. $\tilde{\mathbf{t}}_\varepsilon$) is any vector from the corresponding tangent plane. We denote the boundaries of domains Ω and $\tilde{\Omega}_\varepsilon$ as follows:

$$\begin{gathered}
\Gamma_1 = \partial S \times (0,1),\ \Gamma_2 = S \times \{0,1\}, \\
\tilde{\Gamma}_{1,\varepsilon} = \Theta_\varepsilon\left(\Gamma_1\right),\ \tilde{\Gamma}_{2,\varepsilon} = \Theta_\varepsilon\left(\Gamma_2\right).
\end{gathered}$$

To ensure the well-posedness of our problem [26], we prescribed the set of Navier boundary conditions

$$\tilde{\mathbf{t}}_\varepsilon \cdot \left(P(|\tilde{D}\tilde{\mathbf{u}}_\varepsilon|)\tilde{D}\tilde{\mathbf{u}}_\varepsilon\tilde{\mathbf{n}}_\varepsilon\right) + q\tilde{\mathbf{u}}_\varepsilon \cdot \tilde{\mathbf{t}}_\varepsilon = 0,\ \text{on } \tilde{\Gamma}_{1,\varepsilon} \times (0,T), \tag{4.1}$$

$$\tilde{\mathbf{t}}_\varepsilon \cdot \left(P(|\tilde{D}\tilde{\mathbf{u}}_\varepsilon|)\tilde{D}\tilde{\mathbf{u}}_\varepsilon\tilde{\mathbf{n}}_\varepsilon\right) + h(\varepsilon)\tilde{\mathbf{u}}_\varepsilon \cdot \tilde{\mathbf{t}}_\varepsilon = 0,\ \text{on } \tilde{\Gamma}_{2,\varepsilon} \times (0,T), \tag{4.2}$$

$$\tilde{\mathbf{u}}_\varepsilon \cdot \tilde{\mathbf{n}}_\varepsilon = 0,\ \text{on } \partial\tilde{\Omega}_\varepsilon \times (0,T). \tag{4.3}$$

We suppose that $h(\varepsilon) > 0$ behaves like $O(\varepsilon)$ and $q > 0$. The asymptotic behavior of $h(\varepsilon)$ will be discussed during derivation of weak convergences of density and velocity field (section 4.4.3).

We consider the initial conditions for the density and the momentum

$$\begin{gathered}
\tilde{\rho}_\varepsilon(x,0) = \tilde{\rho}_{0,\varepsilon}(x) \geq 0, \\
\left(\tilde{\rho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon\right)(x,0) = \left(\tilde{\rho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon\right)_0(x,0),\ \text{in } \tilde{\Omega}_\varepsilon.
\end{gathered}$$

Hence, the variational formulation of our problem is

$$\int_0^T \int_{\tilde{\Omega}_\varepsilon} \left(\tilde{\rho}_\varepsilon\partial_t\tilde{\varphi} + \tilde{\rho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon \cdot \tilde{\nabla}\tilde{\varphi}\right) \mathrm{d}\tilde{x}\mathrm{d}t = 0, \tag{4.4}$$

$$\int_0^T \int_{\tilde{\Omega}_\varepsilon} \left(\tilde{\rho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \cdot \partial_t \tilde{\psi} + \tilde{\rho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon : \tilde{D}\tilde{\psi} + \tilde{\rho}_\varepsilon \tilde{\operatorname{div}} \tilde{\psi} \right) \mathrm{d}\tilde{x}\mathrm{d}t$$

$$= \int_0^T \int_{\tilde{\Omega}_\varepsilon} \left(P(|\tilde{D}\tilde{\mathbf{u}}_\varepsilon|)\tilde{D}\tilde{\mathbf{u}}_\varepsilon : \tilde{D}\tilde{\psi} - \tilde{\rho}_\varepsilon \tilde{\mathbf{f}}_\varepsilon \cdot \tilde{\psi} \right) \mathrm{d}x\mathrm{d}t$$

$$+q \int_0^T \int_{\tilde{\Gamma}_{1,\varepsilon}} \tilde{\mathbf{u}}_\varepsilon \cdot \tilde{\psi} \ \mathrm{d}\tilde{\Gamma}\mathrm{d}t + h(\varepsilon) \int_0^T \int_{\tilde{\Gamma}_{2,\varepsilon}} \tilde{\mathbf{u}}_\varepsilon \cdot \tilde{\psi} \ \mathrm{d}\tilde{\Gamma}\mathrm{d}t, \tag{4.5}$$

for any $\tilde{\varphi} \in \mathcal{D}\left(\mathbb{R}^3 \times (0,T)\right)$ and $\tilde{\psi} \in C_0^\infty(0,T; C^\infty(\{\tilde{\Omega}_\varepsilon\}^-)^3)$, where $\{\tilde{\Omega}_\varepsilon\}^-$ stands for the closure of $\tilde{\Omega}_\varepsilon$, satisfying the condition $\tilde{\psi} \cdot \tilde{\mathbf{n}}_\varepsilon|_{\partial\tilde{\Omega}_\varepsilon \times (0,T)} = 0$.

4.2 Curvilinear coordinates

We transform the equations (4.4) and (4.5) to the reference domain Ω by the use of mapping Θ_ε. First, we define the covariant basis (see [7], section 1.2)

$$\mathbf{g}_{1,\varepsilon} = \partial_1 \Theta_\varepsilon = \mathbf{a}_1 + \varepsilon x_3 \partial_1 \mathbf{a}_3, \tag{4.6}$$

$$\mathbf{g}_{2,\varepsilon} = \partial_2 \Theta_\varepsilon = \mathbf{a}_2 + \varepsilon x_3 \partial_2 \mathbf{a}_3, \tag{4.7}$$

$$\mathbf{g}_{3,\varepsilon} = \partial_3 \Theta_\varepsilon = \varepsilon \mathbf{a}_3, \tag{4.8}$$

the covariant metric tensor G_ε

$$[G_\varepsilon]_{ij} = g_{ij,\varepsilon} = \mathbf{g}_{i,\varepsilon} \cdot \mathbf{g}_{j,\varepsilon}, \tag{4.9}$$

and its determinant $g_\varepsilon = \det(G_\varepsilon)$. Further, we also define the contravariant basis by the relations

$$\mathbf{g}^{i,\varepsilon} \cdot \mathbf{g}_{j,\varepsilon} = \delta^i_j. \tag{4.10}$$

It is known from [7], Theorem 1.2-1, that

$$\left[G_\varepsilon^{-1}\right]^{ij} = g^{ij,\varepsilon} = \mathbf{g}^{i,\varepsilon} \cdot \mathbf{g}^{j,\varepsilon}.$$

and also (see [7], proof of Theorem 1.3), that

$$[\mathbf{g}^{i,\varepsilon}(x)]_k = \tilde{\partial}_k \Theta^{-1}_{i,\varepsilon}(\tilde{x}),$$

where $\tilde{x} = \Theta_\varepsilon(x)$.

For convenience, we denote the determinant of submatrix $([G_\varepsilon]_{ij})_{i,j=1}^2$ as d_ε. Relations (4.6)-(4.8) and (4.10) enable us to express the contravariant basis:

$$
\begin{aligned}
\mathbf{g}^{1,\varepsilon} &= d_\varepsilon^{-1}\left(|\mathbf{g}_{2,\varepsilon}|^2\mathbf{g}_{1,\varepsilon} - (\mathbf{g}_{1,\varepsilon}\cdot\mathbf{g}_{2,\varepsilon})\mathbf{g}_{2,\varepsilon}\right),\\
\mathbf{g}^{2,\varepsilon} &= d_\varepsilon^{-1}\left(|\mathbf{g}_{1,\varepsilon}|^2\mathbf{g}_{2,\varepsilon} - (\mathbf{g}_{1,\varepsilon}\cdot\mathbf{g}_{2,\varepsilon})\mathbf{g}_{1,\varepsilon}\right),\\
\mathbf{g}^{3,\varepsilon} &= \varepsilon^{-1}\mathbf{a}_3.
\end{aligned}
\tag{4.11}
$$

We emphasize that subscripts are used for the covariant basis $\{\mathbf{g}_{i,\varepsilon}\}_{i=1}^3$ and superscripts for the contravariant basis $\{\mathbf{g}^{i,\varepsilon}\}_{i=1}^3$ (the same notation as in [7]).

The contravariant basis is well-defined, because $d_\varepsilon > 0$ (see section 4.2.1 for details). For further calculations, we determine explicitly also the matrix G_ε and its inverse:

$$
G_\varepsilon = \begin{pmatrix} g_{11,\varepsilon} & g_{12,\varepsilon} & 0 \\ . & g_{22,\varepsilon} & 0 \\ \text{sym} & . & \varepsilon^2 \end{pmatrix}, \quad G_\varepsilon^{-1} = \begin{pmatrix} g^{11,\varepsilon} & g^{12,\varepsilon} & 0 \\ . & g^{22,\varepsilon} & 0 \\ \text{sym} & . & \varepsilon^{-2} \end{pmatrix},
$$

where

$$
\begin{aligned}
g_{11,\varepsilon} &= |\mathbf{a}_1|^2 + 2\varepsilon x_3\mathbf{a}_1\cdot\partial_1\mathbf{a}_3 + \varepsilon^2x_3^2|\partial_1\mathbf{a}_3|^2,\\
g_{12,\varepsilon} &= \mathbf{a}_1\cdot\mathbf{a}_2 + \varepsilon x_3\left(\mathbf{a}_1\cdot\partial_2\mathbf{a}_3 + \mathbf{a}_2\cdot\partial_1\mathbf{a}_3\right) + \varepsilon^2x_3^2\partial_1\mathbf{a}_3\cdot\partial_2\mathbf{a}_3,\\
g_{22,\varepsilon} &= |\mathbf{a}_2|^2 + 2\varepsilon x_3\mathbf{a}_2\cdot\partial_2\mathbf{a}_3 + \varepsilon^2x_3^2|\partial_2\mathbf{a}_3|^2,\\
g^{11,\varepsilon} &= g_{22,\varepsilon}d_\varepsilon^{-1},\\
g^{12,\varepsilon} &= -g_{12,\varepsilon}d_\varepsilon^{-1},\\
g^{22,\varepsilon} &= g_{11,\varepsilon}d_\varepsilon^{-1}.
\end{aligned}
$$

Terms $g_{13,\varepsilon}$ and $g_{23,\varepsilon}$ are equal to zero, because

$$
\mathbf{g}_{1,\varepsilon}\cdot\mathbf{g}_{3,\varepsilon} = \varepsilon\mathbf{a}_1\cdot\mathbf{a}_3 + \varepsilon^2x_3\mathbf{a}_3\cdot\partial_2\mathbf{a}_3 = 0.
$$

The last equality is due to orthogonality of $\mathbf{a}_1$ and $\mathbf{a}_3$, and equality $\mathbf{a}_3\cdot\partial_2\mathbf{a}_3 = \frac{1}{2}\partial_2|\mathbf{a}_3|^2 = \frac{1}{2}\partial_2 1 = 0$. Similarly, $g_{23,\varepsilon} = 0$ and thus also $g^{13,\varepsilon} = g^{23,\varepsilon} = 0$.

Mapping Θ_ε can be decomposed into two parts: deformation and contraction. Therefore, matrix G_ε, as well as the inverse matrix G_ε^{-1}, can be decomposed into two parts. In the following sections, we need the decomposition of G_ε^{-1}. Thus, we denote

$$E_\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ . & 1 & 0 \\ \text{sym} & . & \varepsilon^{-1} \end{pmatrix},$$

$$R_\varepsilon = \left(\mathbf{r}^{1,\varepsilon}, \mathbf{r}^{2,\varepsilon}, \mathbf{r}^{3,\varepsilon}\right) = \begin{pmatrix} [\mathbf{g}^{1,\varepsilon}]_1 & [\mathbf{g}^{2,\varepsilon}]_1 & [\mathbf{a}_3]_1 \\ [\mathbf{g}^{1,\varepsilon}]_2 & [\mathbf{g}^{2,\varepsilon}]_2 & [\mathbf{a}_3]_2 \\ [\mathbf{g}^{1,\varepsilon}]_3 & [\mathbf{g}^{2,\varepsilon}]_3 & [\mathbf{a}_3]_3 \end{pmatrix}. \tag{4.12}$$

It holds that $G_\varepsilon^{-1} = E_\varepsilon R_\varepsilon^{\mathrm{T}} R_\varepsilon E_\varepsilon$. It is an easy matter to demonstrate that $\det(R_\varepsilon^{\mathrm{T}} R_\varepsilon) = d_\varepsilon^{-1}$, $g_\varepsilon = d_\varepsilon \varepsilon^2$ and

$$R_\varepsilon^{\mathrm{T}} R_\varepsilon = \begin{pmatrix} g^{11,\varepsilon} & g^{12,\varepsilon} & 0 \\ . & g^{22,\varepsilon} & 0 \\ \text{sym} & . & 1 \end{pmatrix}. \tag{4.13}$$

From the relations (4.6)-(4.11), it is simple to prove that $R_\varepsilon^{\mathrm{T}} R_\varepsilon$ is a symmetric positive definite matrix. Hence, $d_\varepsilon^{-1} > 0$ and therefore $\sqrt{d_\varepsilon} > 0$ is well-defined. Furthermore, it stems from Cauchy's inequality that d_ε would equal zero if and only if $\mathbf{g}_{1,\varepsilon} = \mathbf{g}_{2,\varepsilon}$. However, this situation cannot occur because $\mathbf{g}_{1,\varepsilon}$ and $\mathbf{g}_{2,\varepsilon}$ are linearly independent.

4.2.1 Convergence of covariant and contravariant bases

Before we pass to the limit in the variational formulation (4.4)–(4.5), we mention necessary convergences concerning the covariant and contravariant bases. For $\varepsilon \to 0$, we have

$$\mathbf{g}_{1,\varepsilon} \to \mathbf{a}_1, \quad \mathbf{g}_{2,\varepsilon} \to \mathbf{a}_2, \quad \mathbf{g}_{3,\varepsilon} \to \mathbf{o} \quad \text{in } [W^{1,\infty}(\Omega)]^3, \tag{4.14}$$

$$G_\varepsilon \to G = \begin{pmatrix} |\mathbf{a}_1|^2 & \mathbf{a}_1 \cdot \mathbf{a}_2 & 0 \\ . & |\mathbf{a}_2|^2 & 0 \\ \text{sym} & . & 0 \end{pmatrix} \quad \text{in } [L^\infty(\Omega)]^9, \tag{4.15}$$

$$R_\varepsilon^{\mathrm{T}} R_\varepsilon \to R^{\mathrm{T}} R = \begin{pmatrix} d^{-1}|\mathbf{a}_2|^2 & -d^{-1}\mathbf{a}_1 \cdot \mathbf{a}_2 & 0 \\ . & d^{-1}|\mathbf{a}_1|^2 & 0 \\ \text{sym} & . & 1 \end{pmatrix} \quad \text{in } [L^\infty(\Omega)]^9, \tag{4.16}$$

$$g_\varepsilon \to 0, \quad d_\varepsilon \to d = (|\mathbf{a}_1||\mathbf{a}_2|)^2 - (\mathbf{a}_1 \cdot \mathbf{a}_2)^2 \quad \text{in } L^\infty(\Omega). \tag{4.17}$$

Therefore, $d_\varepsilon \geq \delta > 0$ and the contravariant basis is well-defined by relations (4.11). It immediately follows from the above convergences that

$$\mathbf{g}^{1,\varepsilon} \to \mathbf{a}^1 = d^{-1}\left(|\mathbf{a}_2|^2\mathbf{a}_1 - (\mathbf{a}_1 \cdot \mathbf{a}_2)\mathbf{a}_2\right) \quad \text{in } [W^{1,\infty}(\Omega)]^3, \tag{4.18}$$

$$\mathbf{g}^{2,\varepsilon} \to \mathbf{a}^2 = d^{-1}\left(|\mathbf{a}_1|^2\mathbf{a}_2 - (\mathbf{a}_1 \cdot \mathbf{a}_2)\mathbf{a}_1\right) \quad \text{in } [W^{1,\infty}(\Omega)]^3. \tag{4.19}$$

Hence,

$$R = \begin{pmatrix} [\mathbf{a}^1]_1 & [\mathbf{a}^2]_1 & [\mathbf{a}_3]_1 \\ [\mathbf{a}^1]_2 & [\mathbf{a}^2]_2 & [\mathbf{a}_3]_2 \\ [\mathbf{a}^1]_3 & [\mathbf{a}^2]_3 & [\mathbf{a}_3]_3 \end{pmatrix}. \tag{4.20}$$

Let us remark that all these limit functions depend only on x_1 and x_2, because $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ as well as $\mathbf{a}^1$, $\mathbf{a}^2$ are independent of x_3.

4.3 Transformation

4.3.1 Transformation of partial derivatives

For transformed velocity and density, we employ the notation

$$\mathbf{u}_\varepsilon : \Omega \times \langle 0, T\rangle \to \mathbb{R}^3,$$

$$\rho_\varepsilon : \Omega \times \langle 0, T\rangle \to \mathbb{R},$$

where $\mathbf{u}_\varepsilon(x,t) = \tilde{\mathbf{u}}_\varepsilon(\Theta_\varepsilon(x),t)$ and $\rho_\varepsilon(x,t) = \tilde{\rho}_\varepsilon(\Theta_\varepsilon(x),t)$, for all $x \in \Omega$. We denote $\tilde{x} = \Theta_\varepsilon(x)$ and also $x = \Theta_\varepsilon^{-1}(\tilde{x})$. Thus, we can write $\mathbf{u}_\varepsilon(x,t) = \tilde{\mathbf{u}}_\varepsilon(\tilde{x},t)$ and $\rho_\varepsilon(x,t) = \tilde{\rho}_\varepsilon(\tilde{x},t)$.

We express the first spatial partial derivative of a scalar function $\tilde{\varphi}$ according to the chain rule in the following way

$$\tilde{\partial}_j\tilde{\varphi}(\tilde{x},t) = \tilde{\partial}_j\varphi(\Theta_\varepsilon^{-1}(\tilde{x}),t) = \partial_l\varphi(x,t)[\mathbf{g}^{l,\varepsilon}]_j.$$

Similarly, we derive the first spatial partial derivative of a vector function $\tilde{\mathbf{u}}_\varepsilon$ as follows

$$\tilde{\partial}_j\tilde{u}_{i,\varepsilon}(\tilde{x},t) = \tilde{\partial}_j u_{i,\varepsilon}(\Theta_\varepsilon^{-1}(\tilde{x}),t) = \partial_l u_{i,\varepsilon}(x,t)[\mathbf{g}^{l,\varepsilon}]_j = \partial_l\mathbf{u}_\varepsilon(x,t)\cdot\mathbf{g}_{k,\varepsilon}[\mathbf{g}^{k,\varepsilon}]_i[\mathbf{g}^{l,\varepsilon}]_j,$$

where the last equality follows from

$$\partial_l u_{i,\varepsilon} = [\partial_l \mathbf{u}_\varepsilon]_i = \partial_l \mathbf{u}_\varepsilon \cdot \mathbf{g}_{k,\varepsilon}[\mathbf{g}^{k,\varepsilon}]_i.$$

The transformation of the symmetric part of the gradient can be performed in the following way

$$[\tilde{D}\tilde{\mathbf{u}}_\varepsilon]_{ij} = [\bar{\omega}_\varepsilon(\mathbf{u}_\varepsilon)]_{lk}\,[\mathbf{r}^{k,\varepsilon}]_i[\mathbf{r}^{l,\varepsilon}]_j = \left[R_\varepsilon^{\mathrm{T}}\bar{\omega}_\varepsilon(\mathbf{u}_\varepsilon)R_\varepsilon\right]_{ij} \stackrel{\text{def}}{=} [\omega_\varepsilon(\mathbf{u}_\varepsilon)]_{ij}\,, \tag{4.21}$$

where

$$\bar{\omega}_\varepsilon(\mathbf{u}_\varepsilon) = \begin{pmatrix} \partial_1 \mathbf{u}_\varepsilon \cdot \mathbf{g}_{1,\varepsilon} & \frac{1}{2}\left(\partial_1 \mathbf{u}_\varepsilon \cdot \mathbf{g}_{2,\varepsilon} + \partial_2 \mathbf{u}_\varepsilon \cdot \mathbf{g}_{1,\varepsilon}\right) & \frac{1}{2}\left(\frac{\partial_1 \mathbf{u}_\varepsilon \cdot \varepsilon \mathbf{a}_3 + \partial_3 \mathbf{u}_\varepsilon \cdot \mathbf{g}_{1,\varepsilon}}{\varepsilon}\right) \\ \cdot & \partial_2 \mathbf{u}_\varepsilon \cdot \mathbf{g}_{2,\varepsilon} & \frac{1}{2}\left(\frac{\partial_2 \mathbf{u}_\varepsilon \cdot \varepsilon \mathbf{a}_3 + \partial_3 \mathbf{u}_\varepsilon \cdot \mathbf{g}_{2,\varepsilon}}{\varepsilon}\right) \\ \text{sym} & \cdot & \frac{\partial_3 \mathbf{u}_\varepsilon \cdot \varepsilon \mathbf{a}_3}{\varepsilon^2} \end{pmatrix}, \tag{4.22}$$

In the following sections, we need the equivalence of $\bar{\omega}_\varepsilon(\mathbf{u}_\varepsilon)$ and $\omega_\varepsilon(\mathbf{u}_\varepsilon)$ in the L^p-norm, $p \geq 3$, and also in the L_M-norm. Since this equivalence can be proven simultaneously, let us denote any of these norms as $\|\cdot\|_X$. There exist $r_1(X)$, $r_2(X) > 0$ such that for all $\varepsilon \in (0,1)$ the following relation holds

$$r_1(X)\,\|\bar{\omega}_\varepsilon(\mathbf{u}_\varepsilon)\|_X \leq \|\omega_\varepsilon(\mathbf{u}_\varepsilon)\|_X \leq r_2(X)\,\|\bar{\omega}_\varepsilon(\mathbf{u}_\varepsilon)\|_X\,, \tag{4.23}$$

because R_ε is convergent for $\varepsilon \to 0$ in $W^{1,\infty}(\Omega)$ due to (4.18) and (4.19). Furthermore, R_ε does not tend to zero for $\varepsilon \to 0$ as formula (4.20) holds.

The transformation of $\tilde{\mathbf{u}}_\varepsilon \cdot \tilde{\nabla}\tilde{\varphi}$ leads to

$$\tilde{u}_{i,\varepsilon}\tilde{\partial}_i\tilde{\varphi} = u_{i,\varepsilon}\partial_l\varphi[\mathbf{g}^{l,\varepsilon}]_i = \mathbf{u}_\varepsilon^{\mathrm{T}} R_\varepsilon E_\varepsilon \nabla\varphi.$$

The transformation of $\widetilde{\operatorname{div}}\tilde{\psi}$ is done similarly

$$\widetilde{\operatorname{div}}\tilde{\psi} = \tilde{\partial}_i\tilde{\psi}_i = \partial_l\psi_i[\mathbf{g}^{l,\varepsilon}]_i = \nabla\psi : R_\varepsilon E_\varepsilon. \tag{4.24}$$

We remark that term $\nabla\psi : R_\varepsilon E_\varepsilon$ is the trace of $\omega_\varepsilon(\psi)$, because $\widetilde{\operatorname{div}}\tilde{\psi}$ is the trace of $\tilde{D}\tilde{\psi}$.

4.3.2 Transformation of the governing equations

According to [7], we use the following equalities

$$\mathrm{d}\tilde{x} = \sqrt{g_\varepsilon}\ \mathrm{d}x = \varepsilon\sqrt{d_\varepsilon}\ \mathrm{d}x,$$
$$\mathrm{d}\tilde{\Gamma} = |R_\varepsilon E_\varepsilon \mathbf{n}|\sqrt{g_\varepsilon}\ \mathrm{d}\Gamma = |R_\varepsilon E_\varepsilon \mathbf{n}|\varepsilon\sqrt{d_\varepsilon}\ \mathrm{d}\Gamma,$$

to arrive at the transformed equations of the variational formulation (4.4)–(4.5). It holds that $\mathbf{n} = (n_1, n_2, 0)$ on Γ_1, $\mathbf{n} = (0, 0, \pm 1)$ on Γ_2. Therefore,

$$|R_\varepsilon E_\varepsilon \mathbf{n}| = \sqrt{\sum_{i,j=1}^{2} n_i g^{ij,\varepsilon} n_j},\ \text{ on } \Gamma_1,$$
$$|R_\varepsilon E_\varepsilon \mathbf{n}| = \varepsilon^{-1},\ \text{ on } \Gamma_2.$$

Now, we can divide both equations by ε and arrive at the transformed variational formulation

$$\int_0^T \int_\Omega \left(\rho_\varepsilon \partial_t \varphi + \rho_\varepsilon \mathbf{u}_\varepsilon^{\mathrm{T}} R_\varepsilon E_\varepsilon \nabla\varphi\right) \sqrt{d_\varepsilon}\ \mathrm{d}x\mathrm{d}t = 0, \tag{4.25}$$

$$\begin{aligned}\int_0^T \int_\Omega &[\rho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \psi + \rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \omega_\varepsilon(\psi) + \rho_\varepsilon \nabla\psi : R_\varepsilon E_\varepsilon] \sqrt{d_\varepsilon}\ \mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_\Omega [P(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|)\, \omega_\varepsilon(\mathbf{u}_\varepsilon) : \omega_\varepsilon(\psi) - \rho_\varepsilon \mathbf{f}_\varepsilon \cdot \psi] \sqrt{d_\varepsilon}\ \mathrm{d}x\mathrm{d}t + \\ &+ q\int_0^T \int_{\Gamma_1} \mathbf{u}_\varepsilon \cdot \psi |R_\varepsilon E_\varepsilon \mathbf{n}| \sqrt{d_\varepsilon}\ \mathrm{d}\Gamma\mathrm{d}t + \frac{h(\varepsilon)}{\varepsilon}\int_0^T \int_{\Gamma_2} \mathbf{u}_\varepsilon \cdot \psi \sqrt{d_\varepsilon}\ \mathrm{d}\Gamma\mathrm{d}t,\end{aligned} \tag{4.26}$$

for any $\varphi \in \mathcal{D}(\mathbb{R}^3 \times (0,T))$ and $\psi \in C_0^\infty(0,T;[C^\infty(\bar{\Omega})]^3)$, $\psi \cdot \mathbf{n}|_{\partial\Omega\times(0,T)} = 0$. Similarly as in section 3.2.3, we remark that for any $\varepsilon \in (0,1)$, there exists at least one weak solution of equations (4.25)-(4.26).

After imposing the same transformation as for the variational formulation to renormalized continuity equation (see [17] or [19] for its original form), we get

$$\int_0^T \int_\Omega \left[b(\rho_\varepsilon)\partial_t\varphi + b(\rho_\varepsilon)\mathbf{u}_\varepsilon^{\mathrm{T}} R_\varepsilon E_\varepsilon \nabla\varphi\, (b(\rho_\varepsilon) - \rho_\varepsilon b'(\rho_\varepsilon))\, \nabla\mathbf{u}_\varepsilon : R_\varepsilon E_\varepsilon\right] \varphi\sqrt{d_\varepsilon}\ \mathrm{d}x\mathrm{d}t = 0, \tag{4.27}$$

for any $\varphi \in \mathcal{D}\left(\mathbb{R}^3 \times (0,T)\right)$.

4.3.3 Energy equality and its transformation

For any $t \in (0,T)$, we have the energy equality expressed by the following formula [19]

$$\int_{\tilde{\Omega}_\varepsilon} \left(\tilde{\rho}_\varepsilon(t) \frac{|\tilde{\mathbf{u}}_\varepsilon(t)|^2}{2} + \tilde{\rho}_\varepsilon(t) \ln(\tilde{\rho}_\varepsilon(t)) \right) \, \mathrm{d}\tilde{x} +$$

$$+ \int_0^t \int_{\tilde{\Omega}_\varepsilon} P(|\tilde{D}\tilde{\mathbf{u}}_\varepsilon|) \tilde{D}\tilde{\mathbf{u}}_\varepsilon : \tilde{D}\tilde{\mathbf{u}}_\varepsilon \, \mathrm{d}\tilde{x} \, \mathrm{d}s + q \int_0^t \int_{\tilde{\Gamma}_{1,\varepsilon}} |\tilde{\mathbf{u}}_\varepsilon|^2 \, \mathrm{d}\tilde{\Gamma} \, \mathrm{d}s +$$

$$+ h(\varepsilon) \int_0^t \int_{\tilde{\Gamma}_{2,\varepsilon}} |\tilde{\mathbf{u}}_\varepsilon|^2 \, \mathrm{d}\tilde{\Gamma}\mathrm{d}s = \int_0^t \int_{\tilde{\Omega}_\varepsilon} \tilde{\rho}_\varepsilon \tilde{\mathbf{f}}_\varepsilon \cdot \tilde{\mathbf{u}}_\varepsilon \, \mathrm{d}\tilde{x}\mathrm{d}s +$$

$$+ \int_{\tilde{\Omega}_\varepsilon} \left(\frac{|(\tilde{\rho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon)_0|^2}{2\tilde{\rho}_{0,\varepsilon}} + \tilde{\rho}_{0,\varepsilon} \ln(\tilde{\rho}_{0,\varepsilon}) \right) \, \mathrm{d}\tilde{x}. \tag{4.28}$$

By transforming (4.28), we obtain

$$\int_\Omega \left(\rho_\varepsilon(t) \frac{|\mathbf{u}_\varepsilon(t)|^2}{2} + \rho_\varepsilon(t) \ln(\rho_\varepsilon(t)) \right) \sqrt{d_\varepsilon} \, \mathrm{d}x$$

$$+ \int_0^t \int_\Omega P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right) |\omega_\varepsilon(\mathbf{u}_\varepsilon)|^2 \sqrt{d_\varepsilon} \, \mathrm{d}x\mathrm{d}s +$$

$$+ q \int_0^t \int_{\Gamma_1} |\mathbf{u}_\varepsilon|^2 |R_\varepsilon E_\varepsilon \mathbf{n}| \sqrt{d_\varepsilon} \, \mathrm{d}\Gamma\mathrm{d}s + \frac{h(\varepsilon)}{\varepsilon} \int_0^t \int_{\Gamma_2} |\mathbf{u}_\varepsilon|^2 \sqrt{d_\varepsilon} \, \mathrm{d}\Gamma\mathrm{d}s =$$

$$= \int_0^t \int_\Omega \rho_\varepsilon \bar{\mathbf{f}}_\varepsilon \cdot \mathbf{v}_\varepsilon \sqrt{d_\varepsilon} \, \mathrm{d}x\mathrm{d}s +$$

$$+ \int_\Omega \left(\frac{|(\rho_\varepsilon \mathbf{u}_\varepsilon)_0|^2}{2\rho_{0,\varepsilon}} + \rho_{0,\varepsilon} \ln(\rho_{0,\varepsilon}) \right) \sqrt{d_\varepsilon} \, \mathrm{d}x. \tag{4.29}$$

for any $t \in \langle 0,T \rangle$, where

$$\bar{\mathbf{f}}_\varepsilon = (\mathbf{f}_\varepsilon \cdot \mathbf{g}^{1,\varepsilon}, \mathbf{f}_\varepsilon \cdot \mathbf{g}^{2,\varepsilon}, \mathbf{f}_\varepsilon \cdot \mathbf{g}^{3,\varepsilon}),$$

$$\mathbf{v}_\varepsilon = (\mathbf{u}_\varepsilon \cdot \mathbf{g}_{1,\varepsilon}, \mathbf{u}_\varepsilon \cdot \mathbf{g}_{2,\varepsilon}, \mathbf{u}_\varepsilon \cdot \mathbf{g}_{3,\varepsilon}),$$

It is obvious that

$$\begin{aligned}\bar{\mathbf{f}}_\varepsilon \cdot \mathbf{v}_\varepsilon &= \bar{f}_{i,\varepsilon} v_{i,\varepsilon} = (\mathbf{f}_\varepsilon \cdot \mathbf{g}^{i,\varepsilon})(\mathbf{u}_\varepsilon \cdot \mathbf{g}_{i,\varepsilon}) \\ &= (\mathbf{f}_\varepsilon \cdot \mathbf{g}^{i,\varepsilon})\mathbf{g}_{i,\varepsilon} \cdot (\mathbf{u}_\varepsilon \cdot \mathbf{g}_{j,\varepsilon})\mathbf{g}^{j,\varepsilon} = \mathbf{f}_\varepsilon \cdot \mathbf{u}_\varepsilon.\end{aligned}$$

We need to use $\bar{\mathbf{f}}_\varepsilon \cdot \mathbf{v}_\varepsilon$ instead of $\mathbf{f}_\varepsilon \cdot \mathbf{u}_\varepsilon$ for making a priori estimates (see inequality 4.34), because a variant of Korn's inequality holds for $\mathbf{v}_\varepsilon$ (see Theorem 4.1).

It has to be remarked that concerning the existence of a weak solution to equations (4.25)-(4.26) satisfying the energy equality (4.29), we are in a similar situation as in the Chapter 3. Thus, we refer to section 3.2.3 for details connected to this issue.

4.4 Derivation of the limiting 2D equations

The first step of the proof concerns a variant of the first Korn's inequality which is proven in section 4.4.1. This inequality is necessary to perform a prior estimates in section 4.4.2 and subsequently show the boundedness of $\{\rho_\varepsilon\}_{\varepsilon\in(0,1)}$ and $\{\mathbf{v}_\varepsilon\}_{\varepsilon\in(0,1)}$, and perform weak limits. In section 4.4.3, we pass to the limits in the governing equations (4.25)-(4.26). As the last step, we perform the limit passage also for the energy equality (4.29) in section 4.4.4.

4.4.1 Korn's inequality

In this section, we prove a variant of the first Korn's inequality for functions from $[W^{1,p}(\Omega)]^3$, $p > 3$. This inequality is subsequently used to derive prior estimates for ρ_ε and $\mathbf{u}_\varepsilon$ in section 4.4.2.

From [9], we know that there exists $C > 0$ such that

$$\|\mathbf{w}\|_{1,p} \leq C \left(\|D\mathbf{w}\|_p + \|\mathbf{w}\|_p \right) \tag{4.30}$$

holds for any $\mathbf{w} \in [W^{1,p}(\Omega)]^3$, $p \geq 2$.

As a consequence (see Lemma 3.1 and its proof), there exists $\bar{C}(\Omega, p) > 0$ such that

$$\|\mathbf{w}\|_{1,p} \leq \bar{C}(\Omega, p) \left(\|D\mathbf{w}\|_p + \|\mathbf{w}\|_{2,\Gamma} \right). \tag{4.31}$$

From inequality (4.31), we can deduce the following theorem. Although its proof employes similar ideas as the proof of Theorem 3.2, we do not omit it, because it requires also other considerations due to more complex situation induced by the curvilinear coordinates. Without the loss of generality, we denote $\mathbf{u}_\varepsilon = \mathbf{u}_\varepsilon(t)$. Variable $t \in (0,T)$ is arbitrary but fixed.

Theorem 4.1. *Let $\mathbf{u}_\varepsilon \in [W^{1,p}(\Omega)]^3$, $p > 3$, be such that $\mathbf{u}_\varepsilon \cdot \mathbf{n} = \mathbf{u}_\varepsilon \cdot \mathbf{a}_3 = 0$ on $\Gamma = S \times \{0\}$. We define $\mathbf{v}_\varepsilon = (\mathbf{u}_\varepsilon \cdot \mathbf{g}_{1,\varepsilon}, \mathbf{u}_\varepsilon \cdot \mathbf{g}_{2,\varepsilon}, \mathbf{u}_\varepsilon \cdot \varepsilon \mathbf{a}_3)$. Then there exists $C = C(\Omega, p) > 0$, such that*

$$\|\mathbf{v}_\varepsilon\|_{1,p} \le C \left(\|\bar{\omega}_\varepsilon(\mathbf{u}_\varepsilon)\|_p + \|\mathbf{u}_\varepsilon\|_{2,\Gamma} \right), \quad \forall \varepsilon > 0, \tag{4.32}$$

where $\bar{\omega}_\varepsilon(\mathbf{u}_\varepsilon)$ is defined by (4.22).

Proof: Assume the contrary: without loss of generality, there exists a sequence $\{\mathbf{v}_{\varepsilon_n}\}_{n=1}^{+\infty}$ generated by $\{\mathbf{u}_{\varepsilon_n}\}_{n=1}^{+\infty}$, where $\varepsilon_n \to 0$ as n approaches infinity, such that $\|\mathbf{v}_{\varepsilon_n}\|_{1,p} = 1$ and

$$\frac{1}{n} \ge \|\bar{\omega}_{\varepsilon_n}(\mathbf{u}_{\varepsilon_n})\|_p + \|\mathbf{u}_{\varepsilon_n}\|_{2,\Gamma}.$$

Hence,

$$\mathbf{u}_{\varepsilon_n} \to 0, \text{ in } [L^2(\Gamma)]^3, \quad \bar{\omega}_{\varepsilon_n}(\mathbf{u}_{\varepsilon_n}) \to 0, \text{ in } [L^p(\Omega)]^9. \tag{4.33}$$

In addition, from definition of $\mathbf{v}_{\varepsilon_n}$, it follows that $v_{3,\varepsilon_n} = 0$ on Γ. From boundedness of sequence $\{\mathbf{v}_{\varepsilon_n}\}_{n=1}^{+\infty}$ and embedding $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathcal{C}(\bar{\Omega})$, we get the convergences (passing to a subsequence if necessary)

$$\begin{aligned} \mathbf{v}_{\varepsilon_n} &\rightharpoonup \mathbf{v} \quad \text{in } [W^{1,p}(\Omega)]^3, \\ \mathbf{v}_{\varepsilon_n} &\to \mathbf{v} \quad \text{in } [\mathcal{C}(\bar{\Omega})]^3. \end{aligned}$$

We will arrive at a contradiction in three steps:

1. We prove that $\{D\mathbf{v}_{\varepsilon_n}\}_{n=1}^{+\infty}$ is convergent in $[L^p(\Omega)]^9$.

Let us analyze the terms of $\bar{\omega}_{\varepsilon_n}(\mathbf{u}_{\varepsilon_n})$ one by one. We know that

$$\|\partial_3 \mathbf{u}_{\varepsilon_n} \cdot \mathbf{a}_3\|_p \leq \frac{\varepsilon_n}{n}.$$

Hence, $\partial_3 \mathbf{u}_{\varepsilon_n} \cdot \mathbf{a}_3 = \partial_3(\mathbf{u}_{\varepsilon_n} \cdot \mathbf{a}_3) \to \partial_3(\mathbf{u} \cdot \mathbf{a}_3) = 0$ in $L^p(\Omega)$. However, $(\mathbf{u} \cdot \mathbf{a}_3)(x_1, x_2, 0) = 0$ for all $(x_1,x_2) \in S$ (in other words $\mathbf{u} \cdot \mathbf{a}_3 = 0$ on Γ). Thus, $v_3 = \mathbf{u} \cdot \mathbf{a}_3 = 0$ in Ω.

Next, $[\bar{\omega}_{\varepsilon_n}(\mathbf{u}_{\varepsilon_n})]_{11}$ can be written as

$$\partial_1 \mathbf{u}_{\varepsilon_n} \cdot \mathbf{g}_{1,\varepsilon_n} = \partial_1(\mathbf{u}_{\varepsilon_n} \cdot \mathbf{g}_{1,\varepsilon_n}) - \mathbf{u}_{\varepsilon_n} \cdot \partial_1 \mathbf{g}_{1,\varepsilon_n}.$$

From the definition of $\mathbf{g}_{1,\varepsilon_n}$, it follows that $\partial_1 \mathbf{g}_{1,\varepsilon_n} = \partial_1 \mathbf{a}_1 + \varepsilon_n x_3 \partial^2_{11} \mathbf{a_3}$. Therefore, $\partial_1 \mathbf{g}_{1,\varepsilon_n} \in [L^\infty(\Omega)]^3$ can be written as

$$\partial_1 \mathbf{g}_{1,\varepsilon_n} = c_{1,\varepsilon_n} \mathbf{g}_{1,\varepsilon_n} + c_{2,\varepsilon_n} \mathbf{g}_{2,\varepsilon_n} + c_{3,\varepsilon_n} \mathbf{a}_3,$$

where $c_{\alpha,\varepsilon_n} = \partial_1 \mathbf{g}_{1,\varepsilon_n} \cdot \mathbf{g}^{\alpha,\varepsilon_n} \to c_\alpha$ in $L^\infty(\Omega)$, $\alpha = 1, 2$, and $c_{3,\varepsilon_n} = \partial_1 \mathbf{g}_{1,\varepsilon_n} \cdot \mathbf{a}_3 \to c_3$ in $L^\infty(\Omega)$ due to convergences (4.14), (4.18) and (4.19). Hence, $\mathbf{u}_{\varepsilon_n} \cdot \partial_1 \mathbf{g}_{1,\varepsilon_n} \to c_1 v_1 + c_2 v_2$ in $L^\infty(\Omega)$. Together with the convergence $\partial_1 \mathbf{u}_{\varepsilon_n} \cdot \mathbf{g}_{1,\varepsilon_n} \to 0$ in $L^p(\Omega)$, we get

$$\partial_1(\mathbf{u}_{\varepsilon_n} \cdot \mathbf{g}_{1,\varepsilon_n}) = \partial_1 v_{1,\varepsilon_n} \to c_1 v_1 + c_2 v_2 \quad \text{in } L^p(\Omega).$$

Similarly, we show that also the remaining terms of $D\mathbf{v}_{\varepsilon_n}$ converge in $L^p(\Omega)$.

2. We show that $\{\mathbf{v}_{\varepsilon_n}\}_{n=1}^{+\infty}$ is convergent in $[W^{1,p}(\Omega)]^3$.

The Korn's inequality (see Lemma 3.1) can be used in order to prove the statement of this step. We already know that $\mathbf{u}_{\varepsilon_n} \to 0$ in $[L^2(\Gamma)]^3$. Hence, also $\mathbf{v}_{\varepsilon_n} \to 0$ in $[L^2(\Gamma)]^3$. Together with the convergence of $D\mathbf{v}_{\varepsilon_n}$ we get

$$\|\mathbf{v}_{\varepsilon_n} - \mathbf{v}_{\varepsilon_m}\|_{1,p} \leq \bar{C}(\Omega, p) \left(\|D\mathbf{v}_{\varepsilon_n} - D\mathbf{v}_{\varepsilon_m}\|_p + \|\mathbf{v}_{\varepsilon_n} - \mathbf{v}_{\varepsilon_m}\|_{2,\Gamma} \right)$$

which implies the convergence of $\{\mathbf{v}_{\varepsilon_n}\}_{n=1}^{+\infty}$ in $[W^{1,p}(\Omega)]^3$.

3. To arrive at a contradiction, we prove that $\|\mathbf{v}\|_{1,p} = 1$ and simultaneously $\mathbf{v} = 0$.

From $\mathbf{v}_{\varepsilon_n} \to \mathbf{v}$ in $[W^{1,p}(\Omega)]^3$ and $\|\mathbf{v}_{\varepsilon_n}\|_{1,p} = 1$, it stems that $\|\mathbf{v}\|_{1,p} = 1$. According to the definition of $\mathbf{g}_{\alpha,\varepsilon_n}$, $\alpha = 1, 2$, we know that $\partial_3 \mathbf{g}_{\alpha,\varepsilon_n} = \varepsilon_n \partial_\alpha \mathbf{a}_3$. We can write

$$\partial_1 \mathbf{u}_{\varepsilon_n} \cdot \varepsilon_n \mathbf{a}_3 + \partial_3 \mathbf{u}_{\varepsilon_n} \cdot \mathbf{g}_{1,\varepsilon_n} = \partial_1 v_{3,\varepsilon_n} + \partial_3 v_{1,\varepsilon_n} - 2\varepsilon_n \mathbf{u}_{\varepsilon_n} \cdot \partial_1 \mathbf{a}_3.$$

It holds that $\varepsilon_n \mathbf{u}_{\varepsilon_n} \cdot \partial_1 \mathbf{a}_3 = \varepsilon_n (d_{1,\varepsilon_n} \mathbf{u}_{\varepsilon_n} \cdot \mathbf{g}_{1,\varepsilon_n} + d_{2,\varepsilon_n} \mathbf{u}_{\varepsilon_n} \cdot \mathbf{g}_{2,\varepsilon_n}) \to 0$ in $L^\infty(\Omega)$, where $d_{\alpha,\varepsilon_n} = \partial_1 \mathbf{a}_3 \cdot \mathbf{g}^{\alpha,\varepsilon_n} \to d_\alpha$ in $L^\infty(\Omega)$, $\alpha = 1,2$, due to convergences (4.18), (4.19) and the second step of this proof.

Due to $\partial_1 \mathbf{u}_{\varepsilon_n} \cdot \varepsilon_n \mathbf{a}_3 + \partial_3 \mathbf{u}_{\varepsilon_n} \cdot \mathbf{g}_{1,\varepsilon_n} \to 0$ and $\varepsilon_n \mathbf{u}_{\varepsilon_n} \cdot \partial_1 \mathbf{a}_3 \to 0$ in $L^p(\Omega)$, also $\partial_1 v_{3,\varepsilon_n} + \partial_3 v_{1,\varepsilon_n} \to 0$ in $L^p(\Omega)$. In addition,

$$\int_\Omega \partial_1 v_{3,\varepsilon_n} \varphi \, \mathrm{d}x = -\int_\Omega \varepsilon_n (\mathbf{u}_{\varepsilon_n} \cdot \mathbf{a}_3) \partial_1 \varphi \, \mathrm{d}x \to 0,$$

where $\varphi \in \mathcal{D}(\Omega)$. Hence, both $\partial_1 v_{3,\varepsilon_n} \to 0$ and $\partial_3 v_{1,\varepsilon_n} \to 0$ in $\mathcal{D}^*(\Omega)$. In addition with respect to the results of the second step of this proof, we have $\partial_1 v_{3,\varepsilon_n} \to 0$ and $\partial_3 v_{1,\varepsilon_n} \to 0$ in $L^p(\Omega)$. Therefore, $\partial_3 v_1 = 0$ almost everywhere. Similarly, we can show that $\partial_3 v_2 = 0$ almost everywhere. However, relation (4.33) gives us $v_i(x_1, x_2, 0) = 0$, $i = 1,2$, for all $(x_1,x_2) \in S$, which together with $\partial_3 v_i = 0$ gives us $v_i = 0$ in Ω.

Let us remind that in the first part of this proof, we already showed that $v_3 = 0$. To sum it up, $\mathbf{v} = 0$ in Ω and we arrive at a contradiction. ☐

4.4.2 Boundedness and weak limits

First, we make prior estimates. Equation (4.25) implies the conservation of mass, i.e.

$$\int_\Omega \rho_\varepsilon(t) \sqrt{d_\varepsilon} \mathrm{d}x = \int_\Omega \rho_{0,\varepsilon} \sqrt{d_\varepsilon} \mathrm{d}x, \quad \forall t \in (0, T).$$

Therefore due to assumptions of Theorem 4.3 (see section 4.5) on $\bar{\mathbf{f}}_\varepsilon$, the first integral on the right-hand side of the energy equality (4.29) can be estimated as follows

$$\left|\int_0^t \int_\Omega \rho_\varepsilon \bar{\mathbf{f}}_\varepsilon \cdot \mathbf{v}_\varepsilon \sqrt{d_\varepsilon}\ \mathrm{d}x\mathrm{d}s\right| \leq \int_0^t \|\mathbf{v}_\varepsilon(s)\|_\infty \|\bar{\mathbf{f}}_\varepsilon(s)\|_\infty \int_\Omega \rho_\varepsilon(s)\sqrt{d_\varepsilon}\ \mathrm{d}x\mathrm{d}s$$
$$\leq C(\rho_{0,\varepsilon}, \bar{\mathbf{f}}_\varepsilon) \int_0^t \|\mathbf{v}_\varepsilon(s)\|_{1,p}\ \mathrm{d}s.$$

In view of the Young's inequality, property (1.3), inequality (4.32) and estimate (4.23), we arrive at

$$\left|\int_0^t \int_\Omega \rho_\varepsilon \bar{\mathbf{f}}_\varepsilon \cdot \mathbf{v}_\varepsilon\ \mathrm{d}x\mathrm{d}s\right| \leq C\left(C_1 \int_0^t \int_\Omega P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right)|\omega_\varepsilon(\mathbf{u}_\varepsilon)|^2 \mathrm{d}x\mathrm{d}s\right.$$
$$\left. + C_1 \int_0^t \int_{S\times\{0\}} |\mathbf{u}_\varepsilon|^2 \mathrm{d}S\mathrm{d}s + C_2(C_1)\right), \tag{4.34}$$

where $C_1 > 0$ can be made arbitrarily small.

Due to (1.3) and (4.34), we obtain from (4.29) the boundedness

$$\{\sqrt{\rho_\varepsilon}\,|\mathbf{u}_\varepsilon|\}_{\varepsilon\in(0,1)} \text{ in } L^\infty\left(0,T;L^2(\Omega)\right), \tag{4.35}$$
$$\{\rho_\varepsilon\}_{\varepsilon\in(0,1)} \text{ in } L^\infty\left(0,T;L_{\Phi_1}(\Omega)\right), \tag{4.36}$$
$$\{\omega_\varepsilon(\mathbf{u}_\varepsilon)\}_{\varepsilon\in(0,1)} \text{ in } [\tilde{L}_M\left(\Omega\times(0,T)\right)]^9, \tag{4.37}$$
$$\{\mathbf{v}_\varepsilon\}_{\varepsilon\in(0,1)} \text{ in } L^p(0,T;[W^{1,p}(\Omega)]^3) \cap L^2(0,T;[L^2(\partial\Omega)]^3) \tag{4.38}$$

for any $p > 3$. From (4.38), we get the boundedness

$$\{\mathbf{u}_\varepsilon \cdot \mathbf{g}_{\alpha,\varepsilon}\}_{\varepsilon\in(0,1)} \text{ in } L^p\left(0,T;W^{1,p}(\Omega)\right) \cap L^2\left(0,T;L^2(\partial\Omega)\right),\ \alpha = 1,2. \tag{4.39}$$

However, we do not have any information on the boundedness of $\{\mathbf{u}_\varepsilon \cdot \mathbf{a}_3\}_{\varepsilon\in(0,1)}$ yet. Therefore, we prove that

$$\mathbf{u}_\varepsilon \cdot \mathbf{a}_3 \to 0, \quad \text{in } L_M(\Omega\times(0,T)). \tag{4.40}$$

Due to (4.37), we have the boundedness of $\varepsilon^{-1}\partial_3\mathbf{u}_\varepsilon \cdot \mathbf{a}_3$ in $\tilde{L}_M(\Omega\times(0,T))$. It means that $\partial_3\mathbf{u}_\varepsilon \cdot \mathbf{a}_3 = \partial_3(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3) \to 0$. In addition, it holds that

$$|(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3)(x_1,x_2,x_3) - (\mathbf{u}_\varepsilon \cdot \mathbf{a}_3)(x_1,x_2,0)| = \left|\int_0^{x_3} \partial_3(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3)(x_1,x_2,y)\ \mathrm{d}y\right|.$$

According to the boundary conditions, we have

$$(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3)(x_1, x_2, 0) = (\mathbf{u}_\varepsilon \cdot \mathbf{n})(x_1, x_2, 0) = 0.$$

Thus,

$$|(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3)| \leq \int_0^1 |\partial_3(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3)| \, \mathrm{d}x_3.$$

Multiplying this inequality by ε^{-1} and applying norm $\|\cdot\|_{L_M(\Omega\times(0,T))}$ lead to

$$\begin{aligned}
&\left\|\varepsilon^{-1}(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3)\right\|_{L_M(\Omega\times(0,T))} \leq \left\|\int_0^1 \varepsilon^{-1}\partial_3(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3) \, \mathrm{d}x_3\right\|_{L_M(\Omega\times(0,T))} = \\
&= \sup_{\psi_1} \int_0^T \int_\Omega \left|\int_0^1 \varepsilon^{-1}\partial_3(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3) \, \mathrm{d}x_3\right| |\psi_1| \, \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_3\mathrm{d}t \leq \\
&\leq \sup_{\psi_1} \int_0^T \int_\Omega \int_0^1 |\varepsilon^{-1}\partial_3(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3)| \, \mathrm{d}x_3 \; |\psi_1| \, \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_3\mathrm{d}t = \\
&= \sup_{\psi_1} \int_0^1 \int_0^T \int_\Omega |\varepsilon^{-1}\partial_3(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3)||\psi_1| \, \mathrm{d}x\mathrm{d}t\mathrm{d}y_3, \qquad (4.41)
\end{aligned}$$

where $\psi_1 = \psi_1(x_1, x_2, y_3) \in \tilde{L}_N(\Omega \times (0, T))$ satisfies

$$\int_0^T \int_\Omega N(|\psi_1|)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_3\mathrm{d}t \leq 1.$$

Next, we apply Hölder's inequality to the last term in (4.41). It turns out that

$$\begin{aligned}
&\left\|\varepsilon^{-1}(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3)\right\|_{L_M(\Omega\times(0,T))} \leq \\
&\leq \left\|\varepsilon^{-1}\partial_3(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3)\right\|_{L_M(\Omega\times(0,T))} \sup_{\psi_1} \int_0^1 \|\psi_1\|_{L_N(\Omega\times(0,T))} \, \mathrm{d}y_3 \leq \\
&\leq \left\|\varepsilon^{-1}\partial_3(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3)\right\|_{L_M(\Omega\times(0,T))} \sup_{\psi_1} \int_0^1 \left(\int_0^T \int_\Omega N(|\psi_1|) \, \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_3\mathrm{d}t + 1\right) \mathrm{d}x_3 \leq \\
&\leq 2 \left\|\varepsilon^{-1}\partial_3(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3)\right\|_{L_M(\Omega\times(0,T))} \leq C_1 \left\|[\omega_\varepsilon(\mathbf{u}_\varepsilon)]_{33}\right\|_{L_M(\Omega\times(0,T))} \leq \\
&\leq C_1 \int_0^T \int_\Omega M(|[\omega_\varepsilon(\mathbf{u}_\varepsilon)]_{33}|) \, \mathrm{d}x\mathrm{d}t + C_1.
\end{aligned}$$

Hence, we arrive at the boundedness of sequence

$$\{\varepsilon^{-1}(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3)\}_{\varepsilon\in(0,1)} \text{ in } L_M(\Omega \times (0,T)). \tag{4.42}$$

Therefore, the convergence (4.40) holds true.

The boundedness of $\{\rho_\varepsilon\}_{\varepsilon\in(0,1)}$ in $L^\infty(0,T;L_{\Phi_1}(\Omega))$ can be extended to the space $L^\infty(0,T;L_{\Phi_\gamma}(\Omega))$. We remind that $\gamma > 3$ (see Theorem 4.3). We proceed in the following way. First, we test the equation (4.27) by function $\varphi = \varphi(t) \in \mathcal{C}_0^\infty(0,T)$ with $b(z) = \Phi_\gamma(z)$. We arrive at

$$\int_0^T \int_\Omega \Phi_\gamma(\rho_\varepsilon)\varphi'(t) + \left[\left(\Phi_\gamma(\rho_\varepsilon) - \rho_\varepsilon \Phi'_\gamma(\rho_\varepsilon)\right) \nabla \mathbf{u}_\varepsilon : R_\varepsilon E_\varepsilon\right] \varphi(t)\sqrt{d_\varepsilon}\, \mathrm{d}x\mathrm{d}t = 0. \tag{4.43}$$

Function $\Phi_\gamma(z) - z\Phi'_\gamma(z)$ behaves assymptotically as $\Phi_{\gamma-1}(z)$. Furthermore, there exists a positive constant C such that $\Phi_1\left(\Phi_{\gamma-1}(z)\right) \le C\left(\Phi_\gamma(z) + 1\right)$ for $z \ge 0$ [26]. Due to equivalence of Young functions M and Ψ_1, relations (1.3), (4.37) and the Young's inequality, we deduce the estimate

$$\begin{aligned} &\left| \int_0^T \int_\Omega \left[\left(\Phi_\gamma(\rho_\varepsilon) - \rho_\varepsilon \Phi'_\gamma(\rho_\varepsilon)\right) \nabla \mathbf{u}_\varepsilon : R_\varepsilon E_\varepsilon\right] \sqrt{d_\varepsilon}\, \mathrm{d}x\mathrm{d}t \right| \\ &\le C(T) \left(\int_0^T \int_\Omega \left(\Phi_\gamma(\rho_\varepsilon) + P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right) |\omega_\varepsilon(\mathbf{u}_\varepsilon)|^2\right) \sqrt{d_\varepsilon}\, \mathrm{d}x\mathrm{d}t + 1 \right), \end{aligned} \tag{4.44}$$

where $C(T) > 0$. With respect to (4.43), (4.44), (4.80) and the Gronwall's lemma, we obtain the boundedness of

$$\{\rho_\varepsilon\}_{\varepsilon\in(0,1)} \text{ in } L^\infty(0,T;\tilde{L}_{\Phi_\gamma}(\Omega)). \tag{4.45}$$

We focus on the boundedness of $\{\partial_t \rho_\varepsilon\}_{\varepsilon\in(0,1)}$ in the next step. Let us test the equation (4.25) by function $\varphi(x,t) = \varphi_1(t)\psi(x)$, where $\varphi_1 \in L^{p'}(0,T)$, $1/p +$

$1/p' = 1$, $p > 3$, and $\psi \in [W^1 L_{\Psi_{\gamma-1}}(\Omega)]^3$, $\gamma > 3$. We can write

$$\left|\int_0^T \varphi_1' \int_\Omega \rho_\varepsilon \psi \sqrt{d_\varepsilon}\ \mathrm{d}x\mathrm{d}t\right| = \left|\int_0^T \varphi_1 \int_\Omega \rho_\varepsilon \mathbf{u}_\varepsilon^{\mathrm{T}} R_\varepsilon E_\varepsilon \nabla\psi \sqrt{d_\varepsilon}\ \mathrm{d}x\mathrm{d}t\right| =$$

$$= \left|\int_0^T \varphi_1 \int_\Omega \rho_\varepsilon \Big(\left[(\mathbf{u}_\varepsilon \cdot \mathbf{g}_{1,\varepsilon})\mathbf{g}^{1,\varepsilon} + (\mathbf{u}_\varepsilon \cdot \mathbf{g}_{2,\varepsilon})\mathbf{g}^{2,\varepsilon}\right] \cdot \left(\mathbf{g}^{1,\varepsilon}, \mathbf{g}^{2,\varepsilon}\right) \hat{\nabla}\psi + \right.$$

$$\left. + \varepsilon^{-1}(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3)\partial_3\psi\Big) \sqrt{d_\varepsilon}\ \mathrm{d}x\mathrm{d}t\right|, \tag{4.46}$$

where $\hat{\nabla}\psi = (\partial_1\psi, \partial_2\psi)^{\mathrm{T}}$. It is sufficient to estimate only the last term on the right-hand side of (4.46), because it is "the worst term". It holds that

$$\left|\int_0^T \int_\Omega \varphi_1 \rho_\varepsilon \varepsilon^{-1}(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3)\partial_3\psi \sqrt{d_\varepsilon}\ \mathrm{d}x\mathrm{d}t\right| \le$$

$$\le \left\|\varepsilon^{-1}(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3)\right\|_{L_M(\Omega\times(0,T))} \|\rho_\varepsilon \varphi_1 \partial_3\psi\|_{L_N(\Omega\times(0,T))} \|\sqrt{d_\varepsilon}\|_\infty,$$

where the first norm is bounded – see (4.42), and the second norm is less or equal than

$$\int_0^T \int_\Omega N(\rho_\varepsilon|\varphi_1||\partial_3\psi|)\ \mathrm{d}x\mathrm{d}t + 1 \le \int_0^T \int_\Omega |\varphi_1||\partial_3\psi| N(\rho_\varepsilon)\ \mathrm{d}x\mathrm{d}t +$$

$$+ \int_0^T \int_\Omega \rho_\varepsilon |\partial_3\psi| N(|\varphi_1|)\ \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \rho_\varepsilon |\varphi_1| N(|\partial_3\psi|)\ \mathrm{d}x\mathrm{d}t + C. \tag{4.47}$$

Subsequently, the three integrals on the right-hand side of (4.47) can be estimated as follows:

$$\int_0^T \int_\Omega |\varphi_1||\partial_3\psi| N(\rho_\varepsilon)\ \mathrm{d}x\mathrm{d}t \le C_1 \int_0^T |\varphi_1| \left(\int_\Omega \Phi_\gamma(\rho_\varepsilon) + \Psi_{\gamma-1}(|\partial_3\psi|)\ \mathrm{d}x\right) \mathrm{d}t \le$$

$$\le C_1 \|\varphi_1\|_{L^1(0,T)} \left(\left\|\int_\Omega \Phi_\gamma(\rho_\varepsilon)\ \mathrm{d}x\right\|_{L^\infty(0,T)} + \int_\Omega \Psi_{\gamma-1}(|\partial_3\psi|)\ \mathrm{d}x\right),$$

$$\int_0^T \int_\Omega \rho_\varepsilon |\partial_3\psi| N(|\varphi_1|)\ \mathrm{d}x\mathrm{d}t \le C_2 \int_0^T N(|\varphi_1|) \left(\int_\Omega \Phi_\gamma(\rho_\varepsilon) + \Psi_\gamma(|\partial_3\psi|)\mathrm{d}x\right) \mathrm{d}t \le$$

$$\le C_3 \|N(|\varphi_1|)\|_{L^1(0,T)} \left(\left\|\int_\Omega \Phi_\gamma(\rho_\varepsilon)\ \mathrm{d}x\right\|_{L^\infty(0,T)} + \int_\Omega \Psi_\gamma(|\partial_3\psi|)\ \mathrm{d}x\right)$$

and

$$\int_0^T \int_\Omega \rho_\varepsilon |\varphi_1| N(|\partial_3 \psi|) \, \mathrm{d}x\mathrm{d}t \leq C_3 \int_0^T |\varphi_1| \left(\int_\Omega \Phi_\gamma(\rho_\varepsilon) + \Psi_{\gamma'}(|\partial_3 \psi|) \, \mathrm{d}x \right) \mathrm{d}t \leq$$

$$\leq C_3 \left\| \varphi_1 \right\|_{L^1(0,T)} \left(\left\| \int_\Omega \Phi_\gamma(\rho_\varepsilon) \, \mathrm{d}x \right\|_{L^\infty(0,T)} + \int_\Omega \Psi_{\gamma'}(|\partial_3 \psi|) \, \mathrm{d}x \right),$$

where $\gamma > \gamma' \geq \gamma - 1$.

Finally, we conclude that (4.37), (4.39), (4.42) and (4.45) lead to the boundedness of

$$\{\partial_t \rho_\varepsilon\}_{\varepsilon \in (0,1)} \text{ in } L^{p'}\left(0, T; [W^1 L_{\Phi_{\gamma-1}}(\Omega)]^*\right). \tag{4.48}$$

By the use of (4.14), (4.36)-(4.38), (4.45), (4.48) and theorem on compact embedding [23], we get (passing to subsequences if necessary)

$$\rho_\varepsilon \overset{*}{\rightharpoonup} \rho \text{ in } L^\infty\left(0, T; L_{\Phi_\gamma}(\Omega)\right), \tag{4.49}$$

$$\rho_\varepsilon \to \rho \text{ in } \mathcal{C}\left(\langle 0, T\rangle; [W^1 L_{\Phi_\gamma}(\Omega)]^*\right), \tag{4.50}$$

$$\omega_\varepsilon(\mathbf{u}_\varepsilon) \overset{N}{\rightharpoonup} \zeta \tag{4.51}$$

$$\mathbf{u}_\varepsilon \cdot \mathbf{g}_{\alpha,\varepsilon} \rightharpoonup \mathbf{u} \cdot \mathbf{a}_\alpha \text{ in } L^p\left(0, T; W^{1,p}(\Omega)\right) \cap L^2\left(0, T; L^2(\partial\Omega)\right), \quad \alpha = 1, 2. \tag{4.52}$$

Concerning the third projection of $\mathbf{u}_\varepsilon$ into the covariant basis, convergence

$$\mathbf{u}_\varepsilon \cdot \mathbf{a}_3 \to \mathbf{u} \cdot \mathbf{a}_3 = 0 \text{ in } L_M(\Omega \times (0, T)) \tag{4.53}$$

holds true due to (4.42).

From the definition of $\omega_\varepsilon(\mathbf{u}_\varepsilon)$, we can see that

$$\zeta = R^{\mathrm{T}} \begin{pmatrix} \partial_1 \mathbf{u} \cdot \mathbf{a}_1 & \frac{1}{2}\left(\partial_1 \mathbf{u} \cdot \mathbf{a}_2 + \partial_2 \mathbf{u} \cdot \mathbf{a}_1\right) & \zeta_{13} \\ \cdot & \partial_2 \mathbf{u} \cdot \mathbf{a}_2 & \zeta_{23} \\ \text{sym} & \cdot & \zeta_{33} \end{pmatrix} R. \tag{4.54}$$

We prove that the limiting function $\mathbf{u}$ does not depend on the third spatial variable. From (4.37), we know that $\{\varepsilon^{-1}\left(\partial_1 \mathbf{u}_\varepsilon \cdot \varepsilon \mathbf{a}_3 + \partial_3 \mathbf{u}_\varepsilon \cdot \mathbf{g}_{1,\varepsilon}\right)\}_{\varepsilon \in (0,1)}$ is bounded in $\tilde{L}_M(\Omega \times (0, T))$. It holds that

$$\varepsilon^{-1}\left(\partial_1 \mathbf{u}_\varepsilon \cdot \varepsilon \mathbf{a}_3 + \partial_3 \mathbf{u}_\varepsilon \cdot \mathbf{g}_{1,\varepsilon}\right) = \partial_1(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3) + \varepsilon^{-1} \partial_3(\mathbf{u}_\varepsilon \cdot \mathbf{g}_{1,\varepsilon}) - 2\mathbf{u}_\varepsilon \cdot \partial_1 \mathbf{a}_3.$$

After multiplying this equation by ε and by a test function $\varphi \in \mathcal{D}(\Omega)$, and integrating over Ω, we get

$$\int_\Omega \partial_3(\mathbf{u}_\varepsilon \cdot \mathbf{g}_{1,\varepsilon})\varphi \, \mathrm{d}x = \varepsilon \int_\Omega \varepsilon^{-1} \left(\partial_1 \mathbf{u}_\varepsilon \cdot \varepsilon \mathbf{a}_3 + \partial_3 \mathbf{u}_\varepsilon \cdot \mathbf{g}_{1,\varepsilon}\right) \varphi \, \mathrm{d}x$$

$$+\varepsilon \int_\Omega \left(2\mathbf{u}_\varepsilon \cdot \partial_1 \mathbf{a}_3 - \partial_1(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3)\right) \varphi \, \mathrm{d}x.$$

Therefore, the following estimate holds true

$$\left| \int_\Omega (\mathbf{u}_\varepsilon \cdot \mathbf{g}_{1,\varepsilon})\partial_3\varphi \, \mathrm{d}x \right| \leq \varepsilon \left| \int_\Omega \varepsilon^{-1} \left(\partial_1 \mathbf{u}_\varepsilon \cdot \varepsilon \mathbf{a}_3 + \partial_3 \mathbf{u}_\varepsilon \cdot \mathbf{g}_{1,\varepsilon}\right) \varphi \, \mathrm{d}x \right|$$

$$+\varepsilon \left| \int_\Omega (\mathbf{u}_\varepsilon \cdot \mathbf{a}_3)\partial_1\varphi \, \mathrm{d}x \right| + 2\varepsilon \left| \int_\Omega (\mathbf{u}_\varepsilon \cdot \partial_1 \mathbf{a}_3)\varphi \, \mathrm{d}x \right|. \tag{4.55}$$

With respect to (4.37), (4.39) and (4.40), the right-hand side of inequality (4.55) tends to zero for $\varepsilon \to 0$. Finally, we have

$$\left| \int_\Omega (\mathbf{u} \cdot \mathbf{a}_1)\partial_3\varphi \, \mathrm{d}x \right| = 0,$$

and thus $\partial_3(\mathbf{u} \cdot \mathbf{a}_1) = 0$ almost everywhere.

Similarly, we can conclude that $\partial_3(\mathbf{u} \cdot \mathbf{a}_2) = 0$ almost everywhere. In summary and together with (4.40), we arrive at

$$\partial_3 \mathbf{u} = 0, \tag{4.56}$$

almost everywhere, which means that $\mathbf{u}$ is independent of x_3.

As the next step, we pay our attention to convergences of nonlinear terms in equation (4.26). Convergences (passing to subsequences if necessary)

$$\rho_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{g}_{\alpha,\varepsilon}) \xrightarrow{\Psi_\gamma} \rho(\mathbf{u} \cdot \mathbf{a}_\alpha), \ \alpha = 1, 2, \tag{4.57}$$

$$\rho_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3) \to \rho(\mathbf{u} \cdot \mathbf{a}_3) = 0, \text{ in } L_{\Phi_{\gamma-1}}(\Omega \times (0,T)), \tag{4.58}$$

where $\gamma > 3$ (see Theorem 4.3 in section 4.5), are immediate consequences of (4.40), (4.50), (4.52) and theorem concerning compact embedding [23].

We prove convergences (4.57) in two steps. The first step concerns the boundedness of $\{\rho_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{g}_{\alpha,\varepsilon})\}_{\varepsilon\in(0,1)}$ in $L_{\Phi_\gamma}(\Omega\times(0,T))$. Since it holds that

$$\|\rho_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{g}_{\alpha,\varepsilon})\|_{L_{\Phi_\gamma}(\Omega\times(0,T))} \leq \int_0^T \int_\Omega \Phi_\gamma(\rho_\varepsilon|\mathbf{u}_\varepsilon \cdot \mathbf{g}_{\alpha,\varepsilon}|)\ \mathrm{d}x\mathrm{d}t + C_1 \leq$$

$$\leq C_2 \int_0^T \int_\Omega |\mathbf{u}_\varepsilon \cdot \mathbf{g}_{\alpha,\varepsilon}|\Phi_\gamma(\rho_\varepsilon)\ \mathrm{d}x\mathrm{d}t + C_2 \int_0^T \int_\Omega \rho_\varepsilon \Phi_\gamma(|\mathbf{u}_\varepsilon \cdot \mathbf{g}_{\alpha,\varepsilon}|)\ \mathrm{d}x\mathrm{d}t + C_3 \leq$$

$$\leq C_2 \|\mathbf{u}_\varepsilon \cdot \mathbf{g}_{\alpha,\varepsilon}\|_{L^p(0,T;L^\infty(\Omega))} \left\| \int_\Omega \Phi_\gamma(\rho_\varepsilon)\ \mathrm{d}x \right\|_{L^\infty(0,T)} +$$

$$+C_4 \int_0^T \int_\Omega \rho_\varepsilon |\mathbf{u}_\varepsilon \cdot \mathbf{g}_{\alpha,\varepsilon}|^2\ \mathrm{d}x\mathrm{d}t + C_5,$$

where the second term on the right-hand side is less or equal than

$$C_6 \|\rho_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \|\mathbf{u}_\varepsilon \cdot \mathbf{g}_{\alpha,\varepsilon}\|^2_{L^p(0,T;L^\infty(\Omega))},$$

we arrive at (passing to subsequences if necessary) $\rho_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{g}_{\alpha,\varepsilon}) \overset{\Psi_\gamma}{\rightharpoonup} \overline{\rho(\mathbf{u}\cdot\mathbf{a}_\alpha)}$, $\alpha = 1, 2$.

In the second step, we show that $\rho_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{g}_{\alpha,\varepsilon})$ converges to $\rho(\mathbf{u}\cdot\mathbf{a}_\alpha)$, for $\varepsilon \to 0$, in the sense of distributions. We begin with

$$\int_0^T \int_\Omega (\rho_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{g}_{\alpha,\varepsilon}) - \rho(\mathbf{u}\cdot\mathbf{a}_\alpha))\,\varphi\ \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega (\rho_\varepsilon - \rho)(\mathbf{u}_\varepsilon \cdot \mathbf{g}_{\alpha,\varepsilon})\varphi\ \mathrm{d}x\mathrm{d}t -$$

$$- \int_0^T \int_\Omega \rho\,((\mathbf{u}_\varepsilon \cdot \mathbf{g}_{\alpha,\varepsilon}) - (\mathbf{u}\cdot\mathbf{a}_\alpha))\,\varphi\ \mathrm{d}x\mathrm{d}t, \tag{4.59}$$

where $\varphi \in \mathcal{D}(\Omega\times(0,T))$. Since strong convergence (4.50) implies convergence $\rho_\varepsilon \to \rho$ in $C(\langle 0,T\rangle; [W^{1,p}(\Omega)]^*)$ and convergences (4.52) hold, the right-hand side of (4.59) tends to zero for $\varepsilon \to 0$. This concludes the proof of convergences (4.57).

In the following, we demonstrate that also convergence (4.58) holds true. According to the Hölder's inequality, we can start with

$$\|\rho_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3)\|_{L_{\Phi_{\gamma-1}}(\Omega\times(0,T))} = \sup_\varphi \int_0^T \int_\Omega |\rho_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3)\varphi|\ \mathrm{d}x\mathrm{d}t \leq$$

$$\leq \|\mathbf{u}_\varepsilon \cdot \mathbf{a}_3\|_{L_M(\Omega\times(0,T))} \sup_\varphi \|\rho_\varepsilon \varphi\|_{L_N(\Omega\times(0,T))},$$

where the supremum is taken over all functions $\varphi \in \tilde{L}_{\Psi_{\gamma-1}}(\Omega \times (0,T))$ such that

$$\int_0^T \int_\Omega \Psi_{\gamma-1}(|\varphi|)\ \mathrm{d}x\mathrm{d}t \leq 1.$$

From (4.40), we know that $\|\mathbf{u}_\varepsilon \cdot \mathbf{a}_3\|_{L_M(\Omega\times(0,T))} \to 0$. Therefore, it is sufficient to show the boundedness of $\|\rho_\varepsilon \varphi\|_{L_N(\Omega\times(0,T))}$ for proving (4.58). The equivalence of Orlicz spaces L_N and L_{Φ_1} (see Lemma 2.30), and the Young's inequality gives us

$$\begin{aligned} &\|\rho_\varepsilon \varphi\|_{L_N(\Omega\times(0,T))} \leq \int_0^T \int_\Omega \Phi_1(\rho_\varepsilon|\varphi|)\ \mathrm{d}x\mathrm{d}t + C \leq \\ &\leq \int_0^T \int_\Omega \rho_\varepsilon \Phi_1(|\varphi|)\ \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega |\varphi| \Phi_1(\rho_\varepsilon)\ \mathrm{d}x\mathrm{d}t + C. \end{aligned} \tag{4.60}$$

Subsequently due to the Young's inequality, the first integral on the right-hand side of (4.60) is less or equal than

$$\begin{aligned} &\int_0^T \int_\Omega \Phi_\gamma(\rho_\varepsilon)\ \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \Psi_\gamma(\Phi_1(|\varphi|))\ \mathrm{d}x\mathrm{d}t \leq \\ &\leq \int_0^T \int_\Omega \Phi_\gamma(\rho_\varepsilon)\ \mathrm{d}x\mathrm{d}t + C_1 \int_0^T \int_\Omega \Psi_{\gamma'}(|\varphi|)\ \mathrm{d}x\mathrm{d}t + C_2, \end{aligned}$$

where $\gamma > \gamma' \geq \gamma - 1$. The second integral on the right-hand side of (4.60) is less or equal than

$$\begin{aligned} &\int_0^T \int_\Omega \Psi_{\gamma-1}(|\varphi|)\ \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \Phi_{\gamma-1}(\Phi_1(\rho_\varepsilon))\ \mathrm{d}x\mathrm{d}t \leq \\ &\leq \int_0^T \int_\Omega \Psi_{\gamma-1}(|\varphi|)\ \mathrm{d}x\mathrm{d}t + C \int_0^T \int_\Omega \Phi_\gamma(\rho_\varepsilon)\ \mathrm{d}x\mathrm{d}t. \end{aligned}$$

Hence, we conclude that convergence (4.58) holds true.

To overcome the second term on the left-hand side in equation (4.26), some sort of strong convergence of $\rho_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{g}_{\alpha,\varepsilon})$ is needed. We consider "the worst integrals" in (4.26) and prove their boundedness. First, we show that from (4.39), (4.40) and (4.45) follow the boundedness of

$$\int_0^T \int_\Omega \rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \omega_\varepsilon(\psi)\sqrt{d_\varepsilon}\,\mathrm{d}x\mathrm{d}t \tag{4.61}$$

for any $\varepsilon \in (0,1)$ and test function ψ such that $\psi \cdot \mathbf{n}|_{\partial\Omega\times(0,T)} = 0$. Let us use formulas (4.21) and (4.22), and perform the following reasoning:

$$\begin{aligned}\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \omega_\varepsilon(\psi) &= u_{i,\varepsilon}u_{j,\varepsilon}\,[\bar{\omega}_\varepsilon(\psi)]_{lk}\,[\mathbf{r}^{k,\varepsilon}]_i[\mathbf{r}^{l,\varepsilon}]_j = \\ &= (\mathbf{u}_\varepsilon \cdot \mathbf{r}^{k,\varepsilon})(\mathbf{u}_\varepsilon \cdot \mathbf{r}^{l,\varepsilon})\,[\bar{\omega}_\varepsilon(\psi)]_{lk}\,.\end{aligned} \tag{4.62}$$

We remark that $\mathbf{g}_{1,\varepsilon}$ and $\mathbf{g}_{2,\varepsilon}$ determines the same plane as $\mathbf{g}^{1,\varepsilon}$ and $\mathbf{g}^{2,\varepsilon}$ (the normal vector of this plane is $\mathbf{a}_3$). Therefore, the boundedness of sequences $\{\mathbf{u}_\varepsilon \cdot \mathbf{g}_{\alpha,\varepsilon}\}_{\varepsilon\in(0,1)}$ in $L^p(0,T;W^{1,p}(\Omega)) \cap L^2(0,T;L^2(\partial\Omega))$ implies the boundedness of $\{\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon}\}_{\varepsilon\in(0,1)}$ in the same space, for $\alpha = 1,2$.

There are three types of terms in (4.62) and we analyze them one by one. Since $\mathbf{r}^{\alpha,\varepsilon} = \mathbf{g}^{\alpha,\varepsilon}$, for $\alpha = 1,2$, and $\mathbf{r}^{3,\varepsilon} = \mathbf{a}_3$, we can write:

(a) $\rho_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon})(\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\beta,\varepsilon})[\bar{\omega}_\varepsilon(\psi)]_{\alpha\beta}$, $\alpha,\beta = 1,2$

Let us assume that $\psi \in L^q(0,T;[W^1L_{\Psi_\gamma}(\Omega)]^3)$, where $2/p + 1/q = 1$, $\gamma > 3$. The estimate is performed directly as follows:

$$\begin{aligned}&\left|\int_0^T \int_\Omega \rho_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon})(\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\beta,\varepsilon})[\bar{\omega}_\varepsilon(\psi)]_{\alpha\beta}\sqrt{d_\varepsilon}\,\mathrm{d}x\mathrm{d}t\right| \le \\ &\le \|\rho_\varepsilon\|_{L^\infty(0,T;L_{\Phi_\gamma}(\Omega))}\,\|\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon}\|_{L^p(0,T;L^\infty(\Omega))}\,\left\|\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\beta,\varepsilon}\right\|_{L^p(0,T;L^\infty(\Omega))} \\ &\|\psi\|_{L^q(0,T;[W^1L_{\Psi_\gamma}(\Omega)]^3)}\,\|\sqrt{d_\varepsilon}\|_\infty.\end{aligned}$$

(b) $\rho_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon})(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3)[\bar{\omega}_\varepsilon(\psi)]_{\alpha 3}$, $\alpha = 1,2$

We denote $\varepsilon^{-1}\varphi(t)\bar{\psi}(x) = [\bar{\omega}_\varepsilon(\psi(x,t))]_{\alpha 3}$ for convenience, where $\varphi \in L^q(0,T)$, $2/p + 1/q = 1$, and $\bar{\psi} \in [E_{\Psi_{\gamma-1}}(\Omega)]^9$. From Hölder's inequality, we get

$$\begin{aligned}&\left|\int_0^T \int_\Omega \rho_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon})(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3)\varepsilon^{-1}\varphi\bar{\psi}\sqrt{d_\varepsilon}\,\mathrm{d}x\mathrm{d}t\right| \le \\ &\le \|\sqrt{d_\varepsilon}\|_\infty\|\varepsilon^{-1}(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3)\|_{L_M(\Omega\times(0,T))}\|\rho_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon})\varphi\bar{\psi}\|_{L_N(\Omega\times(0,T))},\end{aligned}$$

where (4.42) gives us the boundedness of $\|\varepsilon^{-1}(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3)\|_{L_M(\Omega\times(0,T))}$ and

$$
\begin{aligned}
&\|\rho_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon})\varphi\bar{\psi}\|_{L_N(\Omega\times(0,T))} \leq \int_0^T \int_\Omega \Phi_1(\rho_\varepsilon|\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon}||\varphi||\bar{\psi}|)\ \mathrm{d}x\mathrm{d}t + C_1 \leq \\
&\leq \int_0^T \int_\Omega |\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon}||\varphi||\bar{\psi}|\Phi_1(\rho_\varepsilon)\ \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \rho_\varepsilon|\varphi||\bar{\psi}|\Phi_1(|\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon}|)\ \mathrm{d}x\mathrm{d}t + \\
&+ \int_0^T \int_\Omega \rho_\varepsilon|\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon}|\Phi_1(|\varphi||\bar{\psi}|)\ \mathrm{d}x\mathrm{d}t + C_2.
\end{aligned} \tag{4.63}
$$

Further, we estimate the three integrals on the right-hand side of relation (4.63) as follows:

$$
\begin{aligned}
&\int_0^T |\varphi| \int_\Omega |\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon}||\bar{\psi}|\Phi_1(\rho_\varepsilon)\ \mathrm{d}x\mathrm{d}t \leq \\
&\leq \int_0^T |\varphi|\|(\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon})\|_\infty \left(\int_\Omega |\bar{\psi}|\Phi_1(\rho_\varepsilon)\ \mathrm{d}x\right) \mathrm{d}t \leq \\
&\leq C_3 \int_0^T |\varphi|\|(\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon})\|_\infty \left(\int_\Omega \Phi_\gamma(\rho_\varepsilon)\ \mathrm{d}x + \int_\Omega \Psi_{\gamma-1}(|\bar{\psi}|)\ \mathrm{d}x + C_4\right) \mathrm{d}t \leq \\
&\leq C_3 \left(\|\varphi\|_{L^{p'}(0,T)} + \|\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon}\|_{L^p(0,T;L^\infty(\Omega))}\right) \left(\left\|\int_\Omega \Phi_\gamma(\rho_\varepsilon)\ \mathrm{d}x\right\|_{L^\infty(0,T)} + \right. \\
&\left. + \int_\Omega \Psi_{\gamma-1}(|\bar{\psi}|)\ \mathrm{d}x + C_4\right),
\end{aligned}
$$

where $1/p + 1/p' = 1$ (and thus $p' < q$),

$$
\begin{aligned}
&\int_0^T \int_\Omega \rho_\varepsilon |\varphi||\bar{\psi}|\Phi_1(|\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon}|)\ \mathrm{d}x\mathrm{d}t \leq \\
&\leq \int_0^T |\varphi|\Phi_1(\|\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon}\|_\infty)\left(\int_\Omega \rho_\varepsilon |\bar{\psi}|\ \mathrm{d}x\right)\mathrm{d}t \leq \\
&\leq \int_0^T |\varphi|\Phi_1(\|\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon}\|_\infty)\left(\int_\Omega \Phi_\gamma(\rho_\varepsilon) + \Psi_\gamma(|\bar{\psi}|)\ \mathrm{d}x\right)\mathrm{d}t \leq \\
&\leq \int_0^T |\varphi|\left(\|\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon}\|_\infty^2 + C\right)\left(\int_\Omega \Phi_\gamma(\rho_\varepsilon) + \Psi_\gamma(|\bar{\psi}|)\ \mathrm{d}x\right)\mathrm{d}t \leq \\
&\leq \|\varphi\|_{L^q(0,T)}\left(\|\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon}\|_{L^p(0,T;L^\infty(\Omega))}^2 + C\right)\left(\left\|\int_\Omega \Phi_\gamma(\rho_\varepsilon)\ \mathrm{d}x\right\|_{L^\infty(0,T)} + \right. \\
&\left. + \int_\Omega \Psi_\gamma(|\bar{\psi}|)\ \mathrm{d}x\right),
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^T \int_\Omega |\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon}|\rho_\varepsilon \Phi_1(|\varphi||\bar{\psi}|)\ \mathrm{d}x\mathrm{d}t \leq \\
&\leq \int_0^T \|(\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon})\|_\infty\left(\int_\Omega \rho_\varepsilon \Phi_1(|\varphi||\bar{\psi}|)\ \mathrm{d}x\right)\mathrm{d}t \leq \\
&\leq \int_0^T \|(\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon})\|_\infty\left(\int_\Omega \rho_\varepsilon|\varphi|\Phi_1(|\bar{\psi}|) + \rho_\varepsilon|\bar{\psi}|\Phi_1(|\varphi|)\ \mathrm{d}x\right)\mathrm{d}t + C.
\end{aligned}
$$

We conclude this part by estimating

$$
\begin{aligned}
&\int_0^T |\varphi|\|(\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon})\|_\infty\left(\int_\Omega \rho_\varepsilon \Phi_1(|\bar{\psi}|)\ \mathrm{d}x\right)\mathrm{d}t \leq \\
&\leq C_1 \int_0^T |\varphi|\|(\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon})\|_\infty\left(\int_\Omega \Phi_\gamma(\rho_\varepsilon) + \Psi_{\gamma'}(|\bar{\psi}|)\ \mathrm{d}x + C_2\right)\mathrm{d}t \leq \\
&\leq C_1\|\varphi\|_{L^{p'}(0,T)}\|\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon}\|_{L^p(0,T;L^\infty(\Omega))}\left(\left\|\int_\Omega \Phi_\gamma(\rho_\varepsilon)\ \mathrm{d}x\right\|_{L^\infty(0,T)} + \right. \\
&\left. + \int_\Omega \Psi_{\gamma'}(|\bar{\psi}|)\ \mathrm{d}x + C_2\right),
\end{aligned}
$$

with $\gamma > \gamma' \geq \gamma - 1$ and $1/p + 1/p' = 1$, and

$$\int_0^T \Phi_1(|\varphi|)\|(\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon})\|_\infty \left(\int_\Omega \rho_\varepsilon |\bar{\psi}| \, \mathrm{d}x\right) \mathrm{d}t \leq$$

$$\leq \int_0^T (|\varphi|^2 + C)\|(\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon})\|_\infty \left(\int_\Omega \Phi_\gamma(\rho_\varepsilon) + \Psi_\gamma(|\bar{\psi}|) \, \mathrm{d}x\right) \mathrm{d}t \leq$$

$$\leq \left(\|\varphi\|^2_{L^q(0,T)}\|\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon}\|_{L^p(0,T;L^\infty(\Omega))} + C\|\mathbf{u}_\varepsilon \cdot \mathbf{g}^{\alpha,\varepsilon}\|_{L^1(0,T;L^\infty(\Omega))}\right)$$

$$\left(\left\|\int_\Omega \Phi_\gamma(\rho_\varepsilon) \, \mathrm{d}x\right\|_{L^\infty(0,T)} + \int_\Omega \Psi_\gamma(|\bar{\psi}|) \, \mathrm{d}x\right),$$

(c) $\rho_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3)^2[\bar{\omega}_\varepsilon(\psi)]_{33}$

For convenience, let us denote $\varepsilon^{-1}\varphi(t)\bar{\psi}(x) = [\bar{\omega}_\varepsilon(\psi(x,t))]_{33}$, where $\varphi \in L^q(0,T)$, $2/p + 1/q = 1$, and $\bar{\psi} \in [E_{\Psi_{\gamma-2}}(\Omega)]^9$. By the use of Hölder's inequality, we get

$$\left|\int_0^T \int_\Omega \rho_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3)^2 \varepsilon^{-1}\varphi\bar{\psi}\sqrt{d_\varepsilon} \, \mathrm{d}x\mathrm{d}t\right| \leq$$

$$\leq \|\sqrt{d_\varepsilon}\|_\infty \|\varepsilon^{-1}(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3)^2\|_{L_{\Psi_2}(\Omega\times(0,T))}\|\rho_\varepsilon\varphi\bar{\psi}\|_{L_{\Phi_2}(\Omega\times(0,T))}.$$

Let us remark that M and Ψ_1 are equivalent Young functions (see Lemma 2.30), and $\Psi_2(z^2) \sim \Psi_1(z)$. Thus, the norm $\|\varepsilon^{-1}(\mathbf{u}_\varepsilon \cdot \mathbf{a}_3)^2\|_{L_{\Psi_2}(\Omega\times(0,T))}$ tends to zero, for $\varepsilon \to 0$, due to convergence (4.40) and boundedness (4.42). We estimate the remaining norm $\|\rho_\varepsilon\varphi\bar{\psi}\|_{L_{\Phi_2}(\Omega\times(0,T))}$ in the following way:

$$\|\rho_\varepsilon\varphi\bar{\psi}\|_{L_{\Phi_2}(\Omega\times(0,T))} \leq \int_0^T \int_\Omega \Phi_2(\rho_\varepsilon|\varphi||\bar{\psi}|) \, \mathrm{d}x\mathrm{d}t + C_1 \leq$$

$$\leq \int_0^T \int_\Omega |\varphi||\bar{\psi}|\Phi_2(\rho_\varepsilon) \, \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \rho_\varepsilon \Phi_2(|\varphi||\bar{\psi}|) \, \mathrm{d}x\mathrm{d}t +$$

$$+2\int_0^T \int_\Omega \Phi_1(\rho_\varepsilon)\Phi_1(|\varphi||\bar{\psi}|) \, \mathrm{d}x\mathrm{d}t + C_2,$$

where

$$
\begin{aligned}
&\int_0^T \int_\Omega |\varphi||\bar{\psi}|\Phi_2(\rho_\varepsilon)\ \mathrm{d}x\mathrm{d}t \leq \\
&\leq C\int_0^T |\varphi| \left(\int_\Omega \Phi_\gamma(\rho_\varepsilon) + \Psi_{\gamma-2}(|\bar{\psi}|)\ \mathrm{d}x\right)\mathrm{d}t \leq \\
&\leq C\|\varphi\|_{L^1(0,T)}\left(\left\|\int_\Omega \Phi_\gamma(\rho_\varepsilon)\ \mathrm{d}x\right\|_{L^\infty(0,T)} + \int_\Omega \Psi_{\gamma-2}(|\bar{\psi}|)\ \mathrm{d}x\right),
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^T \int_\Omega \rho_\varepsilon \Phi_2(|\varphi||\bar{\psi}|)\ \mathrm{d}x\mathrm{d}t \leq \int_0^T |\varphi| \left(\int_\Omega \rho_\varepsilon \Phi_2(|\bar{\psi}|)\ \mathrm{d}x\right)\mathrm{d}t + \\
&+\int_0^T \Phi_2(|\varphi|)\left(\int_\Omega \rho_\varepsilon|\bar{\psi}|\ \mathrm{d}x\right)\mathrm{d}t + 2\int_0^T \Phi_1(|\varphi|)\left(\int_\Omega \rho_\varepsilon \Phi_1(|\bar{\psi}|)\ \mathrm{d}x\right)\mathrm{d}t + C \leq \\
&\leq C_1\int_0^T |\varphi|\left(\int_\Omega \Phi_\gamma(\rho_\varepsilon) + \Psi_{\gamma'}(|\bar{\psi}|)\ \mathrm{d}x\right)\mathrm{d}t + \\
&+\int_0^T \Phi_2(|\varphi|)\left(\int_\Omega \Phi_\gamma(\rho_\varepsilon) + \Psi_\gamma(|\bar{\psi}|)\ \mathrm{d}x\right)\mathrm{d}t + \\
&+C_2\int_0^T \Phi_1(|\varphi|)\left(\int_\Omega \Phi_\gamma(\rho_\varepsilon) + \Psi_{\gamma'}(|\bar{\psi}|)\ \mathrm{d}x\right)\mathrm{d}t + C \leq \\
&\leq C_1\|\varphi\|_{L^1(0,T)}\left(\left\|\int_\Omega \Phi_\gamma(\rho_\varepsilon)\ \mathrm{d}x\right\|_{L^\infty(0,T)} + \int_\Omega \Psi_{\gamma'}(|\bar{\psi}|)\ \mathrm{d}x\right) + \\
&+\|\Phi_2(|\varphi|)\|_{L^1(0,T)}\left(\left\|\int_\Omega \Phi_\gamma(\rho_\varepsilon)\ \mathrm{d}x\right\|_{L^\infty(0,T)} + \int_\Omega \Psi_\gamma(|\bar{\psi}|)\ \mathrm{d}x\right) + \\
&+C_2\|\Phi_1(|\varphi|)\|_{L^1(0,T)}\left(\left\|\int_\Omega \Phi_\gamma(\rho_\varepsilon)\ \mathrm{d}x\right\|_{L^\infty(0,T)} + \int_\Omega \Psi_{\gamma'}(|\bar{\psi}|)\ \mathrm{d}x\right) + C,
\end{aligned}
$$

with $\gamma > \gamma' \geq \gamma - 2$, and

$$
\begin{aligned}
&\int_0^T \int_\Omega \Phi_1(\rho_\varepsilon)\Phi_1(|\varphi||\bar{\psi}|)\ \mathrm{d}x\mathrm{d}t \leq \\
&\leq \int_0^T |\varphi| \left(\int_\Omega \Phi_1(\rho_\varepsilon)\Phi_1(|\bar{\psi}|)\ \mathrm{d}x \right) \mathrm{d}t + \\
&+ \int_0^T \Phi_1(|\varphi|) \left(\int_\Omega \Phi_1(\rho_\varepsilon)|\bar{\psi}|\ \mathrm{d}x \right) \mathrm{d}t + C \leq \\
&\leq C_1 \int_0^T |\varphi| \left(\int_\Omega \Phi_\gamma(\rho_\varepsilon) + \Psi_{\gamma'}(|\bar{\psi}|)\ \mathrm{d}x \right) \mathrm{d}t + \\
&+ C_2 \int_0^T \Phi_1(|\varphi|) \left(\int_\Omega \Phi_\gamma(\rho_\varepsilon) + \Psi_{\gamma-1}(|\bar{\psi}|)\ \mathrm{d}x \right) \mathrm{d}t + C \leq \\
&\leq C_1 \|\varphi\|_{L^1(0,T)} \left(\left\| \int_\Omega \Phi_\gamma(\rho_\varepsilon)\ \mathrm{d}x \right\|_{L^\infty(0,T)} + \int_\Omega \Psi_{\gamma'}(|\bar{\psi}|)\ \mathrm{d}x \right) + \\
&+ C_2 \|\Phi_1(|\varphi|)\|_{L^1(0,T)} \left(\left\| \int_\Omega \Phi_\gamma(\rho_\varepsilon)\ \mathrm{d}x \right\|_{L^\infty(0,T)} + \int_\Omega \Psi_{\gamma-1}(|\bar{\psi}|)\ \mathrm{d}x \right) + C,
\end{aligned}
$$

where $\gamma - 1 > \gamma' \geq \gamma - 2$.

We conclude that the integral (4.61) is bounded for any $\varepsilon \in (0,1)$ and ψ such that $\psi \cdot \mathbf{n}|_{\partial\Omega\times(0,T)} = 0$, $\varphi \in L^q(0,T)$ and $\bar{\psi} \in [E_{\Psi_{\gamma-2}}(\Omega)]^9$. Subsequently, we show that also

$$
\int_0^T \int_\Omega P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right) \omega_\varepsilon(\mathbf{u}_\varepsilon) : \omega_\varepsilon(\psi)\sqrt{d_\varepsilon}\ \mathrm{d}x\mathrm{d}t \tag{4.64}
$$

is bounded for any $\varepsilon \in (0,1)$ and $\psi(x,t) = \varphi(t)\bar{\psi}(x)$, $\psi \cdot \mathbf{n}|_{\partial\Omega\times(0,T)} = 0$, where $\varphi \in E_{\Psi_{1/\alpha}}(0,T)$, $\alpha > 2$, and $\bar{\psi} \in [W^1 E_{\Psi_{1/2}}(\Omega)]^3$, $\partial_3\bar{\psi} = 0$. We remark that

$$
\omega_\varepsilon(\bar{\psi}) = R_\varepsilon^{\mathrm{T}} \begin{pmatrix} \partial_1\bar{\psi}\cdot\mathbf{g}_{1,\varepsilon} & \frac{1}{2}\left(\partial_1\bar{\psi}\cdot\mathbf{g}_{2,\varepsilon} + \partial_2\bar{\psi}\cdot\mathbf{g}_{1,\varepsilon}\right) & \frac{1}{2}\left(\partial_1\bar{\psi}\cdot\mathbf{a}_3\right) \\ \cdot & \partial_2\bar{\psi}\cdot\mathbf{g}_{2,\varepsilon} & \frac{1}{2}\left(\partial_2\bar{\psi}\cdot\mathbf{a}_3\right) \\ \mathrm{sym} & \cdot & 0 \end{pmatrix} R_\varepsilon,
$$

which is bounded for $\varepsilon \to 0$ in $[E_{\Psi_{1/2}}(\Omega)]^9$. Due to Young's inequality, it holds that

$$\left|\int_0^T \int_\Omega P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right)\omega_\varepsilon(\mathbf{u}_\varepsilon) : \omega_\varepsilon(\bar{\psi})\varphi\sqrt{d_\varepsilon}\ \mathrm{d}x\mathrm{d}t\right| \leq$$

$$\leq \|\sqrt{d_\varepsilon}\|_\infty \left(|\Omega| \int_0^T \Psi_{1/\alpha}(|\varphi|)\ \mathrm{d}t + \right.$$

$$\left.\int_0^T \int_\Omega \Phi_{1/\alpha}\left(P(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|)|\omega_\varepsilon(\mathbf{u}_\varepsilon)||\omega_\varepsilon(\bar{\psi})|\right) \mathrm{d}x\mathrm{d}t\right), \tag{4.65}$$

where $\alpha > 2$. For simplicity, we denote $w_\varepsilon = P(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|)|\omega_\varepsilon(\mathbf{u}_\varepsilon)|$. It follows from Jensen's inequality and estimate (3.46) that $w_\varepsilon \in L_{\Phi_1}(\Omega \times (0,T))$ implies $w_\varepsilon \in L_{\Phi_{(\alpha-1)/\alpha}}(0,T; L_{\Phi_{1/\alpha}}(\Omega))$. Therefore, the second term on the right-hand side of (4.65) is less or equal than

$$\int_0^T \int_\Omega |\omega_\varepsilon(\bar{\psi})|\Phi_{1/\alpha}(w_\varepsilon) + w_\varepsilon \Phi_{1/\alpha}(|\omega_\varepsilon(\bar{\psi})|)\ \mathrm{d}x\mathrm{d}t + C_1 \leq$$

$$\leq \int_0^T \int_\Omega \Phi_{(\alpha-1)/\alpha}(\Phi_{1/\alpha}(w_\varepsilon)) + \Psi_{(\alpha-1)/\alpha}(|\omega_\varepsilon(\bar{\psi})|) +$$

$$+\Phi_1(w_\varepsilon) + \Psi_1(\Phi_{1/\alpha}(|\omega_\varepsilon(\bar{\psi})|))\ \mathrm{d}x\mathrm{d}t + C_1 \leq$$

$$\leq 3\int_0^T \int_\Omega \Phi_1\left(P(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|)|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right) \mathrm{d}x\mathrm{d}t +$$

$$+\int_0^T \int_\Omega \Psi_{(\alpha-1)/\alpha}\left(|\omega_\varepsilon(\bar{\psi})|\right) \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \Psi_{1/2}\left(|\omega_\varepsilon(\bar{\psi})|\right) \mathrm{d}x\mathrm{d}t + C_2,$$

where $\alpha > 2$. Due to property (1.6), we conclude that integral (4.64) is bounded.

Terms (4.61) and (4.64) represent "the worst integrals" in (4.26). Thus, we omit the estimates of the others and take $\psi(x,t) = \varphi(t)\bar{\psi}(x)$ satisfying $\psi \cdot \mathbf{n}|_{\partial\Omega\times(0,T)} = 0$, where $\varphi \in E_{\Psi_{1/\alpha}}(0,T)$, $\alpha > 2$, and $\bar{\psi} \in [W^1 E_{\Psi_{1/2}}(\Omega)]^3$, $\partial_3\bar{\psi} = 0$, as a test function.

By the use of estimates (4.61) and (4.64), we demonstrate how to perform a limit passage in the second term on the left-hand side of equation (4.26). Let us test the equation (4.26) by function $\psi(x,t) = \bar{\psi}(x)\varphi(t)$, where $\varphi \in \mathcal{C}_0^\infty(0,T)$ and

$\bar{\psi} \in [W^1 E_{\Psi_{1/2}}(\Omega)]^3$, $\partial_3 \bar{\psi} = 0$, $\bar{\psi} \cdot \mathbf{a}^2 = 0$, $\bar{\psi} \cdot \mathbf{a}_3 = 0$ (to control term $\nabla\bar{\psi} : R_\varepsilon E_\varepsilon$) and $\bar{\psi} \cdot \mathbf{n}|_{\partial\Omega} = 0$. Since

$$\bar{\psi} = (\bar{\psi} \cdot \mathbf{a}^1)\mathbf{a}_1 + (\bar{\psi} \cdot \mathbf{a}^2)\mathbf{a}_2 + (\bar{\psi} \cdot \mathbf{a}_3)\mathbf{a}_3 = (\bar{\psi} \cdot \mathbf{a}^1)\mathbf{a}_1,$$

we get

$$\begin{aligned}
&\left| \int_0^T \varphi' \int_\Omega \rho_\varepsilon((\mathbf{u}_\varepsilon \cdot \mathbf{g}_{1,\varepsilon})\mathbf{g}^{1,\varepsilon} + (\mathbf{u}_\varepsilon \cdot \mathbf{g}_{2,\varepsilon})\mathbf{g}^{2,\varepsilon}) \cdot \bar{\psi} \sqrt{d_\varepsilon} \; \mathrm{d}x\mathrm{d}t \right| \leq \\
&\leq \int_0^T |\varphi| \int_\Omega \big(|\rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \omega_\varepsilon(\bar{\psi})| + |\rho_\varepsilon \nabla\bar{\psi} : R_\varepsilon E_\varepsilon| + \\
&+ |P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right) \omega_\varepsilon(\mathbf{u}_\varepsilon) : \omega_\varepsilon(\bar{\psi})| + |\rho_\varepsilon \mathbf{f}_\varepsilon \cdot \bar{\psi}| \big) \sqrt{d_\varepsilon} \; \mathrm{d}x\mathrm{d}t + \\
&+ q \int_0^T |\varphi| \int_{\Gamma_1} |\mathbf{u}_\varepsilon \cdot \bar{\psi}| |R_\varepsilon E_\varepsilon \mathbf{n}| \sqrt{d_\varepsilon} \; \mathrm{d}\Gamma\mathrm{d}t + \\
&\frac{h(\varepsilon)}{\varepsilon} \int_0^T |\varphi| \int_{\Gamma_2} |\mathbf{u}_\varepsilon \cdot \bar{\psi}| \sqrt{d_\varepsilon} \; \mathrm{d}\Gamma\mathrm{d}t,
\end{aligned} \tag{4.66}$$

where

$$\begin{aligned}
&\rho_\varepsilon \left((\mathbf{u}_\varepsilon \cdot \mathbf{g}_{1,\varepsilon})\mathbf{g}^{1,\varepsilon} + (\mathbf{u}_\varepsilon \cdot \mathbf{g}_{2,\varepsilon})\mathbf{g}^{2,\varepsilon} \right) \cdot \bar{\psi} = \\
&\rho_\varepsilon \left((\mathbf{u}_\varepsilon \cdot \mathbf{g}_{1,\varepsilon})\mathbf{g}^{1,\varepsilon} + (\mathbf{u}_\varepsilon \cdot \mathbf{g}_{2,\varepsilon})\mathbf{g}^{2,\varepsilon} \right) \cdot \left((\bar{\psi} \cdot \mathbf{a}^1)\mathbf{g}_{1,\varepsilon} - (\bar{\psi} \cdot \mathbf{a}^1)\varepsilon x_3 \partial_1 \mathbf{a}_3 \right) = \\
&= \rho_\varepsilon (\mathbf{u}_\varepsilon \cdot \mathbf{g}_{1,\varepsilon})\mathbf{g}^{1,\varepsilon} \cdot \bar{\psi} - \varepsilon \rho_\varepsilon (\mathbf{u}_\varepsilon \cdot \mathbf{g}_{2,\varepsilon})(x_3 \mathbf{g}^{2,\varepsilon} \cdot \partial_1 \mathbf{a}_3)\mathbf{a}^1 \cdot \bar{\psi} = \\
&= \left(\rho_\varepsilon (\mathbf{u}_\varepsilon \cdot \mathbf{g}_{1,\varepsilon})\mathbf{g}^{1,\varepsilon} - \varepsilon \mathbf{z}_{1,\varepsilon} \right) \cdot \bar{\psi},
\end{aligned}$$

with $\mathbf{z}_{1,\varepsilon} = \rho_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{g}_{2,\varepsilon})(x_3 \mathbf{g}^{2,\varepsilon} \cdot \partial_1 \mathbf{a}_3)\mathbf{a}^1$. The boundedness of $\{\mathbf{z}_{1,\varepsilon}\}_{\varepsilon \in (0,1)}$ in $L^p(0,T; [L_{\Phi_\gamma}(\Omega)]^3)$ follows from convergences (4.14), (4.18) and (4.19), and from boundedness (4.38) and (4.45). Therefore,

$$\varepsilon \mathbf{z}_{1,\varepsilon} \to 0 \text{ in } L^p(0,T; [L_{\Phi_\gamma}(\Omega)]^3), \tag{4.67}$$

and thus also in $[L_{\Phi_\gamma}(\Omega \times (0,T))]^3$.

Considering the density of $\mathcal{C}_0^\infty(0,T)$ in $E_{\Psi_{1/2}}(0,T)$, embedding $L_{\Psi_{1/\alpha}}(0,T) \hookrightarrow E_{\Psi_{1/2}}(0,T) \subset \tilde{L}_{\Psi_{1/2}}(0,T)$, $\alpha > 2$, and the boundedness of all terms on the right-hand side of the inequality (4.66) – see (4.61) and (4.64), we deduce the boundedness of

$$\{\partial_t \int_0^1 \left(\rho_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{g}_{1,\varepsilon})\mathbf{g}^{1,\varepsilon} - \varepsilon \mathbf{z}_{1,\varepsilon}\right) \sqrt{d_\varepsilon}\ \mathrm{d}x_3\}_{\varepsilon\in(0,1)}$$
$$\text{in } L_{\Phi_{1/\alpha}}(0,T;([W^1 L_{\Psi_{1/2}}(S)]^*)^3),\ \alpha > 2. \tag{4.68}$$

Similarly, testing the equation (4.26) by function $\psi(x,t) = \bar{\psi}(x)\varphi(t)$, where $\varphi \in \mathcal{C}_0^\infty(0,T)$ and $\bar{\psi} \in [W^1 E_{\Psi_{1/2}}(\Omega)]^3$, $\partial_3\bar{\psi} = 0$, $\bar{\psi}\cdot\mathbf{a}^1 = 0$, $\bar{\psi}\cdot\mathbf{a}_3 = 0$ and $\bar{\psi}\cdot\mathbf{n}|_{\partial\Omega} = 0$, leads to the boundedness of

$$\{\partial_t \int_0^1 \left(\rho_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{g}_{2,\varepsilon})\mathbf{g}^{2,\varepsilon} - \varepsilon \mathbf{z}_{2,\varepsilon}\right) \sqrt{d_\varepsilon}\ \mathrm{d}x_3\}_{\varepsilon\in(0,1)}$$
$$\text{in } L_{\Phi_{1/\alpha}}(0,T;([W^1 L_{\Psi_{1/2}}(S)]^*)^3),\ \alpha > 2 \tag{4.69}$$

where $\mathbf{z}_{2,\varepsilon} = \rho_\varepsilon(\mathbf{u}_\varepsilon\cdot\mathbf{g}_{1,\varepsilon})(x_3\mathbf{g}^{1,\varepsilon}\cdot\partial_2\mathbf{a}_3)\mathbf{a}^2$.

Similarly as convergence (4.50), we get (passing to subsequences if necessary)

$$\int_0^1 \left(\rho_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{g}_{\alpha,\varepsilon})\mathbf{g}^{\alpha,\varepsilon} - \varepsilon \mathbf{z}_{\alpha,\varepsilon}\right) \sqrt{d_\varepsilon}\ \mathrm{d}x_3 \to \int_0^1 \rho(\mathbf{u}\cdot\mathbf{a}_\alpha)\mathbf{a}^\alpha\sqrt{d}\ \mathrm{d}x_3$$
$$\text{in } \mathcal{C}\left(\langle 0,T\rangle;([W^1 L_{\Psi_1}(S)]^*)^3\right),\ \alpha = 1,2 \tag{4.70}$$

by the use of (4.14), (4.18), (4.19), (4.57), (4.68), (4.69) and theorem concerning compact embedding [23].

In order to perform a limit passage in the second term on the left-hand side of equation (4.26), we prove the following lemma.

Lemma 4.2. *Let us denote* $\mathbf{v}_\varepsilon = (\mathbf{u}_\varepsilon\cdot\mathbf{g}_{1,\varepsilon}, \mathbf{u}_\varepsilon\cdot\mathbf{g}_{2,\varepsilon}, \mathbf{u}_\varepsilon\cdot\varepsilon\mathbf{a}_3)$. *Assume that* $\{\mathbf{u}_\varepsilon\}_{\varepsilon\in(0,1)}$ *satisfies condition (4.37) and* $\{\mathbf{v}_\varepsilon\}_{\varepsilon\in(0,1)}$ *satisfies condition (4.38). Then for any* $p > 3$ *(passing to a subsequence if necessary), it holds that*

$$\|v_{\alpha,\varepsilon} - \int_0^1 v_{\alpha,\varepsilon}\ \mathrm{d}x_3\|_{L^p(0,T;L^\infty(\Omega))} \to 0,\ \textit{for } \varepsilon\to 0 \textit{ and } \alpha = 1,2. \tag{4.71}$$

Proof: We prove the assertion by a contradiction. Let us suppose the existence of fixed $p > 3$ with a positive constant c_1 and $\{\varepsilon_n\}_{n=1}^{+\infty}$ tending to zero such that

$$\|v_{\alpha,\varepsilon_n} - \int_0^1 v_{\alpha,\varepsilon_n} \, \mathrm{d}x_3\|_{L^p(0,T;L^\infty(\Omega))} \geq c_1 > 0, \ \forall n \in \mathbb{N}. \tag{4.72}$$

Obviously, there exist a nonempty set $I_{\varepsilon_n,c_1} \subset (0,T)$ and fixed $\delta > 0$ sufficiently small such that

$$\|v_{\alpha,\varepsilon_n}(t) - \int_0^1 v_{\alpha,\varepsilon_n}(t) \, \mathrm{d}x_3\|_\infty \geq \frac{c_1}{(1+\delta)T^{1/p}}, \text{ for almost all } t \in I_{\varepsilon_n,c_1}. \tag{4.73}$$

We will arrive at a contradiction in several steps. At the beginning of each step, we emphasize a statement which is proven within a particular step. Finally, the statements are used to demonstrate that there is a contradiction.

(i) There exists a positive constant $c_2 = c_2(c_1)$ such that $|I_{\varepsilon_n,c_1}| \geq c_2 > 0$, for all $n \in \mathbb{N}$.

If not, then (passing to a subsequence if necessary) $|I_{\varepsilon_n,c_1}| \to 0$ for ε_n tending to zero. Let us consider $q \in \mathbb{R}$ such that $q > p$. Due to the boundedness of $\{\mathbf{v}_{\varepsilon_n}\}_{n=1}^{+\infty}$ in $L^q(0,T;[W^{1,p}(\Omega)]^3)$, for any $q > p$, the following inequality contradicts the relation (4.72):

$$\|v_{\alpha,\varepsilon_n} - \int_0^1 v_{\alpha,\varepsilon_n} \, \mathrm{d}x_3\|_{L^p(0,T;L^\infty(\Omega))} =$$

$$= \sqrt[p]{\int\limits_{(0,T)\setminus I_{\varepsilon_n,c_1}} \|v_{\alpha,\varepsilon_n}(t) - \int_0^1 v_{\alpha,\varepsilon_n}(t) \, \mathrm{d}x_3\|_\infty^p \, \mathrm{d}t + \int\limits_{I_{\varepsilon_n,c_1}} \|\cdot\|_\infty^p \, \mathrm{d}t} <$$

$$< \frac{c_1}{1+\delta} + \|v_{\alpha,\varepsilon_n} - \int_0^1 v_{\alpha,\varepsilon_n} \, \mathrm{d}x_3\|^p_{L^q(0,T;L^\infty(\Omega))} |I_{\varepsilon_n,c_1}|^{\frac{q-p}{q}} \overset{n\to+\infty}{\longrightarrow} \frac{c_1}{1+\delta} < c_1.$$

(ii) We show that there exist a nonempty set $J_{\varepsilon_n,c_3} \subset (0,T)$, for $c_3 > 0$ large enough, such that the following inequality holds

$$\|v_{\alpha,\varepsilon_n}(t)\|_{2,\partial\Omega} + \|\mathrm{D}_{12}\mathbf{v}_{\varepsilon_n}(t)\|_p \leq c_3, \text{ for almost all } t \in J_{\varepsilon_n,c_3}, \tag{4.74}$$

where $\mathrm{D}_{12}\mathbf{v}_{\varepsilon_n}$ is 2×2 submatrix of $\mathrm{D}\mathbf{v}_{\varepsilon_n}$ constituted of the first two rows and columns.

If not, then without the loss of generality there exists a sequence $\{c_3(n,t)\}_{n=1}^{+\infty}$, $c_3(n,t) \to +\infty$, for almost all $t \in (0,T)$, such that

$$\|v_{\alpha,\varepsilon_n}(t)\|_{2,\partial\Omega} + \|\mathrm{D}_{12}\mathbf{v}_{\varepsilon_n}(t)\|_p > c_3(n,t),$$

which would be a contradiction with the boundedness of $\{\mathbf{v}_{\varepsilon_n}\}_{n=1}^{+\infty}$ – see condition (4.38).

(iii) It holds that $\sup_{n\in\mathbb{N}} |(0,T) \setminus J_{\varepsilon_n,c_3}| \to 0$ for $c_3 \to +\infty$.

If not, then there exist a sequence $\{c_3(m)\}_{m=1}^{+\infty}$, $c_3(m) \to +\infty$, and a positive constant K_1 such that $\sup_{n\in\mathbb{N}} |(0,T) \setminus J_{\varepsilon_n,c_3(m)}| \geq K_1 > 0$, $\forall c_3(m) \geq c_3(m_0)$, $m_0 \in \mathbb{N}$. It implies (passing to a subsequence of $\{\varepsilon_n\}_{n=1}^{+\infty}$ if necessary)

$$\|v_{\alpha,\varepsilon_n}(t)\|_{2,\partial\Omega} + \|\mathrm{D}_{12}\mathbf{v}_{\varepsilon_n}(t)\|_p > c_3(m), \text{ for almost all } t \in (0,T) \setminus J_{\varepsilon_n,c_3(m)},$$

where $n = n(m)$, and we would get a contradiction with the boundedness of sequence $\{\mathbf{v}_{\varepsilon_n}\}_{n=1}^{+\infty}$ – see condition (4.38).

(iv) For convenience, we simplify the notation $v_{\alpha,\varepsilon_n} = v_{\alpha,\varepsilon_n}(t) \in W^{1,p}(\Omega)$. We justify that

$$\|\mathrm{D}_3\mathbf{v}_{\varepsilon_n}\|_p + \|v_{3,\varepsilon_n}\|_{2,\partial\Omega} \to 0, \text{ for almost all } t \in (0,T), \tag{4.75}$$

where

$$\mathrm{D}_3\mathbf{v}_{\varepsilon_n} = \begin{pmatrix} 0 & 0 & \frac{1}{2}\left(\partial_1 v_{3,\varepsilon_n} + \partial_3 v_{1,\varepsilon_n}\right) \\ \cdot & 0 & \frac{1}{2}\left(\partial_2 v_{3,\varepsilon_n} + \partial_3 v_{2,\varepsilon_n}\right) \\ \text{sym} & \cdot & \partial_3 v_{3,\varepsilon_n} \end{pmatrix}.$$

Comparing $\mathrm{D}_3\mathbf{v}_{\varepsilon_n}$ and term (4.22), the statement of this step follows from definitions of $\mathbf{v}_{\varepsilon_n}$ and $\mathrm{D}_3\mathbf{v}_{\varepsilon_n}$, and boundedness (4.37), (4.38) and (4.42).

(v) According to part (iii), $\sup_{n\in\mathbb{N}} |I_{\varepsilon_n,c_1} \setminus J_{\varepsilon_n,c_3}|$ tends to zero for any c_1 and c_3 approaching $+\infty$ as $I_{\varepsilon_n,c_1} \subset (0,T)$. Therefore, we get $I_{\varepsilon_n,c_1} \cap J_{\varepsilon_n,c_3} \to I_{\varepsilon_n,c_1}$ for $c_3 \to +\infty$. Hence, we can assume that both conditions (4.73) and (4.74) hold for almost all $t \in I_{\varepsilon_n,c_1}$. We prove that

$$\|v_{\alpha,\varepsilon_n}(t) - \int_0^1 v_{\alpha,\varepsilon_n}(t) \, \mathrm{d}x_3\|_\infty \leq c(\|\mathrm{D}_3\mathbf{v}_{\varepsilon_n}(t)\|_p + \|v_{3,\varepsilon_n}(t)\|_{2,\partial\Omega}), \tag{4.76}$$

for almost all $t \in I_{\varepsilon_n,c_1}$, where $c = c(c_1, c_3) > 0$. For simplicity, we denote $v_{\alpha,\varepsilon_n} = v_{\alpha,\varepsilon_n}(t) \in W^{1,p}(\Omega)$ again.

There are two options for the behavior of $\|v_{\alpha,\varepsilon_n} - \int_0^1 v_{\alpha,\varepsilon_n}\, \mathrm{d}x_3\|_\infty$. First, let us assume that

$$\|v_{\alpha,\varepsilon_n} - \int_0^1 v_{\alpha,\varepsilon_n}\, \mathrm{d}x_3\|_\infty \to +\infty, \text{ for } n \to +\infty.$$

For contradiction with (4.76), we further suppose that

$$c_{\varepsilon_n} = \|v_{\alpha,\varepsilon_n} - \int_0^1 v_{\alpha,\varepsilon_n}\, \mathrm{d}x_3\|_\infty > n(\|\mathrm{D}_3\mathbf{v}_{\varepsilon_n}\|_{p,} + \|v_{3,\varepsilon_n}\|_{2,\partial\Omega}).$$

Dividing this inequality by c_{ε_n} leads to

$$1 = \|w_{\alpha,\varepsilon_n} - \int_0^1 w_{\alpha,\varepsilon_n}\, \mathrm{d}x_3\|_\infty > n(\|\mathrm{D}_3\mathbf{w}_{\varepsilon_n}\|_p + \|w_{3,\varepsilon_n}\|_{2,\partial\Omega}),$$

where $\mathbf{w}_{\varepsilon_\mathbf{n}} = c_{\varepsilon_n}^{-1}\mathbf{v}_{\varepsilon_\mathbf{n}}$. We divide also (4.74) by c_{ε_n} and get the convergences $\mathrm{D}\mathbf{w}_{\varepsilon_n} \to 0$ in $L^p(\Omega)$ and $w_{\alpha,\varepsilon_n} \to 0$ in $L^2(\partial\Omega)$. From the Korn's inequality (see Lemma 4.30), we conclude that $w_{\alpha,\varepsilon_n} \to 0$ in $W^{1,p}(\Omega)$ (and also in $L^\infty(\Omega)$ from the compact embedding), which is a contradiction with the unit norm of $w_{\alpha,\varepsilon_n} - \int_0^1 w_{\alpha,\varepsilon_n}\, \mathrm{d}x_3$.

Second, let us suppose that

$$\|v_{\alpha,\varepsilon_n} - \int_0^1 v_{\alpha,\varepsilon_n}\, \mathrm{d}x_3\|_\infty \leq K < +\infty, \ \forall n \in \mathbb{N}.$$

For contradiction with (4.76), we further assume that

$$K \geq \|v_{\alpha,\varepsilon_n} - \int_0^1 v_{\alpha,\varepsilon_n}\, \mathrm{d}x_3\|_\infty > n(\|\mathrm{D}_3\mathbf{v}_{\varepsilon_n}\|_p + \|v_{3,\varepsilon_n}\|_{2,\partial\Omega}). \tag{4.77}$$

Considering inequalities (4.74), (4.77) and classical Korn's inequality, we arrive at the boundedness of $\{\|v_{\alpha,\varepsilon_n}\|_{1,p}\}_{n=1}^{+\infty}$. Therefore (passing to a subsequence if necessary), it follows from the compact embedding that $v_{\alpha,\varepsilon_n} \to v_\alpha$ in $L^\infty(\Omega)$.

Due to (4.75), $\partial_3 v_{3,\varepsilon_n} \to 0$ in $L^p(\Omega)$ which together with the convergence $v_{3,\varepsilon_n} \to 0$ in $L^2(\partial\Omega)$, gives us $v_{3,\varepsilon_n} \to 0$ in $L^p(\Omega)$ (we remind that $\Omega = S \times (0,1)$). Hence, $\partial_\alpha v_{3,\varepsilon_n} \to 0$ in $\mathcal{D}^*(\Omega)$ and also $\partial_3 v_{\alpha,\varepsilon_n} \to 0$ in $\mathcal{D}^*(\Omega)$ due to the convergence (4.75). To conclude, $\partial_3 v_{\alpha,\varepsilon_n} \to 0$ in $\mathcal{D}^*(\Omega)$ implies $v_\alpha = \int_0^1 v_\alpha \mathrm{d}x_3$ which contradicts the inequality (4.73).

(vi) There is a contradiction.

Since convergence (4.75) and inequality (4.76) holds (see steps (iv) and (v)), we arrive at a contradiction with inequality (4.73). It means that the statement of this lemma holds true. $\square$

Let us remind that $\mathbf{v}_\varepsilon = (\mathbf{u}_\varepsilon \cdot \mathbf{g}_{1,\varepsilon}, \mathbf{u}_\varepsilon \cdot \mathbf{g}_{2,\varepsilon}, \mathbf{u}_\varepsilon \cdot \varepsilon \mathbf{a}_3)$. We apply Lemma 4.2 in the following way. First, it holds that

$$
\begin{aligned}
&\int_0^T \int_S \int_0^1 \rho_\varepsilon v_{\alpha,\varepsilon} v_{\beta,\varepsilon} g^{\alpha\beta,\varepsilon} \psi \sqrt{d_\varepsilon}\ \mathrm{d}x_3 \mathrm{d}\hat{x} \mathrm{d}t = \\
&= \int_0^T \int_S \int_0^1 \rho_\varepsilon v_{\alpha,\varepsilon} (v_{\beta,\varepsilon} - \int_0^1 v_{\beta,\varepsilon} \mathrm{d}x_3) g^{\alpha\beta,\varepsilon} \psi \sqrt{d_\varepsilon}\ \mathrm{d}x_3 \mathrm{d}\hat{x} \mathrm{d}t + \\
&+ \int_0^T \int_S \int_0^1 \rho_\varepsilon v_{\alpha,\varepsilon} \left(\int_0^1 v_{\beta,\varepsilon} \mathrm{d}x_3 \right) g^{\alpha\beta,\varepsilon} \psi \sqrt{d_\varepsilon}\ \mathrm{d}x_3 \mathrm{d}\hat{x} \mathrm{d}t = \\
&= \int_0^T \int_S \int_0^1 \rho_\varepsilon v_{\alpha,\varepsilon} (v_{\beta,\varepsilon} - \int_0^1 v_{\beta,\varepsilon} \mathrm{d}x_3) g^{\alpha\beta,\varepsilon} \psi \sqrt{d_\varepsilon}\ \mathrm{d}x_3 \mathrm{d}\hat{x} \mathrm{d}t + \\
&+ \int_0^T \int_S \left(\int_0^1 \rho_\varepsilon v_{\alpha,\varepsilon} g^{\alpha\beta,\varepsilon} \sqrt{d_\varepsilon}\ \mathrm{d}x_3 \right) \left(\int_0^1 v_{\beta,\varepsilon} \mathrm{d}x_3 \right) \psi\ \mathrm{d}\hat{x} \mathrm{d}t
\end{aligned}
$$

where $\psi \in \mathcal{C}_0^\infty(0,T;\mathcal{C}^\infty(\bar{\Omega}))$, $\mathrm{d}\hat{x} = \mathrm{d}x_1 \mathrm{d}x_2$ and α, $\beta = 1,2$. The first integral on the right-hand side tends to zero for $\varepsilon \to 0$ due to Lemma 4.2. Concerning the second integral on the right-hand side, it holds that (due to convergences (4.67) and (4.70))

$$
\begin{aligned}
&\int_0^1 \rho_\varepsilon v_{\alpha,\varepsilon} g^{\alpha\beta,\varepsilon} \sqrt{d_\varepsilon}\ \mathrm{d}x_3 = \int_0^1 (\rho_\varepsilon v_{\alpha,\varepsilon} g^{\alpha\beta,\varepsilon} - \varepsilon \mathbf{z}_{\alpha,\varepsilon} \cdot \mathbf{g}^{\beta,\varepsilon}) \sqrt{d_\varepsilon}\ \mathrm{d}x_3 + \\
&+ \varepsilon \int_0^1 \mathbf{z}_{\alpha,\varepsilon} \cdot \mathbf{g}^{\beta,\varepsilon} \sqrt{d_\varepsilon}\ \mathrm{d}x_3 \to \int_0^1 \rho_\varepsilon v_\alpha g^{\alpha\beta} \sqrt{d}\ \mathrm{d}x_3,
\end{aligned}
$$

where the first integral converges in $\mathcal{C}(\langle 0,T\rangle;[W^1L_{\Psi_1}(S)]^*)$ and the second integral converges in $L^p(0,T;L_{\Phi_\gamma}(S))$, and also

$$\int_0^1 v_{\beta,\varepsilon}\ \mathrm{d}x_3 \rightharpoonup \int_0^1 v_\beta\ \mathrm{d}x_3 \quad \text{in } L^p(0,T;W^{1,p}(S)),$$

which follows from (4.52). In addition, it holds that

$$\int_0^1 \rho_\varepsilon v_\alpha g^{\alpha\beta}\sqrt{d}\ \mathrm{d}x_3 = \hat{\rho} v_\alpha g^{\alpha\beta}\sqrt{d}$$

$$\int_0^1 v_\beta\ \mathrm{d}x_3 = v_\beta,$$

where $\hat{\rho} = \int_0^1 \rho\ \mathrm{d}x_3$, because $\mathbf{v}$ (as well as $\mathbf{u}$, see (4.56)) is independent of x_3. Hence, convergences

$$\begin{aligned}&\int_0^T\int_\Omega \rho_\varepsilon(\mathbf{u}_\varepsilon\cdot\mathbf{g}_{\alpha,\varepsilon})(\mathbf{u}_\varepsilon\cdot\mathbf{g}_{\beta,\varepsilon})g^{\alpha\beta,\varepsilon}\psi\sqrt{d_\varepsilon}\ \mathrm{d}x\mathrm{d}t \to \\ &\to \int_0^T\int_S \hat{\rho}(\mathbf{u}\cdot\mathbf{a}_\alpha)(\mathbf{u}\cdot\mathbf{a}_\beta)g^{\alpha\beta}\psi\sqrt{d}\ \mathrm{d}\hat{x}\mathrm{d}t,\ \alpha,\beta=1,2,\end{aligned} \tag{4.78}$$

are immediate consequences of (4.52), (4.70) and (4.71). Convergences (4.78) are applied in the next section to overcome the nonlinearity in the second term on the left-hand side of (4.26)

4.4.3 Limit of the governing equations

We prescribe the behavior of initial states for $\varepsilon\to 0$ by formulas

$$\int_0^1 \rho_{0,\varepsilon}\ln(\rho_{0,\varepsilon})\sqrt{d_\varepsilon}\ \mathrm{d}x_3 \to \rho_0\ln(\rho_0)\sqrt{d} \quad \text{in } L^1(S), \tag{4.79}$$

$$\int_0^1 \Phi_\gamma(\rho_{0,\varepsilon})\sqrt{d_\varepsilon}\ \mathrm{d}x_3 \to \Phi_\gamma(\rho_0)\sqrt{d} \quad \text{in } L^1(S),\ \gamma>3, \tag{4.80}$$

$$\int_0^1 \frac{|(\rho_\varepsilon\mathbf{u}_\varepsilon)_0|^2}{2\rho_{0,\varepsilon}}\sqrt{d_\varepsilon}\ \mathrm{d}x_3 \to \frac{|(\rho\mathbf{u})_0|^2}{2\rho_0}\sqrt{d} \quad \text{in } L^1(S), \tag{4.81}$$

where all limits on the right-hand sides do not depend on x_3. We remark that the prescribed behavior (4.80) enables us to use the Gronwall's lemma in the proof of boundedness (4.45). Further, we assume that $h(\varepsilon) > 0$ in (4.26) satisfies the condition $h(\varepsilon) \sim O(\varepsilon)$ to assure the convergence of $\frac{h(\varepsilon)}{\varepsilon}$ to a real positive number.

In this section, we denote an integral of a function in the third spatial variable over interval $(0, 1)$ by symbol " $\hat{}$ " over the function. Obviously, these mean values depend only on x_1 and x_2. For example, we write $\hat{\rho} = \int_0^1 \rho \, \mathrm{d}x_3$. Since $\mathbf{u} \cdot \mathbf{a}_3 = 0$ – see (4.53), and $\mathbf{u}$ is independent of x_3 – see (4.56), it holds that

$$\hat{\mathbf{u}} = \int_0^1 \mathbf{u} \, \mathrm{d}x_3 = \mathbf{u} = (\mathbf{u} \cdot \mathbf{a}_1)\mathbf{a}^1 + (\mathbf{u} \cdot \mathbf{a}_2)\mathbf{a}^2.$$

Now, we can perform the limit in (4.25) and (4.26). We use convergences mentioned in section 4.2.1. First, we test the equation (4.25) by function $\varphi \in \mathcal{D}(\mathbb{R}^2 \times (0, T))$. We arrive at

$$\int_0^T \int_\Omega \left(\rho_\varepsilon \partial_t \varphi + \rho_\varepsilon \mathbf{u}_\varepsilon^{\mathrm{T}} (\mathbf{g}^{1,\varepsilon}, \mathbf{g}^{2,\varepsilon}, \mathbf{g}^{3,\varepsilon})(\partial_1 \varphi, \partial_2 \varphi, 0)^{\mathrm{T}} \right) \sqrt{d_\varepsilon} \, \mathrm{d}x \mathrm{d}t = 0.$$

Subsequently, we expand $\mathbf{u}_\varepsilon$ into the covariant basis. Since $\mathbf{g}^{\alpha,\varepsilon} \cdot \mathbf{a_3} = 0$, for $\alpha = 1, 2$, we obtain

$$\int_0^T \int_\Omega \Big(\rho_\varepsilon \partial_t \varphi + \rho_\varepsilon [(\mathbf{u}_\varepsilon \cdot \mathbf{g}_{1,\varepsilon})\mathbf{g}^{1,\varepsilon} + (\mathbf{u}_\varepsilon \cdot \mathbf{g}_{2,\varepsilon})\mathbf{g}^{2,\varepsilon}]^{\mathrm{T}} (\mathbf{g}^{1,\varepsilon}, \mathbf{g}^{2,\varepsilon}) \hat{\nabla} \varphi \Big) \sqrt{d_\varepsilon} \, \mathrm{d}x \mathrm{d}t = 0,$$

where $\hat{\nabla} \varphi = (\partial_1 \varphi, \partial_2 \varphi)$. Afterwards, we perform the limit for $\varepsilon \to 0$, apply convergence (4.57) and get

$$\int_0^T \int_S \left[\hat{\rho} \partial_t \varphi + \hat{\rho} \hat{\mathbf{u}}^{\mathrm{T}} R^{12} \hat{\nabla} \varphi \right] \sqrt{d} \, \mathrm{d}\hat{x} \mathrm{d}t = 0, \tag{4.82}$$

for any $\varphi \in \mathcal{D}(\mathbb{R}^2 \times \langle 0, T \rangle)$, where $R^{12} = (\mathbf{a}^1, \mathbf{a}^2)$ is a submatrix of R and $\mathrm{d}\hat{x} = \mathrm{d}x_1 \mathrm{d}x_2$.

Second, we test the equation (4.26) by function $\psi \in \mathcal{C}_0^\infty(0, T; [\mathcal{C}^\infty(\bar{\Omega})]^3)$ such that $\psi \cdot \mathbf{a}_3 = 0$, $\partial_3 \psi = 0$ and $\psi \cdot \mathbf{n}|_{\partial S \times (0,T)} = 0$. We will show the limit passage for each term in (4.26) independently.

(a) $\rho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \psi$

We expand $\mathbf{u}_\varepsilon$ into the covariant basis. Since $\psi \cdot \mathbf{a}_3 = 0$ and convergences (4.57) hold, we get

$$\int_0^T \int_\Omega \rho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \psi \sqrt{d_\varepsilon}\, \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \rho_\varepsilon [(\mathbf{u}_\varepsilon \cdot \mathbf{g}_{1,\varepsilon})\mathbf{g}^{1,\varepsilon} + (\mathbf{u}_\varepsilon \cdot \mathbf{g}_{2,\varepsilon})\mathbf{g}^{2,\varepsilon}] \cdot \partial_t \psi \sqrt{d_\varepsilon}\, \mathrm{d}x\mathrm{d}t,$$

which converges to

$$\int_0^T \int_S \hat{\rho}\hat{\mathbf{u}} \cdot \partial_t \psi \sqrt{d}\, \mathrm{d}\hat{x}\mathrm{d}t,$$

for $\varepsilon \to 0$, due to (4.57).

(b) $\rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \omega_\varepsilon(\psi)$

As $\partial_3 \psi = 0$ and $\psi \cdot \mathbf{a}_3 = 0$, we know that $[\bar{\omega}_\varepsilon(\psi)]_{33} = 0$ and also that $[\bar{\omega}_\varepsilon(\psi)]_{\alpha 3} = (\partial_\alpha \psi \cdot \mathbf{a}_3)/2$, $\alpha = 1, 2$. After expanding $\mathbf{u}_\varepsilon$ into the covariant basis and applying convergences (4.58) and (4.78), we conclude that

$$\begin{aligned} &\int_0^T \int_\Omega \rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \omega_\varepsilon(\psi) \sqrt{d_\varepsilon}\, \mathrm{d}x\mathrm{d}t = \\ &= \int_0^T \int_\Omega \rho_\varepsilon (\mathbf{u}_\varepsilon \cdot \mathbf{g}_{i,\varepsilon})(\mathbf{u}_\varepsilon \cdot \mathbf{g}_{j,\varepsilon}) g^{ij,\varepsilon} [\omega_\varepsilon(\psi)]_{ij} \sqrt{d_\varepsilon}\, \mathrm{d}x\mathrm{d}t, \end{aligned}$$

where the sum is taken over i, $j = 1, 2, 3$, converges to

$$\begin{aligned} &\int_0^T \int_S \hat{\rho} (\mathbf{u} \cdot \mathbf{a}_\alpha)(\mathbf{u} \cdot \mathbf{a}_\beta) g^{\alpha\beta} [\omega(\psi)]_{\alpha\beta} \sqrt{d}\, \mathrm{d}x\mathrm{d}t = \\ &= \int_0^T \int_S \hat{\rho}\hat{\mathbf{u}} \otimes \hat{\mathbf{u}} : \omega(\psi) \sqrt{d}\, \mathrm{d}\hat{x}\mathrm{d}t, \end{aligned}$$

for $\varepsilon \to 0$ (the sum is taken over α, $\beta = 1, 2$), where

$$\omega(\psi) = R^{\mathrm{T}} \begin{pmatrix} \partial_1 \psi \cdot \mathbf{a}_1 & \frac{1}{2}(\partial_1 \psi \cdot \mathbf{a}_2 + \partial_2 \psi \cdot \mathbf{a}_1) & \frac{1}{2}\partial_1 \psi \cdot \mathbf{a}_3 \\ \cdot & \partial_2 \psi \cdot \mathbf{a}_2 & \frac{1}{2}\partial_2 \psi \cdot \mathbf{a}_3 \\ \mathrm{sym} & \cdot & 0 \end{pmatrix} R.$$

(c) $\rho_\varepsilon \nabla \psi : R_\varepsilon E_\varepsilon$

Since $R_\varepsilon E_\varepsilon = (\mathbf{g}^{1,\varepsilon}, \mathbf{g}^{1,\varepsilon}, \varepsilon^{-1}\mathbf{a}_3)$, we have

$$\int_0^T \int_\Omega \rho_\varepsilon(\partial_1\psi, \partial_2\psi, \partial_3\psi) : (\mathbf{g}^{1,\varepsilon}, \mathbf{g}^{1,\varepsilon}, \varepsilon^{-1}\mathbf{a}_3)\sqrt{d_\varepsilon}\ \mathrm{d}x\mathrm{d}t =$$

$$= \int_0^T \int_\Omega \rho_\varepsilon(\partial_1\psi, \partial_2\psi) : (\mathbf{g}^{1,\varepsilon}, \mathbf{g}^{1,\varepsilon})\sqrt{d_\varepsilon}\ \mathrm{d}x\mathrm{d}t,$$

which tends to (see convergence (4.49))

$$\int_0^T \int_S \hat{\rho}\hat{\nabla}\psi : R^{12}\sqrt{d}\ \mathrm{d}\hat{x}\mathrm{d}t,$$

for $\varepsilon \to 0$.

(d) $P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right)\omega_\varepsilon(\mathbf{u}_\varepsilon) : \omega_\varepsilon(\psi)$

It holds that (see convergence (4.51))

$$\int_0^T \int_\Omega P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right)\omega_\varepsilon(\mathbf{u}_\varepsilon) : \omega_\varepsilon(\psi)\sqrt{d_\varepsilon}\ \mathrm{d}x\mathrm{d}t \to \int_0^T \int_\Omega \overline{P(|\zeta|)\zeta} : \omega(\psi)\sqrt{d}\ \mathrm{d}x\mathrm{d}t,$$

for $\varepsilon \to 0$, where ζ is defined by (4.54). Later, we will show that

$$\int_0^t \int_\Omega \overline{P(|\zeta|)\zeta} : \omega(\psi)\sqrt{d}\ \mathrm{d}x\mathrm{d}s = \int_0^t \int_S P(|\omega(\hat{\mathbf{u}})|)\omega(\hat{\mathbf{u}}) : \omega(\psi)\sqrt{d}\ \mathrm{d}\hat{x}\mathrm{d}s,$$

for any $t \in (0, T)$.

(e) $\rho_\varepsilon\mathbf{f}_\varepsilon \cdot \psi$

After expanding $\mathbf{f}_\varepsilon$ into the contravariant basis and applying the relation $\psi\cdot\mathbf{a}_3 = 0$, we arrive at (see also convergence (4.49) and assumptions of Theorem 4.3 in section 4.5)

$$\int_0^T \int_\Omega \rho_\varepsilon\mathbf{f}_\varepsilon \cdot \psi\sqrt{d_\varepsilon}\ \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \rho_\varepsilon[(\mathbf{f}_\varepsilon \cdot \mathbf{g}^{1,\varepsilon})\mathbf{g}_{1,\varepsilon} + (\mathbf{f}_\varepsilon \cdot \mathbf{g}^{2,\varepsilon})\mathbf{g}_{2,\varepsilon}] \cdot \psi\sqrt{d_\varepsilon}\ \mathrm{d}x\mathrm{d}t$$

which tends to

$$\int_0^T \int_S \widehat{\rho\mathbf{F}} \cdot \psi\sqrt{d}\ \mathrm{d}\hat{x}\mathrm{d}t,$$

for $\varepsilon \to 0$, where $\mathbf{F} = (\mathbf{f} \cdot \mathbf{a}^1)\mathbf{a}_1 + (\mathbf{f} \cdot \mathbf{a}^2)\mathbf{a}_2$ and $\mathbf{f}$ denotes the limit of $\mathbf{f}_\varepsilon$.

(f) $\mathbf{u}_\varepsilon \cdot \psi |R_\varepsilon E_\varepsilon \mathbf{n}|$

Since $\mathbf{n} = (n_1, n_2, 0)^{\mathrm{T}}$ on Γ_1, we have

$$\mathbf{u}_\varepsilon \cdot \psi |R_\varepsilon E_\varepsilon \mathbf{n}| = \mathbf{u}_\varepsilon \cdot \psi |(\mathbf{g}^{1,\varepsilon}, \mathbf{g}^{2,\varepsilon})\hat{\mathbf{n}}|,$$

where $\hat{\mathbf{n}} = (n_1, n_2)$. Due to (4.40) and (4.52), we arrive at

$$\int_0^T \int_{\Gamma_1} \mathbf{u}_\varepsilon \cdot \psi |R_\varepsilon E_\varepsilon \mathbf{n}| \sqrt{d_\varepsilon} \, \mathrm{d}\Gamma \mathrm{d}t \to \int_0^T \int_{\partial S} \hat{\mathbf{u}} \cdot \psi |R^{12} \hat{\mathbf{n}}| \sqrt{d} \, \mathrm{d}S \mathrm{d}t,$$

as ε tends to zero.

(g) $\frac{h(\varepsilon)}{\varepsilon} \mathbf{u}_\varepsilon \cdot \psi$

According to the supposed behavior of $h(\varepsilon)$, i. e. $h(\varepsilon) \sim O(\varepsilon)$, we can use convergences (4.40) and (4.52) and get

$$\varepsilon^{-1} \int_0^T \int_{\Gamma_2} h(\varepsilon) \mathbf{u}_\varepsilon \cdot \psi \sqrt{d_\varepsilon} \, \mathrm{d}\Gamma \mathrm{d}t \to 2h \int_0^T \int_S \hat{\mathbf{u}} \cdot \psi \sqrt{d} \, \mathrm{d}\hat{x} \mathrm{d}t,$$

for $\varepsilon \to 0$, where h is a positive constant.

Finally, we arrive at

$$\begin{aligned} &\int_0^T \int_S \left[\hat{\rho} \hat{\mathbf{u}} \cdot \partial_t \psi + \hat{\rho} \hat{\mathbf{u}} \otimes \hat{\mathbf{u}} : \omega(\psi) + \hat{\rho} \hat{\nabla} \psi : R^{12} \right] \sqrt{d} \, \mathrm{d}\hat{x} \mathrm{d}t = \\ &= \int_0^T \int_\Omega \overline{P(|\zeta|)\zeta} : \omega(\psi) \sqrt{d} \, \mathrm{d}x \mathrm{d}t - \int_0^T \int_S \widehat{\rho \mathbf{F}} \cdot \psi \sqrt{d} \, \mathrm{d}\hat{x} \mathrm{d}t + \\ &+ q \int_0^T \int_{\partial S} \hat{\mathbf{u}} \cdot \psi |R^{12} \hat{\mathbf{n}}| \sqrt{d} \, \mathrm{d}S \mathrm{d}t + 2h \int_0^T \int_S \hat{\mathbf{u}} \cdot \psi \sqrt{d} \, \mathrm{d}\hat{x} \mathrm{d}t, \end{aligned} \tag{4.83}$$

for $\psi \in \mathcal{C}_0^\infty(0, T; [\mathcal{C}^\infty(\bar{\Omega})]^3)$ such that $\psi \cdot \mathbf{a}_3 = 0$, $\partial_3 \psi = 0$ and $\psi \cdot \mathbf{n}|_{\partial S \times (0,T)} = 0$.

4.4.4 Limit of the energy equality

Applying similar approach as in section 4.4.3, convexity and Jensen's inequality, we perform the limit for $\varepsilon \to 0$ also in the energy equality (4.29). We arrive at the following inequality:

$$\begin{aligned}
&\int_S \left(\hat{\rho}\frac{|\hat{\mathbf{u}}|^2}{2} + \hat{\rho}\ln(\hat{\rho}) \right) \sqrt{d}\ \mathrm{d}\hat{x} + \int_0^t \int_\Omega \overline{P\left(|\zeta|\right)|\zeta|^2}\sqrt{d}\ \mathrm{d}x\mathrm{d}s + \\
&+q\int_0^t \int_{\partial S} |\hat{\mathbf{u}}|^2 |R^{12}\hat{\mathbf{n}}|\sqrt{d}\ \mathrm{d}S\mathrm{d}s + 2h\int_0^t \int_S |\hat{\mathbf{u}}|^2\sqrt{d}\ \mathrm{d}\hat{x}\mathrm{d}s \leq \qquad (4.84)\\
&\leq \int_0^t \int_S \widehat{\rho\mathbf{F}}\cdot\hat{\mathbf{u}}\sqrt{d}\ \mathrm{d}\hat{x}\mathrm{d}s + \int_S \frac{|(\rho\mathbf{u})_0|^2}{2\rho_0}\sqrt{d}\ \mathrm{d}x + \int_S \rho_0\ln(\rho_0)\sqrt{d}\ \mathrm{d}x.
\end{aligned}$$

By the use of the same procedure as in [19], Lemmas 3.2 and 3.3, based on the renormalized continuity equation and the Steklov function, we can derive from (4.82) and (4.83) the energy equality

$$\begin{aligned}
&\int_S \left(\hat{\rho}\frac{|\hat{\mathbf{u}}|^2}{2} + \hat{\rho}\ln(\hat{\rho}) \right) \sqrt{d}\ \mathrm{d}\hat{x} + \int_0^t \int_\Omega \overline{P\left(|\zeta|\right)\zeta} : \omega(\hat{\mathbf{u}})\sqrt{d}\ \mathrm{d}x\mathrm{d}s + \\
&+q\int_0^t \int_{\partial S} |\hat{\mathbf{u}}|^2 |R^{12}\hat{\mathbf{n}}|\sqrt{d}\ \mathrm{d}S\mathrm{d}s + 2h\int_0^t \int_S |\hat{\mathbf{u}}|^2\sqrt{d}\ \mathrm{d}\hat{x}\mathrm{d}s = \qquad (4.85)\\
&= \int_0^t \int_S \widehat{\rho\mathbf{F}}\cdot\hat{\mathbf{u}}\sqrt{d}\ \mathrm{d}\hat{x}\mathrm{d}s + \int_S \frac{|(\rho\mathbf{u})_0|^2}{2\rho_0}\sqrt{d}\ \mathrm{d}x + \int_S \rho_0\ln(\rho_0)\sqrt{d}\ \mathrm{d}x.
\end{aligned}$$

Since the function $P(|z|)z$ is monotone, we get

$$\begin{aligned}
0 &\leq \lim_{\varepsilon\to 0}\int_0^t \int_\Omega \left(P(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|)\omega_\varepsilon(\mathbf{u}_\varepsilon) - P(|T|)T\right) : \left(\omega_\varepsilon(\mathbf{u}_\varepsilon) - T\right)\ \mathrm{d}x\mathrm{d}s = \\
&= \lim_{\varepsilon\to 0}\int_0^t \int_\Omega P(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|)|\omega_\varepsilon(\mathbf{u}_\varepsilon)|^2 \mathrm{d}x\mathrm{d}s - \\
&- \int_0^t \int_\Omega \overline{P(|\zeta|)\zeta} : T + P(|T|)T : \zeta + P(|T|)|T|^2\ \mathrm{d}x\mathrm{d}s \qquad (4.86)
\end{aligned}$$

for any symmetric $T \in [\tilde{L}_M(\Omega\times(0,T))]^9$. As a consequence of (4.29), (4.85), convexity and Jensen's inequality, we arrive at

$$\lim_{\varepsilon\to 0}\int_0^t\int_\Omega P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right)|\omega_\varepsilon(\mathbf{u}_\varepsilon)|^2 \mathrm{d}x\mathrm{d}s =$$
$$= \lim_{\varepsilon\to 0}\left(-\int_\Omega\left(\rho_\varepsilon\frac{|\mathbf{u}_\varepsilon|^2}{2}+\rho_\varepsilon\ln(\rho_\varepsilon)\right)\sqrt{d_\varepsilon}\ \mathrm{d}x -\right.$$
$$-q\int_0^t\int_{\Gamma_1}|\mathbf{u}_\varepsilon|^2|R_\varepsilon E_\varepsilon \mathbf{n}|\sqrt{d_\varepsilon}\ \mathrm{d}\Gamma\mathrm{d}s - \frac{h(\varepsilon)}{\varepsilon}\int_0^t\int_{\Gamma_2}|\mathbf{u}_\varepsilon|^2\sqrt{d_\varepsilon}\ \mathrm{d}\Gamma\mathrm{d}s +$$
$$+\int_0^t\int_\Omega \rho_\varepsilon\mathbf{f}_\varepsilon\cdot\mathbf{u}_\varepsilon\sqrt{d_\varepsilon}\ \mathrm{d}x\mathrm{d}s + \int_\Omega\frac{|(\rho_\varepsilon\mathbf{u}_\varepsilon)_0|^2}{2\rho_{0,\varepsilon}}\sqrt{d_\varepsilon}\ \mathrm{d}x +$$
$$\left.+\int_\Omega\rho_{0,\varepsilon}\ln(\rho_{0,\varepsilon})\sqrt{d_\varepsilon}\ \mathrm{d}x\right) \le -\int_S\left(\hat\rho\frac{|\hat{\mathbf{u}}|^2}{2}+\hat\rho\ln(\hat\rho)\right)\sqrt{d}\ \mathrm{d}\hat{x} -$$
$$-q\int_0^t\int_{\partial S}|\hat{\mathbf{u}}|^2|R^{12}\hat{\mathbf{n}}|\sqrt{d}\ \mathrm{d}S\mathrm{d}s - 2h\int_0^t\int_S|\hat{\mathbf{u}}|^2\sqrt{d}\ \mathrm{d}\hat{x}\mathrm{d}s +$$
$$+\int_0^t\int_S\widehat{\rho\mathbf{F}}\cdot\hat{\mathbf{u}}\sqrt{d}\ \mathrm{d}\hat{x}\mathrm{d}s + \int_S\frac{|(\rho\mathbf{u})_0|^2}{2\rho_0}\sqrt{d}\ \mathrm{d}x +$$
$$+\int_S\rho_0\ln(\rho_0)\sqrt{d}\ \mathrm{d}x = \int_0^t\int_\Omega\overline{P\left(|\zeta|\right)\zeta}:\omega(\hat{\mathbf{u}})\sqrt{d}\ \mathrm{d}x\mathrm{d}s. \tag{4.87}$$

Consequently from (4.86), we get

$$0\le\int_0^t\int_\Omega\left(\overline{P(|\zeta|)\zeta}-P(|T|)T\right):\left(\omega(\hat{\mathbf{u}})-T\right)\mathrm{d}x\mathrm{d}s.$$

Taking $T=\omega(\hat{\mathbf{u}})\pm\lambda\omega(\psi)$, for $\lambda>0$, $\psi\in\mathcal{C}_0^\infty(0,T;[\mathcal{C}^\infty(\bar\Omega)]^3)$ such that $\psi\cdot\mathbf{a}_3=0$, $\partial_3\psi=0$ and $\psi\cdot\mathbf{n}|_{\partial S\times(0,T)}=0$, we conclude that

$$\int_0^t\int_\Omega\overline{P(|\zeta|)\zeta}:\omega(\psi)\mathrm{d}x\mathrm{d}s = \int_0^t\int_S P(|\omega(\hat{\mathbf{u}})|)\omega(\hat{\mathbf{u}}):\omega(\psi)\mathrm{d}\hat{x}\mathrm{d}s. \tag{4.88}$$

4.5 Main theorem for the 2D model

To sum it up, the limit equations together with the energy equality are given by the following formulas

$$\int_0^T\int_S\left[\hat\rho\partial_t\varphi+\hat\rho\hat{\mathbf{u}}^{\mathrm{T}}R^{12}\hat\nabla\varphi\right]\sqrt{d}\ \mathrm{d}\hat{x}\mathrm{d}t=0, \tag{4.89}$$

for any $\varphi \in \mathcal{D}\left(\mathbb{R}^2 \times \langle 0,T \rangle\right)$,

$$
\begin{aligned}
&\int_0^T \int_S \left[\hat\rho \hat{\mathbf{u}} \cdot \partial_t \psi + \hat\rho \hat{\mathbf{u}} \otimes \hat{\mathbf{u}} : \omega(\psi) + \hat\rho \hat\nabla \psi : R^{12}\right] \sqrt{d}\ \mathrm{d}\hat{x}\mathrm{d}t = \\
&= \int_0^T \int_S P(|\omega(\hat{\mathbf{u}})|)\omega(\hat{\mathbf{u}}) : \omega(\psi)\sqrt{d}\ \mathrm{d}\hat{x}\mathrm{d}t - \int_0^T \int_S \widehat{\rho \mathbf{F}} \cdot \psi \sqrt{d}\ \mathrm{d}\hat{x}\mathrm{d}t + \\
&+ q \int_0^T \int_{\partial S} \hat{\mathbf{u}} \cdot \psi |R^{12}\hat{\mathbf{n}}| \sqrt{d}\ \mathrm{d}S\mathrm{d}t + 2h \int_0^T \int_S \hat{\mathbf{u}} \cdot \psi \sqrt{d}\ \mathrm{d}\hat{x}\mathrm{d}t,
\end{aligned}
\tag{4.90}
$$

for any $\psi \in \mathcal{C}_0^\infty(0,T;[\mathcal{C}^\infty(\Omega)]^3)$ such that $\partial_3\psi = 0$, $\psi \cdot \mathbf{a}_3 = 0$ in $\Omega \times (0,T)$ and $\psi \cdot \mathbf{n}|_{\partial S \times (0,T)} = 0$,

$$
\begin{aligned}
&\int_S \left(\hat\rho \frac{|\hat{\mathbf{u}}|^2}{2} + \hat\rho \ln(\hat\rho)\right) \sqrt{d}\ \mathrm{d}\hat{x} + \int_0^t \int_S P\left(|\omega(\hat{\mathbf{u}})|\right) |\omega(\hat{\mathbf{u}})|^2 \sqrt{d}\ \mathrm{d}\hat{x}\mathrm{d}s + \\
&+ q \int_0^t \int_{\partial S} |\hat{\mathbf{u}}|^2 |R^{12}\hat{\mathbf{n}}| \sqrt{d}\ \mathrm{d}S\mathrm{d}s + 2h \int_0^t \int_S |\hat{\mathbf{u}}|^2 \sqrt{d}\ \mathrm{d}\hat{x}\mathrm{d}s = \\
&= \int_0^t \int_S \widehat{\rho \mathbf{F}} \cdot \hat{\mathbf{u}} \sqrt{d}\ \mathrm{d}\hat{x}\mathrm{d}s + \int_S \frac{|(\rho\mathbf{u})_0|^2}{2\rho_0} \sqrt{d}\ \mathrm{d}\tilde{x} + \int_S \rho_0 \ln(\rho_0) \sqrt{d}\ \mathrm{d}\tilde{x}.
\end{aligned}
\tag{4.91}
$$

Finally, the main result of this chapter is summarized in the following theorem.

Theorem 4.3. *Let us assume that couples $(\rho_\varepsilon, \mathbf{u}_\varepsilon)$, $\varepsilon \in (0,1)$, satisfying*

$$
\begin{aligned}
&\rho_\varepsilon \in L^\infty(0,T; L_{\Phi_\gamma}(\Omega)), \\
&\mathbf{v}_\varepsilon \in L^p(0,T; [W^{1,p}(\Omega)]^3) \cap L^2(0,T; [L^2(\partial\Omega)]^3)
\end{aligned}
$$

with $\mathbf{v}_\varepsilon = (\mathbf{u}_\varepsilon \cdot \mathbf{g}_{1,\varepsilon}, \mathbf{u}_\varepsilon \cdot \mathbf{g}_{2,\varepsilon}, \mathbf{u}_\varepsilon \cdot \mathbf{g}_{3,\varepsilon})$ for arbitrary but fixed $\gamma > 3$ and $p > 3$, are weak solutions to the transformed equations (4.25)-(4.26) with initial states $\rho_{0,\varepsilon} \in L_{\Phi_\gamma}(\Omega)$ and $\frac{|(\rho_\varepsilon \mathbf{u}_\varepsilon)_0|^2}{2\rho_{0,\varepsilon}} \sqrt{d_\varepsilon} \in L^1(\Omega)$ satisfying (4.79)-(4.81). In addition, we assume that Navier boundary conditions (4.1)-(4.3) hold and $\omega_\varepsilon(\mathbf{u}_\varepsilon) \in [\tilde{L}_M(\Omega \times (0,T))]^9$.

Further, we suppose that function P complies with conditions (1.3)-(1.7), $\mathbf{f}_\varepsilon \to \mathbf{f}$ in $[L^\infty(\Omega \times (0,T))]^3$ and $\mathbf{f}_\varepsilon \cdot \mathbf{g}^{j,\varepsilon} \in [L^\infty(\Omega \times (0,T))]^3$, $j = 1,2,3$, $h(\varepsilon) > 0$ behaves like $O(\varepsilon)$, $q > 0$ and covariant basis $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} \subset [L^\infty(\Omega)]^3$ satisfies conditions

$\partial_\alpha \mathbf{a}_j$ and $\partial^2_{\alpha\beta}\mathbf{a}_3 \in [L^\infty(\Omega)]^3$, where α, $\beta = 1,2$ and $j = 1,2,3$. Then (passing to subsequences if necessary)

$$
\begin{aligned}
\rho_\varepsilon &\overset{*}{\rightharpoonup} \rho \quad \textit{in } L^\infty\left(0,T; L_{\Phi_\gamma}(\Omega)\right), \\
\rho_\varepsilon &\to \rho \quad \textit{in } \mathcal{C}\left(\langle 0,T\rangle; [W^1 L_{\Phi_\gamma}(\Omega)]^*\right), \\
\omega_\varepsilon(\mathbf{u}_\varepsilon) &\overset{N}{\rightharpoonup} \omega(\mathbf{u}) \\
\mathbf{u}_\varepsilon \cdot \mathbf{g}_{\alpha,\varepsilon} &\rightharpoonup \mathbf{u}\cdot\mathbf{a}_\alpha \quad \textit{in } L^p\left(0,T; W^{1,p}(\Omega)\right) \cap L^2\left(0,T; L^2(\partial\Omega)\right), \\
& \qquad \alpha = 1,2, \\
\mathbf{u}_\varepsilon \cdot \mathbf{a}_3 &\to 0 \quad \textit{in } L_M(\Omega \times (0,T)).
\end{aligned}
$$

In addition, couple $(\hat{\rho}, \hat{\mathbf{u}})$, where $\hat{\rho} = \int_0^1 \rho \, \mathrm{d}x_3$ and $\hat{\mathbf{u}} = (\mathbf{u}\cdot\mathbf{a}_1)\mathbf{a}^1 + (\mathbf{u}\cdot\mathbf{a}_2)\mathbf{a}^2$, $\hat{\mathbf{u}}\cdot\hat{\mathbf{n}}|_{\partial S\times(0,T)} = 0$, is a weak solution to the equations (4.89)-(4.90) and complies with the energy equality (4.91).

Chapter 5

Conclusion

Three dimensional model describing fluid motion was considered. In particular, we studied the dynamics of compressible non-Newtonian fluids in thin domains. Thus, we dealt with nonsteady Navier-Stokes equations

$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) &= 0, \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla \rho &= \operatorname{div}(P(|D\mathbf{u}|)D\mathbf{u}) + \rho \mathbf{f} \quad \text{in } \Omega \times (0,T), \end{aligned}$$

where Ω was either a thin pipe (Chapter 3) or a curved three-dimensional domain with only two dominant dimensions (Chapter 4). Our aim was to perform a rigorous derivation of respective lower-dimensional models. New results in the theory of asymptotic analysis were presented in this thesis. Our both main contributions were published in peer-reviewed journals [1, 2].

Prior to the derivation of lower-dimensional models, an introduction to Young functions and Orlicz spaces was given in Chapter 2. Further, we studied Young functions with a logarithmic and an exponential growth. Orlicz spaces defined by the use of these specific Young functions were subsequently applied to prove our main results.

Chapter 3 focused on a rigorous derivation of a one-dimensional model from the three-dimensional Navier-Stokes equations. After proving a variant of the first Korn's inequality and making a priori estimates, we demonstrated boundedness of sequences of densities and rescaled velocity fields. The boundedness allowed us to perform weak limits and pass to the limit in both the governing equations and energy equality. Theorem 3.4 (section 3.4) summarizes our first main result.

Chapter 4 was devoted to an asymptotic analysis of the three-dimensional Navier-Stokes equations acting over a curved domain. We applied a similar approach as in Chapter 4 to arrive at the limit of the governing equations and energy equality. However, the deformation of the domain introduced new difficulties which had to be addressed. Finally, we overcame all the difficulties and presented our second main contribution in Theorem 4.3 (section 4.5).

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PALACKÝ UNIVERSITY OLOMOUC
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Contents

1 Abstract

Governing equations representing mathematical description of continuum mechanics have often three spatial dimensions and one temporal dimension. However, their analytical solution is usually unattainable, and numerical approximation of the solution unduly complicated and computationally demanding. Thus, these models are frequently simplified in various ways. One option of a simplification is a reduction of the number of spatial dimensions. We focused on nonsteady Navier-Stokes equations for compressible nonlinearly viscous fluids in a three-dimensional domain. These equations need a simplification, when possible, to be effectively solved. Therefore, we performed a dimension reduction for this type of model. First, we studied the dynamics of a compressible fluid in thin domains where only one dimension is dominant. We presented a rigorous derivation of a one-dimensional model from the three-dimensional Navier-Stokes equations. Second, we extended the current framework by dealing with nonsteady Navier-Stokes equations for compressible nonlinearly viscous fluids in a deformed three-dimensional domain. We focused on a rigorous derivation of the two-dimensional model. The deformation of a domain introduced new difficulties in the asymptotic analysis, because it affects the limit equations in a non-trivial way.

Key words: Navier-Stokes equations, Compressible fluids, Nonlinear viscosity, Asymptotic analysis, Dimension reduction, Curved domain, Curvilinear coordinates

2 Abstrakt v českém jazyce

Zakladní rovnice, které reprezentují matematický popis mechaniky kontinua, mají často tři prostorové dimenze a jednu časovou. Jejich nevýhodou je, že jejich analytické řešení je často nedosažitelné a jeho numerická aproximace výpočetně velmi náročná. Z těchto důvodů jsou takovéto modely často různými způsoby zjednodušovány. Jednou z možností, jak model zjednodušit, je snížení počtu prostorových dimenzí. Otázkou ovšem zůstává, jak dimenzionální redukci provést matematicky korektně. Zabývali jsme se nestacionárními Navier-Stokesovými rovnicemi pro stlačitelné nelineárně viskózní tekutiny v trojrozměrné oblasti. Nejprve jsme studovali dynamiku stlačitelných tekutin v oblastech, kde dominuje pouze jedna prostorová dimenze. Představili jsme odvození jednorozměrného modelu z trojrozměrných Navier-Stokesových rovnic. Následně jsme rozšířili současný rámec poznání tím, že jsme aplikovali dimenzionální redukci na nestacionární Navier-Stokesovy rovnice pro stlačitelné nelineárně viskóz-ní tekutiny v deformované trojrozměrné oblasti se dvěma dominantními prostorovými dimenzemi. Zjistili jsme, že deformace oblasti netriviálne ovlivňuje výsledné limitní rovnice.

Klíčová slova: Navier-Stokesovy rovnice, Stlačitelné tekutiny, Nelineární viskozita, Asymptotická analýza, Redukce dimenze, Deformovaná oblast, Křivočaré souřadnice

3 Introduction

Governing equations representing mathematical description of continuum mechanics have often three spatial dimensions and one temporal dimension. However, their analytical solution is usually unattainable, and numerical approximation of the solution unduly complicated and computationally demanding. Therefore, these models are frequently simplified in various ways. One option of a simplification is a reduction of the number of spatial dimensions. The thesis is devoted to nonsteady Navier-Stokes equations for compressible nonlinearly viscous fluids in a three-dimensional domain. These equations need a simplification, when possible, to be effectively solved.

The thesis studies the dynamics of a compressible fluid in a thin pipe $\Omega_\varepsilon \subset \mathbb{R}^3$ and in a curved three-dimensional domain $\tilde{\Omega}_\varepsilon$ with two dominant dimensions. The motion of a compressible fluid is described by its velocity $\mathbf{u}$ and density ρ. The time evolution of $\mathbf{u}$ and ρ is governed by the continuity and momentum equations

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1}$$

$$\partial_t (\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \operatorname{div} \mathbb{S} + \rho \mathbf{f} \quad \text{in } \Omega \times (0, T), \tag{2}$$

where $T > 0$, p is the pressure, $\mathbb{S}$ stands for the viscous stress tensor and $\mathbf{f}$ represents the external forces [18].

It is supposed that the fluid is isothermal and non-Newtonian. It means that (without the loss of generality)

$$\mathbb{S} = P(|D\mathbf{u}|) D\mathbf{u}, \quad p = \rho.$$

Similarly as in [26], it is assumed that the function P satisfies, for any U, V belonging to Orlicz class $[\tilde{L}_M(\Omega)]^9$, the following five conditions

$$\int_\Omega P(|U|)|U|^2 \, \mathrm{d}x \geq \int_\Omega M(|U|) \, \mathrm{d}x, \tag{3}$$

$$\int_\Omega (P(|U|)U - P(|V|)V) : (U - V) \, \mathrm{d}x \geq 0, \tag{4}$$

$$P(z)|z|^2 \text{ is a convex function for } z \geq 0, \tag{5}$$

$$\int_\Omega N(P(|U|)|U|)\ \mathrm{d}x \leq C\left(1 + \int_\Omega M(|U|)\ \mathrm{d}x\right), \tag{6}$$

$$P(|U - \lambda V|)(U - \lambda V) \overset{M}{\rightharpoonup} P(|U|)U, \text{ for } \lambda \to 0. \tag{7}$$

4 Recent state summary

The existence of weak solutions for three-dimensional models of fluid dynamics has already been studied. For instance, Pierre-Louis Lions proved the global solvability of Navier-Stokes equations for compressible linearly viscous fluids [17]. Further, Eduard Feireisl extensively studied global existence theory for the full Navier-Stokes-Fourier system [11]. A comprehensive overview on results achieved in the case of Newtonian compressible fluids is given in [22]. Concerning non-Newtonian fluids, Mamontov [18, 19] proved the existence of a global weak solution for compressible Navier-Stokes equations. This knowledge allows us to step forward in finding the solution (or at least its approximation). One possibility to achieve that is by performing a dimension reduction of the equations. Without the proven existence of a weak solution, it would be pointless to study the asymptotic behavior of the equations.

An asymptotic analysis was performed in linear elasticity for rods and beams [13, 14, 24], and for plates and shells [4, 6, 7], at first. Subsequently, rigorous derivation of lower-dimensional models was done also for fluids. An asymptotic analysis of three-dimensional steady Navier-Stokes equations based on the asymptotic expansion was presented in [21]. For comparison, the same result was achieved directly in [28] without the need to apply any asymptotic expansion. Regarding nonsteady Navier-Stokes equations for incompressible fluids, they were simplified into a lower-dimensional model in [12]. Further, a three-dimensional system for barotropic Navier-Stokes equations was asymptotically analyzed and the resulting one-dimensional and two-dimensional models were presented in [27] and [20], respectively. It was also shown that weak solutions of both three-dimensional Navier-Stokes equations for barotropic flows and three-dimensional full Navier-Stokes-Fourier equations tend to strong solutions of the respective one-dimensional system as the three-dimensional model tends to the one-dimensional model [3, 5]. Recently, Ducomet et al. [8] presented a rigorous derivation of a two-dimensional model from the three-dimensional compressible barotropic Navier-Stokes-Poisson system with radiation.

New difficulties arise by considering non-Newtonian fluids (i. e. fluids having nonlinear viscous stress tensor). This problem was tackled for the first time in [26], where a two-dimensional model was derived by a suitable scaling from nonsteady Navier-Stokes equations for compressible fluids.

5 Thesis objectives

The thesis is aimed on obtaining new contributions to the theory of rigorous asymptotic analysis of nonsteady Navier-Stokes equations for compressible nonlinearly viscous fluids in a three dimensional domain. There are two main objectives of the thesis. First, the motion of a compressible fluid in a thin pipe is studied. We present a derivation of a one-dimensional model from equations (1)-(2) under Navier boundary conditions [1].The second aim is to investigate the motion of a compressible fluid in a thin deformed domain. We focus on a derivation of a two-dimensional model from equations (1)-(2) under Navier boundary conditions [2].

Since nonsteady Navier-Stokes equations for compressible nonlinearly viscous fluids need a simplification, when possible, to be effectively solved, achieving the objectives of the thesis is particularly valuable. Obtained lower-dimensional models can be subsequently used in real-world applications by practitioners.

6 Theoretical framework

The theoretical framework can be summarized into three main parts. First, Young functions and Orlicz spaces along with their properties are studied in the thesis. Afterwards, both problems in question are separately described in detail and transformation of governing equations is performed.

6.1 Young functions and Orlicz spaces

A brief introduction to Young functions and Orlicz spaces is presented in the thesis. Let $u : Q \to \mathbb{R}$, $Q \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a measurable function and let Φ, Ψ be a pair of complementary Young functions. The set $L_\Phi(Q)$ of all u such that $\|u\|_{L_\Phi(Q)} < +\infty$ is called the **Orlicz space**. The positive number $\|u\|_{L_\Phi(Q)}$ is defined as

$$\|u\|_{L_\Phi(Q)} = \sup_v \int_Q |u(x)v(x)| \mathrm{d}x,$$

where the supremum is taken over all functions $v \in \tilde{L}_\Psi(Q)$ satisfying condition $\int_Q \Psi(|v(x)|)\mathrm{d}x \leq 1$. The Orlicz space $L_\Phi(Q)$ is a Banach space and $\|\cdot\|_{L_\Phi(Q)}$ is the norm on $L_\Phi(Q)$.

Orlicz spaces and their properties are described in the thesis. In addition, the thesis is focused on a special class of Young functions with an exponential growth and their complementary functions, because the theory concerning these Young functions and respective Orlicz spaces is needed in the subsequent derivation of lower-dimensional models.

Let us define Young functions needed for reaching the objectives of the thesis as $\Phi_\gamma(z) = (1+z)\ln^\gamma(1+z)$, with $\gamma > 1$, and $\Phi_1(z) = z\ln(z+1)$. Functions Ψ_γ, $\gamma \geq 1$, denote the complementary functions to Φ_γ, $\gamma \geq 1$. Subsequently, we define $M(z) = \mathrm{e}^z - z - 1$ and its complementary function $N(z) = (1+z)\ln(1+z) - z$. Further, we denote $\Phi_{1/\alpha}(z)$, $\alpha \in (1, +\infty)$, the Young functions with growth $z\ln^{1/\alpha} z$, $z \geq z_0 > 0$, and their complementary functions $\Psi_{1/\alpha}(z)$.

6.2 Fluid flow in a thin pipe

Let us employ notation $\bar{\mathbf{u}}_\varepsilon$ and $\bar{\rho}_\varepsilon$ for the velocity and the density, respectively, in equations (1)-(2) to highlight the connection to Ω_ε. Similar notation (subscript ε and accent "$^-$") is applied also for other functions connected to Ω_ε.

Domain $\Omega_\varepsilon \subset \mathbb{R}^3$ is defined by the use of a referential domain $\Omega = (0,1) \times S$ with $S \subset \mathbb{R}^2$, $|S| = 1$ and $\partial S \in C^{0,1}$, and mapping $\mathbf{R}_\varepsilon : \Omega \to \Omega_\varepsilon$ so that

$$\mathbf{R}_\varepsilon : (x_1, x_2, x_3) \longmapsto (x_1, \varepsilon x_2, \varepsilon x_3).$$

It means that $\Omega_\varepsilon = (0,1) \times \varepsilon S$. As well as in [22], section 4.17.2.4, we suppose that Ω is not axially symmetric. Axial symmetry would mean that the appearance of Ω remains unchanged if rotated around an axis along the first spatial dimension.

Symbols $\mathbf{n}$ and $\bar{\mathbf{n}}_\varepsilon$ stand for unit outward normals to Ω and Ω_ε, respectively. Similarly, $\mathbf{t}$ and $\bar{\mathbf{t}}_\varepsilon$ are vectors from the corresponding tangent planes. We employ the following notation for the borders of domains Ω and Ω_ε:

$$\Gamma_1 = (0,1) \times \partial S,\ \ \Gamma_2 = \{0,1\} \times S,$$
$$\Gamma_{1,\varepsilon} = \mathbf{R}_\varepsilon\left(\Gamma_1\right),\ \ \Gamma_{2,\varepsilon} = \mathbf{R}_\varepsilon\left(\Gamma_2\right).$$

To ensure the well-posedness of our problem [26], we prescribe Navier boundary conditions

$$\bar{\mathbf{t}}_\varepsilon \cdot \left(P(|\bar{D}\bar{\mathbf{u}}_\varepsilon|)\bar{D}\bar{\mathbf{u}}_\varepsilon \bar{\mathbf{n}}_\varepsilon\right) + h(\varepsilon)\bar{\mathbf{u}}_\varepsilon \cdot \bar{\mathbf{t}}_\varepsilon = 0 \quad \text{on } \Gamma_{1,\varepsilon} \times (0,T), \tag{8}$$

$$\bar{\mathbf{t}}_\varepsilon \cdot \left(P(|\bar{D}\bar{\mathbf{u}}_\varepsilon|)\bar{D}\bar{\mathbf{u}}_\varepsilon \bar{\mathbf{n}}_\varepsilon\right) + q\bar{\mathbf{u}}_\varepsilon \cdot \bar{\mathbf{t}}_\varepsilon = 0 \quad \text{on } \Gamma_{2,\varepsilon} \times (0,T), \tag{9}$$

$$\bar{\mathbf{u}}_\varepsilon \cdot \bar{\mathbf{n}}_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon \times (0,T). \tag{10}$$

It is supposed that $h(\varepsilon) > 0$ behaves like $O(\varepsilon)$ and $q > 0$.

We consider the initial conditions for the density and the momentum

$$\bar{\rho}_\varepsilon(\bar{x}, 0) = \bar{\rho}_{0,\varepsilon}(\bar{x}) \geq 0, \quad \forall \bar{x} \in \Omega_\varepsilon$$
$$\left(\bar{\rho}_\varepsilon \bar{\mathbf{u}}_\varepsilon\right)(\bar{x}, 0) = \left(\bar{\rho}_\varepsilon \bar{\mathbf{u}}_\varepsilon\right)_0(\bar{x}, 0), \quad \forall \bar{x} \in \Omega_\varepsilon.$$

The variational formulation of our problem is

$$\int_0^T \int_{\Omega_\varepsilon} \left(\bar{\rho}_\varepsilon \partial_t \bar{\varphi} + \bar{\rho}_\varepsilon \bar{\mathbf{u}}_\varepsilon \cdot \bar{\nabla}\bar{\varphi}\right) \mathrm{d}\bar{x}\mathrm{d}t = 0, \tag{11}$$

$$\int_0^T \int_{\Omega_\varepsilon} \left(\bar{\rho}_\varepsilon \bar{\mathbf{u}}_\varepsilon \cdot \partial_t \bar{\psi} + \bar{\rho}_\varepsilon \bar{\mathbf{u}}_\varepsilon \otimes \bar{\mathbf{u}}_\varepsilon : \bar{D}\bar{\psi} + \bar{\rho}_\varepsilon \bar{\mathrm{div}} \bar{\psi}\right) \mathrm{d}\bar{x}\mathrm{d}t$$

$$= \int_0^T \int_{\Omega_\varepsilon} \left(P(|\bar{D}\bar{\mathbf{u}}_\varepsilon|)\bar{D}\bar{\mathbf{u}}_\varepsilon : \bar{D}\bar{\psi} - \bar{\rho}_\varepsilon \bar{\mathbf{f}}_\varepsilon \cdot \bar{\psi}\right) \mathrm{d}x\mathrm{d}t$$

$$+h(\varepsilon) \int_0^T \int_{\Gamma_{1,\varepsilon}} \bar{\mathbf{u}}_\varepsilon \cdot \bar{\psi} \, \mathrm{d}\bar{\Gamma}\mathrm{d}t + q \int_0^T \int_{\Gamma_{2,\varepsilon}} \bar{\mathbf{u}}_\varepsilon \cdot \bar{\psi} \, \mathrm{d}\bar{\Gamma}\mathrm{d}t, \tag{12}$$

for any $\bar{\varphi} \in \mathcal{D}(\mathbb{R}^3 \times (0,T))$ and $\bar{\psi} \in C_0^\infty(0,T; C^\infty(\bar{\Omega}_\varepsilon)^3)$ satisfying condition $\bar{\psi} \cdot \bar{\mathbf{n}}_\varepsilon|_{\partial\Omega_\varepsilon \times (0,T)} = 0$.

Variational formulation (11)–(12) can be transformed into

$$\int_0^T \int_\Omega (\rho_\varepsilon \partial_t \varphi + \rho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_\varepsilon \varphi) \, \mathrm{d}x\mathrm{d}t = 0, \tag{13}$$

$$\int_0^T \int_\Omega \left[\rho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \psi + \rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \omega_\varepsilon(\psi) + \rho_\varepsilon \mathrm{div}_\varepsilon \psi\right] \, \mathrm{d}x\mathrm{d}t$$

$$= \int_0^T \int_\Omega \left[P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right) \omega_\varepsilon(\mathbf{u}_\varepsilon) : \omega_\varepsilon(\psi) - \rho_\varepsilon \mathbf{f}_\varepsilon \cdot \psi\right] \, \mathrm{d}x\mathrm{d}t$$

$$+ \frac{h(\varepsilon)}{\varepsilon} \int_0^T \int_{\Gamma_1} \mathbf{u}_\varepsilon \cdot \psi \, \mathrm{d}\Gamma\mathrm{d}t + q \int_0^T \int_{\Gamma_2} \mathbf{u}_\varepsilon \cdot \psi \, \mathrm{d}\Gamma\mathrm{d}t, \tag{14}$$

for any $\varphi \in \mathcal{D}(\mathbb{R}^3 \times (0,T))$ and $\psi \in C_0^\infty(0,T; [C^\infty(\bar{\Omega})]^3)$, $\psi \cdot \mathbf{n}|_{\partial\Omega \times (0,T)} = 0$.

By transforming the energy equality (see [19] for its original form), we obtain

$$\int_\Omega \left(\rho_\varepsilon(t) \frac{|\mathbf{u}_\varepsilon(t)|^2}{2} + \rho_\varepsilon(t) \ln(\rho_\varepsilon(t))\right) \, \mathrm{d}x$$

$$+ \int_0^t \int_\Omega P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right) |\omega_\varepsilon(\mathbf{u}_\varepsilon)|^2 \, \mathrm{d}x\mathrm{d}s +$$

$$+ \frac{h(\varepsilon)}{\varepsilon} \int_0^t \int_{\Gamma_1} |\mathbf{u}_\varepsilon|^2 \, \mathrm{d}\Gamma\mathrm{d}s + q \int_0^t \int_{\Gamma_2} |\mathbf{u}_\varepsilon|^2 \, \mathrm{d}\Gamma\mathrm{d}s =$$

$$= \int_0^t \int_\Omega \rho_\varepsilon \mathbf{g}_\varepsilon \cdot \mathbf{v}_\varepsilon \, \mathrm{d}x\mathrm{d}s + \int_\Omega \left(\frac{|(\rho_\varepsilon \mathbf{u}_\varepsilon)_0|^2}{2\rho_{0,\varepsilon}} + \rho_{0,\varepsilon} \ln(\rho_{0,\varepsilon})\right) \, \mathrm{d}x. \tag{15}$$

for any $t \in \langle 0, T\rangle$, where $\mathbf{g}_\varepsilon = (f_{1,\varepsilon}, \varepsilon^{-1} f_{2,\varepsilon}, \varepsilon^{-1} f_{3,\varepsilon})$, $\mathbf{v}_\varepsilon = (u_{1,\varepsilon}, \varepsilon u_{2,\varepsilon}, \varepsilon u_{3,\varepsilon})$.

6.3 Fluid flow in a thin deformed domain

Let us employ notation $\tilde{\mathbf{u}}_\varepsilon$ and $\tilde{\rho}_\varepsilon$ for the velocity and the density, respectively, in equations (1)-(2) to highlight the connection to $\tilde{\Omega}_\varepsilon$. Similarly, we denote also other functions connected to $\tilde{\Omega}_\varepsilon$ with subscript ε and accent " ˜ ".

The domain $\tilde{\Omega}_\varepsilon \subset \mathbb{R}^3$ is defined by the use of a reference domain $\Omega = S \times (0,1)$, $S \subset \mathbb{R}^2$, $\partial S \in C^{0,1}$, and the mapping $\Theta_\varepsilon : \Omega \to \tilde{\Omega}_\varepsilon$ so that

$$\Theta_\varepsilon : (x_1, x_2, x_3) \longmapsto \theta(x_1, x_2) + \varepsilon x_3 \mathbf{a}_3(x_1, x_2),$$

where $\theta : S \to \mathbb{R}^3$ and

$$\begin{aligned}
\mathbf{a}_1 &= (\partial_1\theta_1, \partial_1\theta_2, \partial_1\theta_3)^{\mathrm{T}},\\
\mathbf{a}_2 &= (\partial_2\theta_1, \partial_2\theta_2, \partial_2\theta_3)^{\mathrm{T}},\\
\mathbf{a}_3 &= \frac{\mathbf{a}_1 \times \mathbf{a}_2}{\|\mathbf{a}_1 \times \mathbf{a}_2\|}.
\end{aligned}$$

We suppose that $\mathbf{a}_j$, $\partial_\alpha \mathbf{a}_j$ and $\partial^2_{\alpha\beta}\mathbf{a}_3 \in [L^\infty(\Omega)]^3$, where α, $\beta = 1, 2$ and $j = 1, 2, 3$.

Symbols $\mathbf{n}$ and $\tilde{\mathbf{n}}_\varepsilon$ stand for unit outward normals to Ω and $\tilde{\Omega}_\varepsilon$, respectively. Similarly, $\mathbf{t}$ (resp. $\tilde{\mathbf{t}}_\varepsilon$) is any vector from the corresponding tangent plane. We denote the boundaries of domains Ω and $\tilde{\Omega}_\varepsilon$ as follows:

$$\begin{gathered}
\Gamma_1 = \partial S \times (0,1),\ \Gamma_2 = S \times \{0,1\},\\
\tilde{\Gamma}_{1,\varepsilon} = \Theta_\varepsilon(\Gamma_1),\ \tilde{\Gamma}_{2,\varepsilon} = \Theta_\varepsilon(\Gamma_2).
\end{gathered}$$

To ensure the well-posedness of our problem [26], we prescribed the set of Navier boundary conditions

$$\tilde{\mathbf{t}}_\varepsilon \cdot \Big(P(|\tilde{D}\tilde{\mathbf{u}}_\varepsilon|)\tilde{D}\tilde{\mathbf{u}}_\varepsilon \tilde{\mathbf{n}}_\varepsilon \Big) + q\tilde{\mathbf{u}}_\varepsilon \cdot \tilde{\mathbf{t}}_\varepsilon = 0, \text{ on } \tilde{\Gamma}_{1,\varepsilon} \times (0,T), \tag{16}$$

$$\tilde{\mathbf{t}}_\varepsilon \cdot \Big(P(|\tilde{D}\tilde{\mathbf{u}}_\varepsilon|)\tilde{D}\tilde{\mathbf{u}}_\varepsilon \tilde{\mathbf{n}}_\varepsilon \Big) + h(\varepsilon)\tilde{\mathbf{u}}_\varepsilon \cdot \tilde{\mathbf{t}}_\varepsilon = 0, \text{ on } \tilde{\Gamma}_{2,\varepsilon} \times (0,T), \tag{17}$$

$$\tilde{\mathbf{u}}_\varepsilon \cdot \tilde{\mathbf{n}}_\varepsilon = 0, \text{ on } \partial\tilde{\Omega}_\varepsilon \times (0,T). \tag{18}$$

We suppose that $h(\varepsilon) > 0$ behaves like $O(\varepsilon)$ and $q > 0$. The asymptotic behavior of $h(\varepsilon)$ will be discussed during derivation of weak convergences of density and velocity field.

We consider the initial conditions for the density and the momentum

$$\tilde{\rho}_\varepsilon(x,0) = \tilde{\rho}_{0,\varepsilon}(x) \geq 0,$$
$$\left(\tilde{\rho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon\right)(x,0) = \left(\tilde{\rho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon\right)_0(x,0), \text{ in } \tilde{\Omega}_\varepsilon.$$

Hence, the variational formulation of our problem is

$$\int_0^T \int_{\tilde{\Omega}_\varepsilon} \left(\tilde{\rho}_\varepsilon \partial_t \tilde{\varphi} + \tilde{\rho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \cdot \tilde{\nabla} \tilde{\varphi}\right) \mathrm{d}\tilde{x}\mathrm{d}t = 0, \tag{19}$$

$$\begin{aligned}
&\int_0^T \int_{\tilde{\Omega}_\varepsilon} \left(\tilde{\rho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \cdot \partial_t \tilde{\psi} + \tilde{\rho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon : \tilde{D}\tilde{\psi} + \tilde{\rho}_\varepsilon \tilde{\operatorname{div}} \tilde{\psi}\right) \mathrm{d}\tilde{x}\mathrm{d}t \\
&= \int_0^T \int_{\tilde{\Omega}_\varepsilon} \left(P(|\tilde{D}\tilde{\mathbf{u}}_\varepsilon|)\tilde{D}\tilde{\mathbf{u}}_\varepsilon : \tilde{D}\tilde{\psi} - \tilde{\rho}_\varepsilon \tilde{\mathbf{f}}_\varepsilon \cdot \tilde{\psi}\right) \mathrm{d}x\mathrm{d}t \\
&+q \int_0^T \int_{\tilde{\Gamma}_{1,\varepsilon}} \tilde{\mathbf{u}}_\varepsilon \cdot \tilde{\psi} \ \mathrm{d}\tilde{\Gamma}\mathrm{d}t + h(\varepsilon) \int_0^T \int_{\tilde{\Gamma}_{2,\varepsilon}} \tilde{\mathbf{u}}_\varepsilon \cdot \tilde{\psi} \ \mathrm{d}\tilde{\Gamma}\mathrm{d}t,
\end{aligned} \tag{20}$$

for any $\tilde{\varphi} \in \mathcal{D}\left(\mathbb{R}^3 \times (0,T)\right)$ and $\tilde{\psi} \in C_0^\infty(0,T; C^\infty(\{\tilde{\Omega}_\varepsilon\}^-)^3)$, where $\{\tilde{\Omega}_\varepsilon\}^-$ stands for the closure of $\tilde{\Omega}_\varepsilon$, satisfying the condition $\tilde{\psi} \cdot \tilde{\mathbf{n}}_\varepsilon|_{\partial\tilde{\Omega}_\varepsilon \times (0,T)} = 0$.

After transforming equations (19)-(20), we get

$$\int_0^T \int_\Omega \left(\rho_\varepsilon \partial_t \varphi + \rho_\varepsilon \mathbf{u}_\varepsilon^{\mathrm{T}} R_\varepsilon E_\varepsilon \nabla \varphi\right) \sqrt{d_\varepsilon} \ \mathrm{d}x\mathrm{d}t = 0, \tag{21}$$

$$\begin{aligned}
&\int_0^T \int_\Omega \left[\rho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \psi + \rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \omega_\varepsilon(\psi) + \rho_\varepsilon \nabla \psi : R_\varepsilon E_\varepsilon\right] \sqrt{d_\varepsilon} \ \mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_\Omega \left[P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right) \omega_\varepsilon(\mathbf{u}_\varepsilon) : \omega_\varepsilon(\psi) - \rho_\varepsilon \mathbf{f}_\varepsilon \cdot \psi\right] \sqrt{d_\varepsilon} \ \mathrm{d}x\mathrm{d}t + \\
&+q \int_0^T \int_{\Gamma_1} \mathbf{u}_\varepsilon \cdot \psi |R_\varepsilon E_\varepsilon \mathbf{n}| \sqrt{d_\varepsilon} \ \mathrm{d}\Gamma\mathrm{d}t + \frac{h(\varepsilon)}{\varepsilon} \int_0^T \int_{\Gamma_2} \mathbf{u}_\varepsilon \cdot \psi \sqrt{d_\varepsilon} \ \mathrm{d}\Gamma\mathrm{d}t,
\end{aligned} \tag{22}$$

for any $\varphi \in \mathcal{D}\left(\mathbb{R}^3 \times (0,T)\right)$ and $\psi \in C_0^\infty(0,T; [C^\infty(\bar{\Omega})]^3)$, $\psi \cdot \mathbf{n}|_{\partial\Omega \times (0,T)} = 0$.

Similarly, transforming the energy equality [19] leads to

$$
\begin{aligned}
&\int_\Omega \left(\rho_\varepsilon(t) \frac{|\mathbf{u}_\varepsilon(t)|^2}{2} + \rho_\varepsilon(t) \ln(\rho_\varepsilon(t)) \right) \sqrt{d_\varepsilon}\ \mathrm{d}x \\
&+ \int_0^t \int_\Omega P\left(|\omega_\varepsilon(\mathbf{u}_\varepsilon)|\right) |\omega_\varepsilon(\mathbf{u}_\varepsilon)|^2 \sqrt{d_\varepsilon}\ \mathrm{d}x\mathrm{d}s + \\
&+ q \int_0^t \int_{\Gamma_1} |\mathbf{u}_\varepsilon|^2 |R_\varepsilon E_\varepsilon \mathbf{n}| \sqrt{d_\varepsilon}\ \mathrm{d}\Gamma\mathrm{d}s + \frac{h(\varepsilon)}{\varepsilon} \int_0^t \int_{\Gamma_2} |\mathbf{u}_\varepsilon|^2 \sqrt{d_\varepsilon}\ \mathrm{d}\Gamma\mathrm{d}s = \\
&= \int_0^t \int_\Omega \rho_\varepsilon \bar{\mathbf{f}}_\varepsilon \cdot \mathbf{v}_\varepsilon \sqrt{d_\varepsilon}\ \mathrm{d}x\mathrm{d}s + \\
&+ \int_\Omega \left(\frac{|(\rho_\varepsilon \mathbf{u}_\varepsilon)_0|^2}{2\rho_{0,\varepsilon}} + \rho_{0,\varepsilon} \ln(\rho_{0,\varepsilon}) \right) \sqrt{d_\varepsilon}\ \mathrm{d}x.
\end{aligned} \tag{23}
$$

for any $t \in \langle 0, T \rangle$, where

$$
\begin{aligned}
\bar{\mathbf{f}}_\varepsilon &= (\mathbf{f}_\varepsilon \cdot \mathbf{g}^{1,\varepsilon}, \mathbf{f}_\varepsilon \cdot \mathbf{g}^{2,\varepsilon}, \mathbf{f}_\varepsilon \cdot \mathbf{g}^{3,\varepsilon}), \\
\mathbf{v}_\varepsilon &= (\mathbf{u}_\varepsilon \cdot \mathbf{g}_{1,\varepsilon}, \mathbf{u}_\varepsilon \cdot \mathbf{g}_{2,\varepsilon}, \mathbf{u}_\varepsilon \cdot \mathbf{g}_{3,\varepsilon}),
\end{aligned}
$$

7 Original results and summary

Three dimensional model describing fluid motion is considered. In particular, we study the dynamics of compressible non-Newtonian fluids in thin domains. Thus, we deal with nonsteady Navier-Stokes equations

$$
\begin{aligned}
\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) &= 0, \\
\partial_t (\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla \rho &= \operatorname{div}(P(|D\mathbf{u}|) D\mathbf{u}) + \rho \mathbf{f} \quad \text{in } \Omega \times (0, T),
\end{aligned}
$$

where Ω is either a thin pipe or a curved three-dimensional domain with only two dominant dimensions. New results in the theory of asymptotic analysis are presented in this thesis. Both main contributions were published in peer-reviewed journals [1, 2].

First, the thesis is focused on a rigorous derivation of a one-dimensional model from the three-dimensional Navier-Stokes equations. After proving a variant of the first Korn's inequality and making a priori estimates, we demonstrated boundedness of sequences of densities and rescaled velocity fields. The boundedness allowed us to perform weak limits and pass to the limit in both the governing equations and energy equality. The limit equations together with the energy equality are given by:

$$\int_0^T \int_0^1 \hat{\rho}\partial_t\varphi + \hat{\rho}u_1\partial_1\varphi \;\mathrm{d}x_1\mathrm{d}t = 0, \tag{24}$$

for any $\varphi \in \mathcal{D}(\mathbb{R} \times \langle 0, T\rangle)$,

$$\begin{aligned} &\int_0^T \int_0^1 \hat{\rho}u_1\partial_t\psi_1 + \hat{\rho}u_1^2\partial_1\psi_1 + \hat{\rho}\partial_1\psi_1 \mathrm{d}x_1\mathrm{d}t = \\ &= |S| \int_0^T \int_0^1 P(|\partial_1 u_1|)\partial_1 u_1 \partial_1\psi_1 \;\mathrm{d}x_1\mathrm{d}t - \int_0^T \int_0^1 \widehat{\rho f_1}\psi_1 \mathrm{d}x_1\mathrm{d}t + \\ &+ |\partial S| h \int_0^T \int_0^1 u_1\psi_1 \mathrm{d}x_1\mathrm{d}t, \end{aligned} \tag{25}$$

for any $\psi = (\psi_1(x_1), 0, 0)$, where $\psi_1 \in \mathcal{C}_0^\infty(0, T; \mathcal{C}^\infty(\langle 0, 1\rangle))$ complies with condition $\psi_1(0, t) = \psi_1(1, t) = 0$, for all $t \in (0, T)$, and

$$\begin{aligned} &\int_0^1 \hat{\rho}\frac{|u_1|^2}{2} + \hat{\rho}\ln(\hat{\rho}) \;\mathrm{d}x_1 + |S| \int_0^t \int_\Omega P\left(|\partial_1 u_1|\right) |\partial_1 u_1|^2 \;\mathrm{d}x_1\mathrm{d}s + \\ &+ |\partial S| h \int_0^t \int_0^1 |u_1|^2 \;\mathrm{d}x_1\mathrm{d}s = \int_0^t \int_0^1 \widehat{\rho f_1} u_1 \;\mathrm{d}x_1\mathrm{d}s + \qquad (26) \\ &+ \int_0^1 \frac{|(\rho\mathbf{u})_0|^2}{2\rho_0} \;\mathrm{d}x_1 + \int_0^1 \rho_0 \ln(\rho_0) \;\mathrm{d}x_1. \qquad (27) \end{aligned}$$

The following theorem summarizes our first main result.

Theorem 1. *Let us assume that couples* $(\rho_\varepsilon, \mathbf{u}_\varepsilon)$*,* $\varepsilon \in (0, 1)$*, satisfying*

$$\begin{aligned} &\rho_\varepsilon \in L^\infty(0, T; L_{\Phi_\gamma}(\Omega)), \\ &\mathbf{v}_\varepsilon \in L^p(0, T; [W^{1,p}(\Omega)]^3) \cap L^2(0, T; [L^2(\partial\Omega)]^3), \end{aligned}$$

with $\mathbf{v}_\varepsilon = (u_{1,\varepsilon}, \varepsilon u_{2,\varepsilon}, \varepsilon u_{3,\varepsilon})$ *and* Ω *being not axially symmetric,* $\partial\Omega \in \mathcal{C}^{0,1}$*, are weak solutions to the equations (13)-(14), complying with energy equality (15),* *with initial states* $\rho_{0,\varepsilon} \in L_{\Phi_\gamma}(\Omega)$ *and* $\frac{|(\rho_\varepsilon \mathbf{u}_\varepsilon)_0|^2}{2\rho_{0,\varepsilon}} \in L^1(\Omega)$ *satisfying*

$$\int_S \rho_{0,\varepsilon} \ln(\rho_{0,\varepsilon})\ \mathrm{d}x_2 \mathrm{d}x_3 \to \rho_0 \ln(\rho_0) \quad \text{in } L^1(0,1), \tag{28}$$

$$\int_S \Phi_\gamma(\rho_{0,\varepsilon})\ \mathrm{d}x_2 \mathrm{d}x_3 \to \Phi_\gamma(\rho_0) \quad \text{in } L^1(0,1), \tag{29}$$

$$\int_S \frac{|(\rho_\varepsilon \mathbf{u}_\varepsilon)_0|^2}{2\rho_{0,\varepsilon}}\ \mathrm{d}x_2 \mathrm{d}x_3 \to \frac{|(\rho \mathbf{u})_0|^2}{2\rho_0} \quad \text{in } L^1(0,1), \tag{30}$$

for $\varepsilon \to 0$*, and for arbitrary but fixed* $\gamma > 3$ *and* $p > 3$*. In addition, we assume that Navier boundary conditions (8)-(10) hold and* $\omega_\varepsilon(\mathbf{u}_\varepsilon) \in [\tilde{L}_M(\Omega \times (0,T))]^9$*.*

Further, we suppose that function P *complies with conditions (3)-(7),* $\mathbf{f}_\varepsilon \to \mathbf{f}$ *in* $[L^\infty(\Omega \times (0,T))]^3$*,* $h(\varepsilon) > 0$ *behaves like* $O(\varepsilon)$*, see (8), and* $q > 0$*, see (9). Then (passing to subsequences if necessary)*

$$\begin{aligned}
\rho_\varepsilon &\overset{*}{\rightharpoonup} \rho \quad \text{in } L^\infty\left(0,T; L_{\Phi_\gamma}(\Omega)\right),\\
\rho_\varepsilon &\to \rho \quad \text{in } \mathcal{C}\left(\langle 0,T\rangle; [W^{1,p}(\Omega)]^*\right),\\
\omega_\varepsilon(\mathbf{u}_\varepsilon) &\overset{N}{\rightharpoonup} \omega(\mathbf{u})\\
u_{1,\varepsilon} &\rightharpoonup u_1 \quad \text{in } L^p\left(0,T; W^{1,p}(\Omega)\right) \cap L^2\left(0,T; L^2(\partial\Omega)\right),\\
u_{\alpha,\varepsilon} &\to 0 \quad \text{in } L_M(\Omega \times (0,T)),\ \alpha = 2,3.
\end{aligned}$$

In addition, couple $(\hat{\rho}, u_1)$*, where* $u_1 = u_1(x_1)$ *and* $\hat{\rho} = \int_S \rho\ \mathrm{d}x_2 \mathrm{d}x_3$*, is a weak solution to the equations (24)-(25) and complies with the energy equality (26).*

Second, the thesis is devoted to a rigorous asymptotic analysis of the three-dimensional Navier-Stokes equations acting over a curved domain. We applied a similar approach as in [1] to arrive at the limit of the governing equations and energy equality. However, the deformation of the domain introduced new difficulties which had to be addressed. Finally, we overcame all the difficulties and presented the limit equations and energy equality as

$$\int_0^T \int_S \left[\hat{\rho}\partial_t \varphi + \hat{\rho}\hat{\mathbf{u}}^{\mathrm{T}} R^{12} \hat{\nabla}\varphi\right] \sqrt{d}\ \mathrm{d}\hat{x}\mathrm{d}t = 0, \tag{31}$$

for any $\varphi \in \mathcal{D}\left(\mathbb{R}^2 \times \langle 0, T\rangle\right)$,

$$\int_0^T \int_S \left[\hat{\rho}\hat{\mathbf{u}} \cdot \partial_t \psi + \hat{\rho}\hat{\mathbf{u}} \otimes \hat{\mathbf{u}} : \omega(\psi) + \hat{\rho}\hat{\nabla}\psi : R^{12}\right] \sqrt{d}\ \mathrm{d}\hat{x}\mathrm{d}t =$$

$$= \int_0^T \int_S P(|\omega(\hat{\mathbf{u}})|)\omega(\hat{\mathbf{u}}) : \omega(\psi)\sqrt{d}\ \mathrm{d}\hat{x}\mathrm{d}t - \int_0^T \int_S \widehat{\rho\mathbf{F}} \cdot \psi\sqrt{d}\ \mathrm{d}\hat{x}\mathrm{d}t +$$

$$+q\int_0^T \int_{\partial S} \hat{\mathbf{u}} \cdot \psi |R^{12}\hat{\mathbf{n}}|\sqrt{d}\ \mathrm{d}S\mathrm{d}t + 2h\int_0^T \int_S \hat{\mathbf{u}} \cdot \psi\sqrt{d}\ \mathrm{d}\hat{x}\mathrm{d}t, \tag{32}$$

for any $\psi \in \mathcal{C}_0^\infty(0, T; [\mathcal{C}^\infty(\Omega)]^3)$ such that $\partial_3\psi = 0$, $\psi \cdot \mathbf{a}_3 = 0$ in $\Omega \times (0, T)$ and $\psi \cdot \mathbf{n}|_{\partial S\times(0,T)} = 0$, and

$$\int_S \left(\hat{\rho}\frac{|\hat{\mathbf{u}}|^2}{2} + \hat{\rho}\ln(\hat{\rho})\right)\sqrt{d}\ \mathrm{d}\hat{x} + \int_0^t \int_S P\left(|\omega(\hat{\mathbf{u}})|\right)|\omega(\hat{\mathbf{u}})|^2\sqrt{d}\ \mathrm{d}\hat{x}\mathrm{d}s +$$

$$+q\int_0^t \int_{\partial S} |\hat{\mathbf{u}}|^2 |R^{12}\hat{\mathbf{n}}|\sqrt{d}\ \mathrm{d}S\mathrm{d}s + 2h\int_0^t \int_S |\hat{\mathbf{u}}|^2\sqrt{d}\ \mathrm{d}\hat{x}\mathrm{d}s = \tag{33}$$

$$= \int_0^t \int_S \widehat{\rho\mathbf{F}} \cdot \hat{\mathbf{u}}\sqrt{d}\ \mathrm{d}\hat{x}\mathrm{d}s + \int_S \frac{|(\rho\mathbf{u})_0|^2}{2\rho_0}\sqrt{d}\ \mathrm{d}\tilde{x} + \int_S \rho_0\ln(\rho_0)\sqrt{d}\ \mathrm{d}\tilde{x}.$$

Our second main contribution is summarized in the following theorem.

Theorem 2. *Let us assume that couples $(\rho_\varepsilon, \mathbf{u}_\varepsilon)$, $\varepsilon \in (0, 1)$, satisfying*

$$\rho_\varepsilon \in L^\infty(0, T; L_{\Phi_\gamma}(\Omega)),$$
$$\mathbf{v}_\varepsilon \in L^p(0, T; [W^{1,p}(\Omega)]^3) \cap L^2(0, T; [L^2(\partial\Omega)]^3)$$

with $\mathbf{v}_\varepsilon = (\mathbf{u}_\varepsilon \cdot \mathbf{g}_{1,\varepsilon}, \mathbf{u}_\varepsilon \cdot \mathbf{g}_{2,\varepsilon}, \mathbf{u}_\varepsilon \cdot \mathbf{g}_{3,\varepsilon})$ for arbitrary but fixed $\gamma > 3$ and $p > 3$, are weak solutions to the transformed equations (21)-(22), complying with energy equality (23), with initial states $\rho_{0,\varepsilon} \in L_{\Phi_\gamma}(\Omega)$ and $\frac{|(\rho_\varepsilon\mathbf{u}_\varepsilon)_0|^2}{2\rho_{0,\varepsilon}}\sqrt{d_\varepsilon} \in L^1(\Omega)$ satisfying

$$\int_0^1 \rho_{0,\varepsilon}\ln(\rho_{0,\varepsilon})\sqrt{d_\varepsilon}\ \mathrm{d}x_3 \to \rho_0\ln(\rho_0)\sqrt{d} \quad \text{in } L^1(S), \tag{34}$$

$$\int_0^1 \Phi_\gamma(\rho_{0,\varepsilon})\sqrt{d_\varepsilon}\ \mathrm{d}x_3 \to \Phi_\gamma(\rho_0)\sqrt{d} \quad \text{in } L^1(S),\ \gamma > 3, \tag{35}$$

$$\int_0^1 \frac{|(\rho_\varepsilon\mathbf{u}_\varepsilon)_0|^2}{2\rho_{0,\varepsilon}}\sqrt{d_\varepsilon}\ \mathrm{d}x_3 \to \frac{|(\rho\mathbf{u})_0|^2}{2\rho_0}\sqrt{d} \quad \text{in } L^1(S), \tag{36}$$

for $\varepsilon \to 0$. In addition, we assume that Navier boundary conditions (16)-(18) hold and $\omega_\varepsilon(\mathbf{u}_\varepsilon) \in [\tilde{L}_M(\Omega \times (0,T))]^9$.

Further, we suppose that function P complies with conditions (3)-(7), $\mathbf{f}_\varepsilon \to \mathbf{f}$ in $[L^\infty(\Omega \times (0,T))]^3$ and $\mathbf{f}_\varepsilon \cdot \mathbf{g}^{j,\varepsilon} \in [L^\infty(\Omega \times (0,T))]^3$, $j = 1,2,3$, $h(\varepsilon) > 0$ behaves like $O(\varepsilon)$, $q > 0$ and covariant basis $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} \subset [L^\infty(\Omega)]^3$ satisfies conditions $\partial_\alpha \mathbf{a}_j$ and $\partial^2_{\alpha\beta}\mathbf{a}_3 \in [L^\infty(\Omega)]^3$, where α, $\beta = 1,2$ and $j = 1,2,3$. Then (passing to subsequences if necessary)

$$\begin{aligned}
\rho_\varepsilon &\overset{*}{\rightharpoonup} \rho \quad \text{in } L^\infty\left(0,T; L_{\Phi_\gamma}(\Omega)\right), \\
\rho_\varepsilon &\to \rho \quad \text{in } \mathcal{C}\left(\langle 0,T\rangle; [W^1 L_{\Phi_\gamma}(\Omega)]^*\right), \\
\omega_\varepsilon(\mathbf{u}_\varepsilon) &\overset{N}{\rightharpoonup} \omega(\mathbf{u}) \\
\mathbf{u}_\varepsilon \cdot \mathbf{g}_{\alpha,\varepsilon} &\rightharpoonup \mathbf{u} \cdot \mathbf{a}_\alpha \quad \text{in } L^p\left(0,T; W^{1,p}(\Omega)\right) \cap L^2\left(0,T; L^2(\partial\Omega)\right), \\
&\qquad \alpha = 1,2, \\
\mathbf{u}_\varepsilon \cdot \mathbf{a}_3 &\to 0 \quad \text{in } L_M(\Omega \times (0,T)).
\end{aligned}$$

In addition, couple $(\hat{\rho}, \hat{\mathbf{u}})$, where $\hat{\rho} = \int_0^1 \rho \, \mathrm{d}x_3$ and $\hat{\mathbf{u}} = (\mathbf{u} \cdot \mathbf{a}_1)\mathbf{a}^1 + (\mathbf{u} \cdot \mathbf{a}_2)\mathbf{a}^2$, $\hat{\mathbf{u}} \cdot \hat{\mathbf{n}}|_{\partial S \times (0,T)} = 0$, is a weak solution to the equations (31)-(32) and complies with the energy equality (33).

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- Andrášik, R., Vodák, R.: Compressible nonlinearly viscous fluids: Asymptotic analysis in a 3D curved domain, J. Math. Fluid Mech. 21: 13 (2019), https://doi.org/10.1007/s00021-019-0412-y.
- Favilli, F., Bíl, M., Sedoník, J., Andrášik, R., Kasal, P., Agreiter, A., Streifeneder, T.: Application of KDE+ software to identify collective risk hotspots of ungulate-vehicle collisions in South Tyrol, Northern Italy, European Journal of Wildlife Research 64(5), 59 (2018).
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List of conferences

- Roadkill and wildlife-vehicle collisions on transportation infrastructure: Causes, outcomes and mitigation, Brno (CZ): "WVC hotspots with large mammals: Which factors make them dangerous?", November 15, 2018.
- ICSC 2018 International Cycling Safety Conference, Barcelona (ES): "An analysis of bicycle crash time-series, Czechia, 1995 – 2017" (poster), October 10 – 11, 2018.
- IENE 2018 International Conference, Eindhoven (NL): "Animal-vehicle collisions: Improvement of regression models with the use of cluster analysis", "Which factors are different between WVC hotspots with large mammals and randomly selected sites along roads?", "KDE+ workshop: new approaches to WVC hotspot identification" (workshop co-organizer), September 10 – 14, 2018.
- Olomoucian Days of Applied Mathematics ODAM 2017, Olomouc (CZ): "Compressible nonlinearly viscous fluids: Asymptotic analysis in a 3D curved domain", May 31 – June 2, 2017.
- International Conference on Ecology and Transportation ICOET 2017, Salt Lake City (US): "What are the traffic intensities of crashes with animals?" (poster), May 14 – 18, 2017.
- European Safety and Reliability Conference ESREL 2016, Glasgow (UK): "How (not) to work with small probabilities: Evaluating the individual risk of railway transport", "Traffic accident hotspots: Identifying the boundary between the signal and the noise", September 25 – 29, 2016.
- IENE 2016 International Conference, Lyon (FR): "A comparison of certain methods for spatial analysis of animal-vehicle collisions", "A workshop on KDE+: A method for identification of animal-vehicle-collision hotspots" (workshop co-organizer), August 30 – September 2, 2016.
- EUSTO Final International Conference, Dresden (DE): "Vulnerability of network systems", Dresden, May 26 – 27, 2016.

- European Geosciences Union General Assembly 2016, Vienna (AT): "RUPOK – a web-based application for assessment of impacts of natural hazards on the transportation infrastructure" (poster), April 17 – 22, 2016.
- Mathematics and Computer Science in Practice: Potential and Reality, Prague (CZ), December 9 – 11, 2015 (participant).
- European Safety and Reliability Conference ESREL 2015, Zürich (CH): "Traffic accidents: Random or pattern occurrence?", September 7 – 10, 2015.
- Young Researchers Seminar 2015, Rome (IT): "KDE+ method: Clustering of traffic accidents", June 17 – 19, 2015.
- ICSC 2014 International Cycling Safety Conference, Göteborg (SE): "Fatal cycling accidents in the Czech Republic: Factors and causes of death" (poster), November 18 – 19, 2014.
- IENE 2014 International Conference, Malmö (SE): "A new method for identification of clusters of animal-vehicle collisions on road network" (poster, awarded for the best poster), September 15 – 19, 2014.
- Particles in Flows, Summer School and Workshop, Prague (CZ), August 25 – 31, 2014 (participant).
- International Symposium on Modern Mathematics and Mechanics in Olomouc, Olomouc (CZ), June 23 – 27, 2014 (co-organizer).
- Modern mathematical methods in engineering (3mi), Horní Lomná (CZ): "Inverse problems with application to the derivative reconstruction and the inverse Laplace transform from noisy data", June 2 – 4, 2014.
- Equadiff 2013, Prague (CZ): "Primal-dual nonlinear rescaling method with dynamic scaling parameter update for optimization arising from 3D contact problems", August 26 – 30, 2013.
- Olomoucian Days of Applied Mathematics ODAM 2013, Olomouc (CZ): "Nonlinear rescaling method and self-concordant functions", June 12 – 14, 2013.
- Modern mathematical methods in engineering (3mi), Horní Lomná (CZ): "Primal-dual nonlinear rescaling method with dynamic scaling parameter update for convex optimization problems", June 3 – 5, 2013.
- Mathematical Theory in Fluid Mechanics, the 13th School, Kácov (CZ), May 24 – 31, 2013 (participant).

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